ALMOST CONSTRAINED SUBSPACES OF BANACH SPACES

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Abstract. In this paper, we obtain some sufficient conditions for an almost constrained subspace to be constrained (in fact, by a unique norm 1 projection), which improves significantly upon all existing conditions of similar type with significantly simpler proofs.

1. Introduction

Let $X$ be a real Banach space. We will denote by $B_X[x,r]$ the closed ball of radius $r > 0$ around $x \in X$. We will identify any element $x \in X$ with its canonical image in $X^{**}$. Unless otherwise specified, all subspaces we consider are norm closed. Our notations are otherwise standard. Any unexplained terminology can be found in either [4] or [9].

Recall that a subspace $Y$ of $X$ is called 1-complemented or constrained if there is a norm 1 projection on $X$ with range $Y$.

Definition 1.1 ([7]). A Banach space $X$ is said to have the finite-infinite intersection property ($IP_{f,\infty}$) if every family of closed balls in $X$ with empty intersection contains a finite subfamily with empty intersection.

It is well known that dual spaces and their constrained subspaces have $IP_{f,\infty}$. By $w^*$-compactness of the dual ball and the Principle of Local Reflexivity, it can be shown (see e.g., [7]) that $X$ has the $IP_{f,\infty}$ if and only if any family of closed balls centred at points of $X$ that intersects in $X^{**}$ also intersects in $X$. With this in mind, we define

Definition 1.2 ([1]). A subspace $Y$ of $X$ is said to be an almost constrained ($AC$) subspace of $X$ if any family of closed balls centred at points of $Y$ that intersects in $X$ also intersects in $Y$.

Thus, $X$ has the $IP_{f,\infty}$ if and only if $X$ is an $AC$-subspace of $X^{**}$. Clearly, any constrained subspace is an $AC$-subspace. In the case of $IP_{f,\infty}$, whether the converse is also true remains an open question (see [12, Remark 2, page 60], also [6, X(10)]). However, we will give an example to show that an $AC$-subspace need not, in general, be constrained.

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In addition, we apply some tools and techniques developed in [1] to obtain sufficient conditions for an AC-subspace to be constrained, much in the spirit of [6, 7]. Our condition is in terms of functionals with “locally unique” Hahn-Banach (i.e., norm-preserving) extensions, which improves significantly upon all existing conditions of similar type, as noted in [3, 8], and has significantly simpler proof. As in [6, 7], these conditions actually imply the existence of a unique norm 1 projection.

**Definition 1.3.** Let $Y$ be subspace of $X$.

(a) For $y^* \in Y^*$, $HB(y^*) = \{ x^* \in X^* : x^*|_Y = y^* \text{ and } \|x^*\| = \|y^*\| \}$.

(b) $Y$ is a $U$-subspace of $X$ if for any $y^* \in Y^*$, $HB(y^*)$ is a singleton. $X$ is said to be Hahn-Banach smooth if $X$ is a $U$-subspace of $X^{**}$.

(c) The duality mapping $D$ for $X$ is the set-valued map from $S(X)$ to $S(X^*)$ defined by

$$D(x) = \{ x^* \in S(X^*) : x^*(x) = 1 \}, \quad x \in S(X).$$

(d) $x \in S(X)$ is a smooth point of $B(X)$ if $D(x)$ is a singleton.

(e) $Y$ is a weakly $U$-subspace of $X$ if for every $y^* \in D(S(Y))$, $HB(y^*)$ is a singleton.

$X$ is weakly Hahn-Banach smooth if $X$ is a weakly $U$-subspace of $X^{**}$.

If $Y$ is a $U$-subspace, or even a weakly $U$-subspace of $X$, then it satisfies our sufficient condition. It is shown in [3, Theorem 2] that an AC-subspace $Y$ is constrained in $X$ if every point of $S(Y)$ is a smooth point of $B(X)$. We show that this happens if and only if every subspace $Z$ of $Y$ is a weakly $U$-subspace of $X$. Thus, our condition is weaker.

It follows from our result that $X$ is smooth if and only if every subspace of $X$ is a weakly $U$-subspace. This parallels the classical result of Taylor-Foguel [15] that $X^*$ is strictly convex if and only if every subspace of $X$ is a $U$-subspace.

2. Some characterizations and a counterexample

We will use the following notation:

**Notation.** Let $Y$ be a subspace of $X$. For all $x \in X$,

$$\mathfrak{P}(x) = \bigcap_{y \in Y} B_Y[y, \|x - y\|].$$

Clearly, $\mathfrak{P}(y) = \{ y \}$ for all $y \in Y$. Also, $Y$ is an AC-subspace of $X$ if and only if $\mathfrak{P}(x) \neq \emptyset$ for all $x \in X$.

We recall a definition from [7].

**Definition 2.1.** Let $Y$ be a subspace of $X$. We define

$$O(Y, X) = \{ x \in X : \|x - y\| \geq \|y\| \text{ for all } y \in Y \}.$$  

$O(X, X^{**})$ is denoted by $O(X)$.

The following proposition characterizes AC-subspaces.

**Proposition 2.2.** For a subspace $Y$ of $X$, the following are equivalent:

(a) $Y$ is an AC-subspace of $X$.

(b) For all $x \in X$, there exists $y \in Y$ and $z \in O(Y, X)$ such that $x = y + z$.

(c) For every subspace $Z$ of $X$ such that $Y \subseteq Z$ and $\dim(Z/Y) = 1$, $Y$ is constrained in $Z$. 


Proof. (a) ⇒ (b). Let \( x_0 \in X \). By (a), there exists \( y_0 \in \mathcal{P}(x_0) \). This implies \( \|y_0 - y\| \leq \|x_0 - y\| \) for all \( y \in Y \). Or, putting \( u = y_0 - y \), \( \|u\| \leq \|x_0 - y_0 + u\| \) for all \( u \in Y \). That is, \( z_0 = x_0 - y_0 \in O(Y, X) \) and \( x_0 = y_0 + z_0 \).

(b) ⇒ (c). Let \( Z \) be as in (c). Then one can write \( Z = \text{span}(Y \cup \{x_0\}) \) for some \( x_0 \in X \). By (b), there exists \( y_0 \in Y \) and \( z_0 \in O(Y, X) \) such that \( x_0 = y_0 + z_0 \). It follows that \( Z = Y \oplus \mathbb{R}z_0 \). But then, by definition of \( O(Y, X) \), \( \alpha z_0 + y \mapsto y \) is a norm 1 projection from \( Z \) onto \( Y \).

(c) ⇒ (a). By (c), for every \( x \in X \), there is a norm 1 projection \( P_x \) from \( Z_x = \text{span}(Y \cup \{x\}) \) onto \( Y \). Clearly, \( P_x(x) \in \mathcal{P}(x) \).

Recall that a hyperplane \( H \) in \( X \) is a subspace such that \( H = \ker(x^*) \) for some \( x^* \in S(X^*) \). Since \( \dim(X/H) = 1 \), we get

**Corollary 2.3.** Suppose \( H \) is a hyperplane in \( X \). Then \( H \) is an AC-subspace if and only if \( H \) is constrained in \( X \).

**Corollary 2.4.** A subspace \( Y \) is an AC-subspace of \( X \) if and only if there is a (not necessarily linear) map \( P \) from \( X \) onto \( Y \) satisfying the following properties:

(a) \( P^2 = P \);
(b) \( P(x) = \lambda P(x) \) for all \( x \in X \), \( \lambda \in \mathbb{R} \);
(c) \( P(x + y) = P(x) + y \) for all \( x \in X \), \( y \in Y \);
(d) \( \|P(x)\| \leq \|x\| \) for all \( x \in X \).

Proof. If \( P \) is as above, then clearly for any \( x \in X \), \( P(x) \in \mathcal{P}(x) \). Thus, \( Y \) is an AC-subspace of \( X \).

Conversely, let \( Y \) be an AC-subspace of \( X \). For \( z \in O(Y, X) \), let \( Z_z = Y \oplus \mathbb{R}z \) and \( P_z \) be a norm 1 projection from \( Y_z \) onto \( Y \). Observe that for \( z_1, z_2 \in O(Y, X) \), either \( Y_{z_1} \cap Y_{z_2} = Y \) or \( Y_{z_1} = Y_{z_2} \). By Proposition 2.2(b), \( \bigcup_{z \in O(Y, X)} Y_z = X \).

Define \( P : X \rightarrow Y \) by \( P(x) = P_z(x) \), if \( x \in Y_z \). Then \( P \) is well-defined and satisfies all the listed properties.

**Remark 2.5.** Proposition 2.2(a) ⇔ (c) for the case of \( IP_{f, \infty} \) was noted in [12] Theorem 5.9. Corollary 2.3 was also noted in [11]. Corollary 2.4 for the case of \( IP_{f, \infty} \) was noted in [8] Theorem 2. In all these cases, our proof is simpler.

Let us note that in Proposition 2.2(b), the representation \( x = y + z \) with \( y \in Y \) and \( z \in O(Y, X) \) need not be unique.

**Example 2.6.** We now give an example to show that an AC-subspace need not, in general, be constrained. We need the following result (we thank Professor T.S.S.R.K. Rao of ISI, Bangalore, for drawing our attention to this result).

**Theorem 2.7 ([11]).** There exist Banach spaces \( Z \supseteq X \) with \( \dim(Z/X) = 2 \) satisfying

(i) There is no projection with norm 1 from \( Z \) onto \( X \).
(ii) For every \( \varepsilon > 0 \), there is a projection with norm \( \leq 1 + \varepsilon \) from \( Z \) onto \( X \).
(iii) For every \( Y \) with \( Z \supseteq Y \supseteq X \) and \( \dim(Y/X) = 1 \), there is a projection with norm 1 from \( Y \) onto \( X \).

By Proposition 2.2 (iii) implies that \( X \) is an AC-subspace of \( Z \), while by (i), there is no norm 1 projection from \( Z \) onto \( X \).

**Definition 2.8.** (a) [10] A Banach space \( X \) such that \( X^* \) is isometrically isomorphic to \( L^1(\mu) \) for some positive measure \( \mu \) is called an \( L^1 \)-predual.

(b) A Banach space is a \( P_1 \)-space if it is constrained in every superspace.
Remark 2.9. (a) From the results of [11] Chapter 3, it follows that $X$ is a real $L^1$-predual with $IP_{f,\infty}$ if and only if $X$ is a real $\mathcal{P}_1$-space. In particular, $X$ is constrained in $X^{**}$.

(b) It can be shown that the space $X$ in Example 2.6 is not constrained in $X^{**}$. Therefore, it could have been a possible counterexample to the $IP_{f,\infty}$ question as well. But, from the construction in [11], it is clear that the space $X$ is a real $L^1$-predual, but not a real $\mathcal{P}_1$-space. Thus it lacks the $IP_{f,\infty}$.

3. Some sufficient conditions

We now obtain sufficient conditions for an AC-subspace to be constrained. Some preliminaries first. As we introduce the following notation.

**Definition 3.1.** Let $Y$ be a subspace of $X$. For $x \in X$ and $y^* \in Y^*$, put

$$U(x, y^*) = \inf \{ y^*(y) + \|x - y\| : y \in Y \},$$

$$L(x, y^*) = \sup \{ y^*(y) - \|x - y\| : y \in Y \}.$$

For $x^* \in X^*$, we will write $U(x, x^*)$ for $U(x, x^*|_Y)$. Let $C(x) = \{ x^* \in B(X^*) : U(x, x^*) = L(x, x^*) \}$, for $x \in X$, and $C = \bigcap_{x \in X} C(x)$.

The following result is immediate from the proof of the Hahn-Banach Theorem (see, e.g., [11] Section 48).

**Lemma 3.2.** Let $Y$ be a subspace of $X$, $x_0 \notin Y$ and $y^* \in S(Y^*)$. Then $L(x_0, y^*) \leq U(x_0, y^*)$ and $\alpha$ lies between these two numbers if and only if there exists $x^* \in HB(y^*)$ with $x^*(x_0) = \alpha$.

**Remark 3.3.** It is clear that for any $x^* \in B(X^*)$ and $x \in X$, $L(x, x^*) \leq x^*(x) \leq U(x, x^*)$ and for any $y^* \in S(Y^*)$, $HB(y^*)$ is singleton if and only if for all $x \in X$, $L(x, y^*) = U(x, y^*)$.

The next three results are from [11]. We include the proofs for the sake of completeness.

**Lemma 3.4.** Let $Y$ be a subspace of $X$. For $x_1, x_2 \in X$, the following are equivalent:

(a) $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$.

(b) For all $x^* \in B(X^*)$, $U(x_2, x^*) \leq U(x_1, x^*)$.

**Proof.** Clearly, $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$ if and only if $\|x_2 - y\| \leq \|x_1 - y\|$, for all $y \in Y$.

(a) $\implies$ (b). If for all $y \in Y$, $\|x_2 - y\| \leq \|x_1 - y\|$, then for all $x^* \in B(X^*)$, $x^*(y) + \|x_2 - y\| \leq x^*(y) + \|x_1 - y\|$. Therefore, $U(x_2, x^*) \leq U(x_1, x^*)$.

(b) $\implies$ (a). Suppose $\|x_2 - y_0\| > \|x_1 - y_0\|$ for some $y_0 \in Y$. Then there exists $\varepsilon > 0$ such that $\|x_2 - y_0\| - \varepsilon \geq \|x_1 - y_0\|$. Choose $x^* \in B(X^*)$ such that $\|x_1 - y_0\| \leq \|x_2 - y_0\| - \varepsilon < x^*(x_2 - y_0) - \varepsilon/2$. Thus $U(x_1, x^*) \leq x^*(y_0) + \|x_1 - y_0\| < x^*(x_2) - \varepsilon/2 < U(x_2, x^*)$. \hfill $\square$

**Proposition 3.5.** Let $Y$ be a subspace of $X$, $x^* \in B(X^*)$ and $x_0 \in X \setminus Y$. The following are equivalent:

(a) $x^* \in C(x_0)$.

(b) $\|x^*|_Y\| = 1$ and every $x^*_1 \in HB(x^*|_Y)$ takes the same value at $x_0$. 

Hence and $O$ can be written as
(a) Proof. Since $x \to y$ there is $\lim_{x \to y} x_n(x_0) = x^*(x_0)$. For a subspace $Y$ Proposition 3.7.
(b) The result now follows from Lemma 3.2.
(c) Since $x_n(x_0) = x^*(x_0)$ with $x^*(x_0) \neq x_1^*(x_0)$, then the constant sequence $x_n = x_1^*$ clearly satisfies $\lim_n x_n^*(y) = x^*(y)$ for all $y \in Y$, but $\{x_n^*(x_0)\}$ cannot converge to $x^*(x_0)$.

Proposition 3.6. Let $Y$ be a subspace of $X$. For $x^* \in B(X^*)$, the following are equivalent:
(a) $x^* \in C$.
(b) $\|x^*\| = 1$ and $HB(x^*|Y) = \{x^*\}$.
(c) $\|x^*|Y\| = 1$ and if $\{x^*_\alpha\} \subseteq S(X^*)$ is a net such that $x^*_\alpha|Y \to x^*|Y$ in the $w^*$-topology of $X^*$, then $x^*_\alpha \to x^*$ in the $w^*$-topology of $X^*$.
(d) $\|x^*|Y\| = 1$ and if $\{x^*_\alpha\} \subseteq S(X^*)$ is such that $x^*_\alpha|Y \to x^*|Y$ in the $w^*$-topology of $Y^*$, then $x^*_\alpha \to x^*$ in the $w^*$-topology of $Y^*$.

Here is our first sufficient condition for an AC-subspace to be constrained.

Proposition 3.7. For a subspace $Y$ of $X$, the following are equivalent:
(a) $Y$ is an AC-subspace of $X$ and $O(Y, X)$ is a closed subspace of $X$.
(b) $Y$ is an AC-subspace of $X$ and $O(Y, X)$ is a linear subspace of $X$.
(c) $Y$ is constrained in $X$ and for all $x \in X$, $\Psi(x)$ is a singleton.

Moreover, in this case, $Y$ is constrained by a unique norm 1 projection.

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). Since $Y$ is an AC-subspace of $X$, by Proposition 2.2 any $x \in X$ can be written as $x = y + z$, where $y \in Y$ and $z \in O(Y, X)$. Since both $Y$ and $O(Y, X)$ are linear subspaces and $Y \cap O(Y, X) = \{0\}$, this representation is unique and $x \mapsto y$ is a well-defined linear map. Since $z \in O(Y, X)$, this map is of norm 1. Hence $Y$ is constrained in $X$. Moreover, since $y \in \Psi(x)$, $\Psi(x)$ is single-valued.

(c) $\Rightarrow$ (a). Let $Y$ be constrained in $X$ by a norm 1 projection $P$ and for all $x \in X$, let $\Psi(x)$ be a singleton. Clearly, $Y$ is an AC-subspace of $X$ and for all
$x \in X$, $\Psi(x) = \{P(x)\}$. It is easy to see that $\ker(P) \subseteq O(Y, X)$ and since for all $x \in X$, $\Psi(x) = \{P(x)\}$, $\ker(P) \supseteq O(Y, X)$. Thus, $O(Y, X) = \ker(P)$ is a closed subspace of $X$. □

**Remark 3.8.** (a) Even in the case of $IP_{f, \infty}$, this observation is new. References [6] and [7] discuss more complicated situations when $O(X)$, being a linear subspace of $X^{**}$, automatically implies that it is a $w^*$-closed subspace of $X^{**}$.

(b) We do not know if (c) can be replaced by “$Y$ is constrained by a unique norm 1 projection”.

(c) It follows from the proof that

$$\bigcup \{\ker(P) : P \text{ is a norm 1 projection onto } Y\} \subseteq O(Y, X).$$

Are these two sets equal?

The following result significantly improves [3, Lemma 2], which was also the key tool in [3].

**Lemma 3.9.** Let $Y$ be a subspace of $X$. Let $x_1, x_2 \in X$ be such that $x_1 \in \bigcap_{y \in Y} B_X[y, \|x_2 - y\|]$. Then for any $x^* \in C(x_2)$, $x^*(x_1 - x_2) = 0$.

**Proof.** Let $x_1, x_2 \in X$ be such that $x_1 \in \bigcap_{y \in Y} B_X[y, \|x_2 - y\|]$. Then, by Lemma 3.4 for all $x^* \in B(X^*)$,

$$L(x_2, x^*) \leq L(x_1, x^*) \leq U(x_1, x^*) \leq U(x_2, x^*).$$

Thus for $x^* \in C(x_2)$, equality holds. By Lemma 3.2 the result follows. □

Here is our main theorem.

**Theorem 3.10.** Let $Y$ be a subspace of $X$. Suppose

1. for every $x_1, x_2 \in X$, $C(x_1) \cap C(x_2)$ separates points of $Y$.

If $Y$ is an AC-subspace of $X$, then $Y$ is constrained in $X$. Moreover, the projection is unique and $O(Y, X)$ is a closed subspace of $X$.

**Proof.** Since $Y$ is an AC-subspace of $X$, $\Psi(x) \neq \emptyset$ for all $x \in X$. By Lemma 3.9 for all $x \in X$,

$$x^*(x - y) = 0 \quad \text{for any } x^* \in C(x), \quad y \in \Psi(x).$$

Now if $y_1, y_2 \in \Psi(x)$, then for any $x^* \in C(x)$, $x^*(x - y_1) = x^*(x - y_2) = 0$. Therefore, $x^*(y_1 - y_2) = 0$. By (1), $y_1 = y_2$. That is, $\Psi(x)$ is single-valued. Let $\Psi(x) = \{P(x)\}$. Then, $P$ satisfies all the properties listed in Corollary 2.4. So, it only remains to show that $P$ is additive.

Let $x_1, x_2 \in X$. If $x^* \in C(x_1) \cap C(x_2)$, then by Proposition 3.5 $x^* \in C(x_1 + x_2)$ and by (2), $x^*(x_1 - P(x_1)) = x^*(x_2 - P(x_2)) = x^*((x_1 + x_2) - P(x_1 + x_2)) = 0$. Therefore, $x^*(P(x_1 + x_2) - P(x_1) - P(x_2)) = 0$. By (1), $P(x_1 + x_2) = P(x_1) + P(x_2)$.

The rest of the result follows from Proposition 3.4. □

By Theorem 3.10 the condition “$C$ separates points of $Y$” is sufficient for an AC-subspace to be constrained by a unique norm 1 projection. This condition is clearly satisfied if $Y$ is a $U$-subspace, or even a weakly $U$ subspace of $X$.

It is shown in [3, Theorem 2] that an AC-subspace $Y$ is constrained in $X$ by a unique norm 1 projection if every point of $S(Y)$ is a smooth point of $B(X)$. By the following result, our condition is much weaker.
Proposition 3.11. Every point of $S(Y)$ is a smooth point of $B(X)$ if and only if every subspace $Z$ of $Y$ is a weakly $U$-subspace of $X$. In particular, $X$ is smooth if and only if every subspace of $X$ is a weakly $U$-subspace of $X$.

Proof. Suppose every point of $S(Y)$ is a smooth point of $B(X)$. Let $Z$ be any subspace of $Y$. Suppose $z^* \in S(Z^*)$ attains its norm at $z_0 \in S(Z)$. By assumption, $z_0$ is a smooth point of $B(X)$. Now, $z^* \in D_Z(z_0)$ and $\text{HB}(z^*) \in D_X(z_0)$. Since $D_X(z_0)$ is a singleton, so is $\text{HB}(z^*)$. Thus, $Z$ is a weakly $U$-subspace of $X$.

Conversely, suppose there exists $y_0 \in S(Y)$ such that $D_X(y_0)$ is not a singleton. Suppose $\{x_1, x_2\} \subseteq D_X(y_0)$ and $x_1 \neq x_2$. Let $Z = \{x \in Y : x_1(x) = x_2(x)\}$. Then $y_0 \in S(Z)$ and therefore, $\|x_1^*|_Z\| = \|x_2^*|_Z\| = 1$. Thus, $z^* = x_1^*|_Z \in S(Z^*)$ attains its norm at $y_0 \in S(Z)$, but $\{x_1^*, x_2^*\} \subseteq \text{HB}(z^*)$. □

Example 3.12. As noted in [6], the space $X = L^\infty$ gives an example of a dual space such that there are infinitely many norm 1 projections from $X^{**}$ onto $X$. This produces an example of a space with $IP_{f,\infty}$ that is constrained in $X^{**}$, but $O(X)$ is not a closed subspace of $X^{**}$. This also shows that our sufficient condition, although weaker than the known ones, is still not necessary for an AC-subspace to be constrained.

We conclude the paper with some necessary and/or sufficient conditions for $O(Y, X)$ to be a closed subspace of $X$. First we need a characterization of $O(Y, X)$.

This is a slight improvement over that in [1].

Definition 3.13. We say $A \subseteq B(X^*)$ is a norming set for $X$ if $\|x\| = \sup\{x^*(x) : x^* \in A\}$ for all $x \in X$.

A subspace $F$ of $X^*$ is called a norming subspace if $B(F)$ is a norming set for $X$.

Lemma 3.14. Let $Y$ be a subspace of $X$. For $x \in X$, the following are equivalent:

(a) $x \in O(Y, X)$.
(b) $\ker(x)|_Y \subseteq Y^*$ is a norming subspace for $Y$.
(c) $0 \in \bigcap_{y \in Y} B_Y[y, \|x - y\|]$.
(d) For every $x^* \in B(X^*)$, $L(x, x^*) \leq 0 \leq U(x, x^*)$.
(e) For every $y^* \in B(Y^*)$, $L(x, y^*) \leq 0 \leq U(x, y^*)$.

Further, for a w*-closed subspace $F \subseteq X^*$, $F|_Y$ is a norming subspace for $Y$ if and only if $F_{\perp} \subseteq O(Y, X)$, where $F_{\perp} = \{x \in X : f(x) = 0 \text{ for all } f \in F\}$.

Proof. Let $F \subseteq X^*$ be a w*-closed subspace such that $F_{\perp} \subseteq O(Y, X)$. Then $F = (X/F_{\perp})^*$ and therefore, it suffices to show that $\|y\| = \|y + F_{\perp}\| = d(y, F_{\perp})$.

Clearly, $\|y\| \geq d(y, F_{\perp})$. Also, since $F_{\perp} \subseteq O(Y, X)$, for any $y \in Y$ and $z \in F_{\perp}$, $\|y + z\| \geq \|y\|$. Thus, $d(y, F_{\perp}) \geq \|y\|$.

Specializing to $F = \ker(x)$, we get (a) ⇒ (b).

(b) ⇒ (a). Since $\ker(x)|_Y$ norms $Y$, $\|y\| = \|y|_{\ker(x)}\| = d(y, \mathbb{R}x)$ for all $y \in Y$. Hence $\|x - y\| \geq \inf_{\lambda \in \mathbb{R}} \|y - \lambda x\| = \|y\|$ for all $y \in Y$. Thus, $x \in O(Y, X)$.

Now suppose $F \subseteq X^*$ is a w*-closed subspace such that $F|_Y$ is a norming subspace for $Y$. If $x \in F_{\perp}$, then $F \subseteq \ker(x)$ and therefore, $x \in O(Y, X)$. That is, $F_{\perp} \subseteq O(Y, X)$.

(a) ⇔ (c) and (d) ⇒ (e) are immediate from definition, while (c) ⇒ (d) follows from Lemma [3.3].
Let $\mathcal{N} = \{F : F$ is a $\ast$-closed subspace of $X^*$ and $F|_Y$ is a norming subspace for $Y\}$ and $N = \bigcap \mathcal{N}$. Similar to [2], we observe

**Proposition 3.15.** Let $Y$ be a subspace of $X$. $O(Y, X)$ is a closed subspace of $X$ if and only if $N|_Y$ is a norming subspace for $Y$. In particular, this happens if $C|_Y$ is a norming set for $Y$.

**Proof.** By Lemma 3.14, $F \in \mathcal{N}$ if and only if $F_\perp \subseteq O(Y, X)$. Thus if $N|_Y$ norms $Y$, then $N \in \mathcal{N}$ and hence, $N_\perp \subseteq O(Y, X)$. On the other hand, if $x \in O(Y, X)$, then $\ker(x) \in \mathcal{N}$, and hence, $N \subseteq \ker(x)$. That is, $x \in N_\perp$. Therefore, $O(Y, X) = N_\perp$, and $O(Y, X)$ is a closed subspace of $X$.

Conversely, if $O(Y, X)$ is a closed subspace of $X$ and $M = O(Y, X)_\perp$, then $M_\perp = O(Y, X)$ and therefore, $M \in \mathcal{N}$. Moreover, for every $F \in \mathcal{N}$, $F_\perp \subseteq O(Y, X) = M_\perp$, and hence, $M \subseteq F$. This shows $N = M$ and $N \in \mathcal{N}$.

Now, if $C|_Y$ is a norming set for $Y$, then as above, $C_\perp \subseteq O(Y, X)$.

Conversely let $x \in O(Y, X)$. Let $x^* \in C$. By Lemmas 3.12 and 3.14, there exists $z^* \in \text{HB}(x^*|_Y)$ such that $z^*(x) = 0$. Since $x^* \in C$, $\text{HB}(x^*|_Y) = \{x^*\}$, and we have $x^*(x) = 0$. Thus, $C_\perp = O(Y, X)$. \hfill $\square$

**Definition 3.16.** (a) [16] Let $Y$ be a subspace of $X$. Let $A(Y) = \{x^* \in B(X^*) : x^*|_Y$ is an extreme point of $B(Y^*)\}$.

(b) [9] A subspace $Y \subseteq X$ is said to be an $M$-ideal if there exists a subspace $N \subseteq X^*$ such that $X^* = Y^\perp \oplus_1 N$.

**Proposition 3.17.** In each of the following cases, $O(Y, X)$ is a closed subspace of $X$, a fortiori, if $Y$ is an $AC$-subspace, then $Y$ is constrained by a unique norm 1 projection.

(a) $Y$ is a weakly separating subspace of $X$.

(b) $Y$ is an $M$-ideal in $X$.

(c) $Y$ is a subspace of $X = C(K)$ containing the constants and separating points of $K$.

**Proof.** (a) A careful examination of the proof of [16] Lemma 1 actually shows that $A(Y) \subseteq C$. It is easy to see that $A(Y)$ is a norming set for $Y$. The result follows from Proposition 3.15.

(b) [9] Theorem I.1.12 observes that an $M$-ideal is a $U$-subspace.

(c) As observed in [16], such a $Y$ is weakly separating. \hfill $\square$

**Remark 3.18.** (a) In [16], it is shown that for a weakly separating subspace in $C(K)$, if there is a norm 1 projection, it must be unique. Clearly, our conclusion is stronger.

(b) In [13], it is shown that an $M$-ideal with the $IP_{f, \infty}$ is an $M$-summand. An argument similar to [2] Proposition 2.8 shows that an $M$-ideal $Y$ in $X$ with the
$IP_{\infty}$ is an $AC$-subspace of $X$. Thus, Proposition 3.17(b) improves the result in [13].

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