ALMOST CONSTRAINED SUBSPACES OF BANACH SPACES

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Abstract. In this paper, we obtain some sufficient conditions for an almost constrained subspace to be constrained (in fact, by a unique norm 1 projection), which improves significantly upon all existing conditions of similar type with significantly simpler proofs.

1. Introduction

Let $X$ be a real Banach space. We will denote by $B_X[x,r]$ the closed ball of radius $r > 0$ around $x \in X$. We will identify any element $x \in X$ with its canonical image in $X^{**}$. Unless otherwise specified, all subspaces we consider are norm closed. Our notations are otherwise standard. Any unexplained terminology can be found in either [4] or [9].

Recall that a subspace $Y$ of $X$ is called 1-complemented or constrained if there is a norm 1 projection on $X$ with range $Y$.

Definition 1.1 ([7]). A Banach space $X$ is said to have the finite-infinite intersection property ($IP_{f,\infty}$) if every family of closed balls in $X$ with empty intersection contains a finite subfamily with empty intersection.

It is well known that dual spaces and their constrained subspaces have $IP_{f,\infty}$. By w*-compactness of the dual ball and the Principle of Local Reflexivity, it can be shown (see e.g., [7]) that $X$ has the $IP_{f,\infty}$ if and only if any family of closed balls centred at points of $X$ that intersects in $X$ also intersects in $X^{**}$. With this in mind, we define

Definition 1.2 ([1]). A subspace $Y$ of $X$ is said to be an almost constrained ($AC$) subspace of $X$ if any family of closed balls centred at points of $Y$ that intersects in $X$ also intersects in $Y$.

Thus, $X$ has the $IP_{f,\infty}$ if and only if $X$ is an $AC$-subspace of $X^{**}$. Clearly, any constrained subspace is an $AC$-subspace. In the case of $IP_{f,\infty}$, whether the converse is also true remains an open question (see [12, Remark 2, page 60], also [6, X(10)]). However, we will give an example to show that an $AC$-subspace need not, in general, be constrained.
In addition, we apply some tools and techniques developed in [1] to obtain sufficient conditions for an AC-subspace to be constrained, much in the spirit of [6, 7]. Our condition is in terms of functionals with “locally unique” Hahn-Banach (i.e., norm-preserving) extensions, which improves significantly upon all existing conditions of similar type, as noted in [3, 8], and has significantly simpler proof. As in [6, 7], these conditions actually imply the existence of a unique norm 1 projection.

**Definition 1.3.** Let $Y$ be subspace of $X$.

1. For $y^* \in Y^*$, $\text{HB}(y^*) = \{ x^* \in X^* : x^*|_Y = y^* \text{ and } \|x^*\| = \|y^*\| \}$.
2. $Y$ is a $U$-subspace of $X$ if for any $y^* \in Y^*$, $\text{HB}(y^*)$ is a singleton. $X$ is said to be Hahn-Banach smooth if $X$ is a $U$-subspace of $X^{**}$.
3. The duality mapping $D$ for $X$ is the set-valued map from $S(X)$ to $S(X^*)$ defined by

$$D(x) = \{ x^* \in S(X^*) : x^*(x) = 1 \}, \quad x \in S(X).$$

4. $x \in S(X)$ is a smooth point of $B(X)$ if $D(x)$ is a singleton.
5. $Y$ is a weakly $U$-subspace of $X$ if for every $y^* \in D(S(Y))$, $\text{HB}(y^*)$ is a singleton.

If $Y$ is a $U$-subspace, or even a weakly $U$-subspace of $X$, then it satisfies our sufficient condition. It is shown in [8, Theorem 2] that an AC-subspace $Y$ is constrained in $X$ if every point of $S(Y)$ is a smooth point of $B(X)$. We show that this happens if and only if every subspace $Z$ of $Y$ is a weakly $U$-subspace of $X$. Thus, our condition is weaker.

It follows from our result that $X$ is smooth if and only if every subspace of $X$ is a weakly $U$-subspace. This parallels the classical result of Taylor-Foguel [15, 6] that $X^*$ is strictly convex if and only if every subspace of $X$ is a $U$-subspace.

2. **Some characterizations and a counterexample**

We will use the following notation:

**Notation.** Let $Y$ be a subspace of $X$. For all $x \in X$,

$$\mathcal{P}(x) = \bigcap_{y \in Y} B_Y[y, \|x - y\|].$$

Clearly, $\mathcal{P}(y) = \{ y \}$ for all $y \in Y$. Also, $Y$ is an AC-subspace of $X$ if and only if $\mathcal{P}(x) \neq \emptyset$ for all $x \in X$.

We recall a definition from [9].

**Definition 2.1.** Let $Y$ be a subspace of $X$. We define

$$O(Y, X) = \{ x \in X : \|x - y\| \geq \|y\| \text{ for all } y \in Y \}.$$ 

$O(X, X^{**})$ is denoted by $O(X)$.

The following proposition characterizes AC-subspaces.

**Proposition 2.2.** For a subspace $Y$ of $X$, the following are equivalent:

1. $Y$ is an AC-subspace of $X$.
2. For all $x \in X$, there exists $y \in Y$ and $z \in O(Y, X)$ such that $x = y + z$.
3. For every subspace $Z$ of $X$ such that $Y \subseteq Z$ and $\text{dim}(Z/Y) = 1$, $Y$ is constrained in $Z$. 

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\textit{Proof.} (a) $\implies$ (b). Let $x_0 \in X$. By (a), there exists $y_0 \in \mathfrak{P}(x_0)$. This implies $\|y_0 - y\| \leq \|x_0 - y\|$ for all $y \in Y$. Or, putting $u = y_0 - y$, $\|u\| \leq \|x_0 - y_0 + u\|$ for all $u \in Y$. That is, $z_0 = x_0 - y_0 \in O(Y, X)$ and $x_0 = y_0 + z_0$.

(b) $\implies$ (c). Let $Z$ be as in (c). Then one can write $Z = \overline{\text{span}}(Y \cup \{x_0\})$ for some $x_0 \in X$. By (b), there exists $y_0 \in Y$ and $z_0 \in O(Y, X)$ such that $x_0 = y_0 + z_0$. It follows that $Z = Y \oplus \mathbb{R}z_0$. But then, by definition of $O(Y, X)$, $\alpha z_0 + y \mapsto y$ is a norm 1 projection from $Z$ onto $Y$.

(c) $\implies$ (a). By (c), for every $x \in X$, there is a norm 1 projection $P_x$ from $Z_x = \overline{\text{span}}(Y \cup \{x\})$ onto $Y$. Clearly, $P_x(x) \in \mathfrak{P}(x)$. \hfill $\square$

Recall that a hyperplane $H$ in $X$ is a subspace such that $H = \ker(x^*)$ for some $x^* \in S(X^*)$. Since $\dim(X/H) = 1$, we get

\textbf{Corollary 2.3.} Suppose $H$ is a hyperplane in $X$. Then $H$ is an AC-subspace if and only if $H$ is constrained in $X$.

\textbf{Corollary 2.4.} A subspace $Y$ is an AC-subspace of $X$ if and only if there is a (not necessarily linear) map $P$ from $X$ onto $Y$ satisfying the following properties:

(a) $P^2 = P$;
(b) $P(x) = \lambda P(x)$ for all $x \in X$, $\lambda \in \mathbb{R}$;
(c) $P(x + y) = P(x) + y$ for all $x \in X$, $y \in Y$;
(d) $\|P(x)\| \leq \|x\|$ for all $x \in X$.

\textit{Proof.} If $P$ is as above, then clearly for any $x \in X$, $P(x) \in \mathfrak{P}(x)$. Thus, $Y$ is an AC-subspace of $X$.

Conversely, let $Y$ be an AC-subspace of $X$. For $z \in O(Y, X)$, let $Y_z = Y \oplus \mathbb{R}z$ and $P_z$ be a norm 1 projection from $Y_z$ onto $Y$. Observe that for $z_1, z_2 \in O(Y, X)$, either $Y_{z_1} \cap Y_{z_2} = Y$ or $Y_{z_1} = Y_{z_2}$. By Proposition 2.2(b), $\bigcup_{z \in O(Y, X)} Y_z = X$. Define $P : X \to Y$ by $P(x) = P_z(x)$, if $x \in Y_z$. Then $P$ is well-defined and satisfies all the listed properties. \hfill $\square$

\textbf{Remark 2.5.} Proposition 2.2(a) $\iff$ (c) for the case of $IP_{f, \infty}$ was noted in \cite{12} Theorem 5.9. Corollary 2.3 was also noted in \cite{11}. Corollary 2.4 for the case of $IP_{f, \infty}$ was noted in \cite{8} Theorem 2]. In all these cases, our proof is simpler.

Let us note that in Proposition 2.2(b), the representation $x = y + z$ with $y \in Y$ and $z \in O(Y, X)$ need not be unique.

\textbf{Example 2.6.} We now give an example to show that an AC-subspace need not, in general, be constrained. We need the following result (we thank Professor T.S.S.R.K. Rao of ISI, Bangalore, for drawing our attention to this result).

\textbf{Theorem 2.7 (\cite{11}).} There exist Banach spaces $Z \supseteq X$ with $\dim(Z/X) = 2$ satisfying

(i) There is no projection with norm 1 from $Z$ onto $X$.
(ii) For every $\varepsilon > 0$, there is a projection with norm $\leq 1 + \varepsilon$ from $Z$ onto $X$.
(iii) For every $Y$ with $Z \supseteq Y \supseteq X$ and $\dim(Y/X) = 1$, there is a projection with norm 1 from $Y$ onto $X$.

By Proposition 2.2 (iii) implies that $X$ is an AC-subspace of $Z$, while by (i), there is no norm 1 projection from $Z$ onto $X$.

\textbf{Definition 2.8.} (a) \cite{10} A Banach space $X$ such that $X^*$ is isometrically isomorphic to $L^1(\mu)$ for some positive measure $\mu$ is called an $L^1$-predual.

(b) A Banach space is a $P_1$-space if it is constrained in every superspace.
Remark 2.9. (a) From the results of [11] Chapter 3, it follows that $X$ is a real $L^1$-predual with $IP_{f, \infty}$ if and only if $X$ is a real $\mathcal{P}_1$-space. In particular, $X$ is constrained in $X^{**}$.

(b) It can be shown that the space $X$ in Example 2.6 is not constrained in $X^{**}$. Therefore, it could have been a possible counterexample to the $IP_{f, \infty}$ question as well. But, from the construction in [11], it is clear that the space $X$ is a real $L^1$-predual, but not a real $\mathcal{P}_1$-space. Thus it lacks the $IP_{f, \infty}$.

3. Some sufficient conditions

We now obtain sufficient conditions for an $AC$-subspace to be constrained. Some preliminaries first. As in [11], we introduce the following notation.

Definition 3.1. Let $Y$ be a subspace of $X$. For $x \in X$ and $y^* \in Y^*$, put
\[
U(x, y^*) = \inf \{ y^*(y) + \|x - y\| : y \in Y \},
\]
\[
L(x, y^*) = \sup \{ y^*(y) - \|x - y\| : y \in Y \}.
\]
For $x^* \in X^*$, we will write $U(x, x^*)$ for $U(x, x^*|_Y)$. Let $C(x) = \{ x^* \in B(X^*) : U(x, x^*) = L(x, x^*) \}$, for $x \in X$, and $C = \bigcap_{x \in X} C(x)$.

The following result is immediate from the proof of the Hahn-Banach Theorem (see, e.g., [11] Section 48).

Lemma 3.2. Let $Y$ be a subspace of $X$, $x_0 \not\in Y$ and $y^* \in S(Y^*)$. Then $L(x_0, y^*) \leq U(x_0, y^*)$ and $\alpha$ lies between these two numbers if and only if there exists $x^* \in HB(y^*)$ with $x^*(x_0) = \alpha$.

Remark 3.3. It is clear that for any $x^* \in B(X^*)$ and $x \in X$, $L(x, x^*) \leq x^*(x) \leq U(x, x^*)$ and for any $y^* \in S(Y^*)$, $HB(y^*)$ is singleton if and only if for all $x \in X$, $L(x, y^*) = U(x, y^*)$.

The next three results are from [11]. We include the proofs for the sake of completeness.

Lemma 3.4. Let $Y$ be a subspace of $X$. For $x_1, x_2 \in X$, the following are equivalent:

(a) $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$.

(b) For all $x^* \in B(X^*)$, $U(x_2, x^*) \leq U(x_1, x^*)$.

Proof. Clearly, $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$ if and only if $\|x_2 - y\| \leq \|x_1 - y\|$, for all $y \in Y$.

(a) $\Rightarrow$ (b). If for all $y \in Y$, $\|x_2 - y\| \leq \|x_1 - y\|$, then for all $x^* \in B(X^*)$, $x^*(y) + \|x_2 - y\| \leq x^*(y) + \|x_1 - y\|$. Therefore, $U(x_2, x^*) \leq U(x_1, x^*)$.

(b) $\Rightarrow$ (a). Suppose $\|x_2 - y_0\| > \|x_1 - y_0\|$ for some $y_0 \in Y$. Then there exists $\varepsilon > 0$ such that $\|x_2 - y_0\| - \varepsilon \geq \|x_1 - y_0\|$. Choose $x^* \in B(X^*)$ such that $\|x_1 - y_0\| \leq \|x_2 - y_0\| - \varepsilon < x^*(x_2 - y_0) - \varepsilon/2$. Thus $U(x_1, x^*) \leq x^*(y_0) + \|x_1 - y_0\| < x^*(x_2) - \varepsilon/2 < U(x_2, x^*)$. \(\square\)

Proposition 3.5. Let $Y$ be a subspace of $X$, $x^* \in B(X^*)$ and $x_0 \in X \setminus Y$. The following are equivalent:

(a) $x^* \in C(x_0)$.

(b) $\|x^*|_Y\| = 1$ and every $x_1^* \in HB(x^*|_Y)$ takes the same value at $x_0$. 

Hence \( y \) can be written as 

\[(a) \quad \text{Proof.} \]

\[\begin{align*}
\text{Since } x^n \text{ converge to } y, \quad \lim_{n \to \infty} x^n(x_0) &= x^*(x_0). \\
\text{Therefore, } \lim_{n \to \infty} x^n(x_0) &= x^*(x_0).
\end{align*}\]

(b) \( \Leftrightarrow \) (c). Let \( x^n \in S(X^*) \) be a net such that \( \lim_n x^n(y) = x^*(y) \) for all \( y \in Y \). It follows that any \( w^* \)-cluster point of \( \{x^n\} \) is in \( \text{HB}(x^*|Y) \). By (b), therefore, \( \lim_n x^n(x_0) = x^*(x_0) \).

(c) \( \Rightarrow \) (d). If \( x^n \in \text{HB}(x^*|Y) \) with \( x^*(x_0) \neq x^n(x_0) \), then the constant sequence \( x^n = x^n(x_0) \) clearly satisfies \( \lim_n x^n(y) = x^*(y) \) for all \( y \in Y \), but \( \{x^n(x_0)\} \) cannot converge to \( x^*(x_0) \).

Proposition 3.6. Let \( Y \) be a subspace of \( X \). For \( x^* \in B(X^*) \), the following are equivalent:

(a) \( x^* \in C \).
(b) \( \|x^*\| = 1 \) and \( \text{HB}(x^*|Y) = \{x^*\} \).
(c) \( \|x^*\| = 1 \) and if \( \{x^n\} \subseteq S(X^*) \) is a net such that \( x^n|Y \to x^*|Y \) in the \( w^* \)-topology of \( Y^* \), then \( x^n \to x^* \) in the \( w^* \)-topology of \( X^* \).
(d) \( \|x^*\| = 1 \) and if \( \{x^n\} \subseteq S(X^*) \) is such that \( x^n|Y \to x^*|Y \) in the \( w^* \)-topology of \( Y^* \), then \( x^n \to x^* \) in the \( w^* \)-topology of \( X^* \).

Here is our first sufficient condition for an \( AC \)-subspace to be constrained.

Proposition 3.7. For a subspace \( Y \) of \( X \), the following are equivalent:

(a) \( Y \) is an \( AC \)-subspace of \( X \) and \( O(Y, X) \) is a closed subspace of \( X \).
(b) \( Y \) is an \( AC \)-subspace of \( X \) and \( O(Y, X) \) is a linear subspace of \( X \).
(c) \( Y \) is constrained in \( X \) and for all \( x \in X \), \( \mathcal{P}(x) \) is a singleton.

Moreover, in this case, \( Y \) is constrained by a unique norm 1 projection.

Proof. (a) \( \Rightarrow \) (b) is trivial.

(b) \( \Rightarrow \) (c). Since \( Y \) is an \( AC \)-subspace of \( X \), by Proposition 2.7 any \( x \in X \) can be written as \( x = y + z \), where \( y \in Y \) and \( z \in O(Y, X) \). Since both \( Y \) and \( O(Y, X) \) are linear subspaces and \( Y \cap O(Y, X) = \{0\} \), this representation is unique and \( x \mapsto y \) is a well-defined linear map. Since \( z \in O(Y, X) \), this map is of norm 1. Hence \( Y \) is constrained in \( X \). Moreover, since \( y \in \mathcal{P}(x) \), \( \mathcal{P}(x) \) is single-valued.

(c) \( \Rightarrow \) (a). Let \( Y \) be constrained in \( X \) by a norm 1 projection \( P \) and for all \( x \in X \), let \( \mathcal{P}(x) \) be a singleton. Clearly, \( Y \) is an \( AC \)-subspace of \( X \) and for all
Let \( x \in X \), \( \Psi(x) = \{ P(x) \} \). It is easy to see that \( \ker(P) \subseteq O(Y, X) \) and since for all \( x \in X \), \( \Psi(x) = \{ P(x) \} \), \( \ker(P) \supseteq O(Y, X) \). Thus, \( O(Y, X) = \ker(P) \) is a closed subspace of \( X \).

Remark 3.8. (a) Even in the case of \( IPf, \infty \), this observation is new. References \([6]\) and \([7]\) discuss more complicated situations when \( O(X) \), being a linear subspace of \( X^{**} \), automatically implies that it is a \( w^* \)-closed subspace of \( X^{**} \).

(b) We do not know if (c) can be replaced by “\( Y \) is constrained by a unique norm 1 projection”.

(c) It follows from the proof that
\[
\bigcup \{ \ker(P) : P \text{ is a norm 1 projection onto } Y \} \subseteq O(Y, X).
\]
Are these two sets equal?

The following result significantly improves \([3, \text{Lemma 2}]\), which was also the key tool in \([8]\).

**Lemma 3.9.** Let \( Y \) be a subspace of \( X \). Let \( x_1, x_2 \in X \) be such that \( x_1 \in \bigcap_{y \in Y} B_X[y, \| x_2 - y \|] \). Then for any \( x^* \in C(x_2) \), \( x^*(x_1 - x_2) = 0 \).

**Proof.** Let \( x_1, x_2 \in X \) be such that \( x_1 \in \bigcap_{y \in Y} B_X[y, \| x_2 - y \|] \). Then, by Lemma 3.3, for all \( x^* \in B(X^*) \),
\[
L(x_2, x^*) \leq L(x_1, x^*) \leq U(x_1, x^*) \leq U(x_2, x^*).
\]
Thus for \( x^* \in C(x_2) \), equality holds. By Lemma 3.2, the result follows.

Here is our main theorem.

**Theorem 3.10.** Let \( Y \) be a subspace of \( X \). Suppose

1. for every \( x_1, x_2 \in X \), \( C(x_1) \cap C(x_2) \) separates points of \( Y \).

If \( Y \) is an \( AC \)-subspace of \( X \), then \( Y \) is constrained in \( X \). Moreover, the projection is unique and \( O(Y, X) \) is a closed subspace of \( X \).

**Proof.** Since \( Y \) is an \( AC \)-subspace of \( X \), \( \Psi(x) \neq \emptyset \) for all \( x \in X \). By Lemma 3.9, for all \( x \in X \),
\[
x^*(x - y) = 0 \quad \text{for any } x^* \in C(x), \ y \in \Psi(x).
\]
Now if \( y_1, y_2 \in \Psi(x) \), then for any \( x^* \in C(x) \), \( x^*(y_1 - y_2) = x^*(x - y_2) = 0 \). Therefore, \( x^*(y_1 - y_2) = 0 \). By (1), \( y_1 = y_2 \). That is, \( \Psi(x) \) is single-valued. Let \( \Psi(x) = \{ P(x) \} \). Then, \( P \) satisfies all the properties listed in Corollary 2.1. So, it only remains to show that \( P \) is additive.

Let \( x_1, x_2 \in X \). If \( x^* \in C(x_1) \cap C(x_2) \), then by Proposition 3.3, \( x^* \in C(x_1 + x_2) \) and by (2), \( x^*(x_1 - P(x_1)) = x^*(x_2 - P(x_2)) = x^*((x_1 + x_2) - P(x_1 + x_2)) = 0 \). Therefore, \( x^*(P(x_1 + x_2) - P(x_1) - P(x_2)) = 0 \). By (1), \( P(x_1 + x_2) = P(x_1) + P(x_2) \).

The rest of the result follows from Proposition 3.4.

By Theorem 3.10, the condition “\( C \) separates points of \( Y \)” is sufficient for an \( AC \)-subspace to be constrained by a unique norm 1 projection. This condition is clearly satisfied if \( Y \) is a \( U \)-subspace, or even a weakly \( U \) subspace of \( X \).

It is shown in \([8, \text{Theorem 2}]\) that an \( AC \)-subspace \( Y \) is constrained in \( X \) by a unique norm 1 projection if every point of \( S(Y) \) is a smooth point of \( B(X) \). By the following result, our condition is much weaker.
Proposition 3.11. Every point of $S(Y)$ is a smooth point of $B(X)$ if and only if every subspace $Z$ of $Y$ is a weakly $U$-subspace of $X$. In particular, $X$ is smooth if and only if every subspace of $X$ is a weakly $U$-subspace of $X$.

Proof. Suppose every point of $S(Y)$ is a smooth point of $B(X)$. Let $Z$ be any subspace of $Y$. Suppose $z^* \in S(Z^*)$ attains its norm at $z_0 \in S(Z)$. By assumption, $z_0$ is a smooth point of $B(X)$. Now, $z^* \in D_z(z_0)$ and $\text{HB}(z^*) \in D_X(z_0)$. Since $D_X(z_0)$ is a singleton, so is $\text{HB}(z^*)$. Thus, $Z$ is a weakly $U$-subspace of $X$.

Conversely, suppose there exists $y_0 \in S(Y)$ such that $D_X(y_0)$ is not a singleton. Suppose $\{x_1, x_2\} \subseteq D_X(y_0)$ and $x_1 \neq x_2$. Let $Z = \{x \in Y : x_1(x) = x_2(x)\}$. Then $y_0 \in S(Z)$ and therefore, $\|x_1^*[Z] = \|x_2^*[Z]\| = 1$. Thus, $z^* = x_1^*[Z] \in S(Z^*)$ attains its norm at $y_0 \in S(Z)$, but $\{x_1, x_2\} \subseteq \text{HB}(z^*)$. 

Example 3.12. As noted in [3], the space $X = L^\infty$ gives an example of a dual space such that there are infinitely many norm 1 projections from $X^{**}$ onto $X$. This produces an example of a space with $IP_{1,\infty}$ that is constrained in $X^{**}$, but $O(X)$ is not a closed subspace of $X^{**}$. This also shows that our sufficient condition, although weaker than the known ones, is still not necessary for an $AC$-subspace to be constrained.

We conclude the paper with some necessary and/or sufficient conditions for $O(Y, X)$ to be a closed subspace of $X$. First we need a characterization of $O(Y, X)$. This is a slight improvement over that in [1].

Definition 3.13. We say $A \subseteq B(X^*)$ is a norming set for $X$ if $\|x\| = \sup\{x^*(x) : x^* \in A\}$ for all $x \in X$.

A subspace $F$ of $X^*$ is called a norming subspace if $B(F)$ is a norming set for $X$.

Lemma 3.14. Let $Y$ be a subspace of $X$. For $x \in X$, the following are equivalent:

(a) $x \in O(Y, X)$.
(b) $\ker(x)|_Y \subseteq Y^*$ is a norming subspace for $Y$.
(c) $0 \in \bigcap_{y \in Y} B_Y[y, \|x - y\|]$.
(d) For every $x^* \in B(X^*)$, $L(x, x^*) \leq 0 \leq U(x, x^*)$.
(e) For every $y^* \in B(Y^*)$, $L(x, y^*) \leq 0 \leq U(x, y^*)$.

Further, for a w*-closed subspace $F \subseteq X^*$, $F|_Y$ is a norming subspace for $Y$ if and only if $F_\perp \subseteq O(Y, X)$, where $F_\perp = \{x \in X : f(x) = 0 \forall f \in F\}$.

Proof. Let $F \subseteq X^*$ be a w*-closed subspace such that $F_{\perp} \subseteq O(Y, X)$. Then $F = (X/F_{\perp})^*$ and therefore, it suffices to show that $\|y\| = \|y + F_{\perp}\| = d(y, F_{\perp})$.

Clearly, $\|y\| \geq d(y, F_{\perp})$. Also, since $F_{\perp} \subseteq O(Y, X)$, for any $y \in Y$ and $z \in F_{\perp}$, $\|y + z\| \geq \|y\|$. Thus, $d(y, F_{\perp}) \geq \|y\|$.

Specializing to $F = \ker(x)$, we get (a) $\Rightarrow$ (b).
(b) $\Rightarrow$ (a). Since $\ker(x)|_Y$ norms $X$, $\|y\| = \|y|_{\ker(x)}\| = d(y, R(x))$ for all $y \in Y$. Hence $\|x - y\| \geq \inf_{\lambda \in \mathbb{R}} \|y - \lambda x\| = \|y\|$ for all $y \in Y$. Thus, $x \in O(Y, X)$.

Now suppose $F \subseteq X^*$ is a w*-closed subspace such that $F|_Y$ is a norming subspace for $Y$. If $x \in F_{\perp}$, then $F \subseteq \ker(x)$ and therefore, $x \in O(Y, X)$. That is, $F_{\perp} \subseteq O(Y, X)$.

(a) $\Leftrightarrow$ (c) and (d) are immediate from definition, while (c) $\Rightarrow$ (d) follows from Lemma 3.3.

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(e) ⇒ (a). For every \( y^* \in B(Y^*) \), \( 0 \leq U(x, y^*) \) implies for all \( y^* \in B(Y^*) \) and \( y \in Y \),
\[
0 \leq y^*(y) + \|x - y\| \implies y^*(-y) \leq \|x - y\|.
\]
Since this is true for all \( y^* \in B(Y^*)\), \( \|y\| \leq \|x - y\| \) for all \( y \in Y \). That is, \( x \in O(Y, X) \). 

Let \( \mathcal{N} = \{ F : F \text{ is a w*-closed subspace of } X^* \text{ and } F|_Y \text{ is a norming subspace for } Y \} \) and \( N = \bigcap \mathcal{N} \). Similar to [9], we observe

**Proposition 3.15.** Let \( Y \) be a subspace of \( X \). \( O(Y, X) \) is a closed subspace of \( X \) if and only if \( N|_Y \) is a norming subspace for \( Y \). In particular, this happens if \( C|_Y \) is a norming set for \( Y \).

**Proof.** By Lemma 3.14 \( F \in \mathcal{N} \) if and only if \( F_\perp \subseteq O(Y, X) \). Thus if \( N|_Y \) norms \( Y \), then \( N \in \mathcal{N} \) and hence, \( N_\perp \subseteq O(Y, X) \). On the other hand, if \( x \in O(Y, X) \), then \( \ker(x) \subseteq N_\perp \), and hence, \( N \subseteq \ker(x) \). That is, \( x \in N_\perp \). Therefore, \( O(Y, X) = N_\perp \), and \( O(Y, X) \) is a closed subspace of \( X \).

Conversely, if \( O(Y, X) \) is a closed subspace of \( X \) and \( M = O(Y, X)_\perp \), then \( M_\perp = O(Y, X) \) and therefore, \( M \in \mathcal{N} \). Moreover, for every \( F \in \mathcal{N} \), \( F_\perp \subseteq O(Y, X) = M_\perp \), and hence, \( M \subseteq F \). This shows \( N = M \) and \( N \in \mathcal{N} \).

Now, if \( C|_Y \) is a norming set for \( Y \), then as above, \( C_\perp \subseteq O(Y, X) \).

Conversely let \( x \in O(Y, X) \). Let \( x^* \in C \). By Lemmas 3.2 and 3.14 there exists \( z^* \in HB(x^*|_Y) \) such that \( z^*(x) = 0 \). Since \( x^* \in C \), \( HB(x^*|_Y) = \{x^*\} \), and we have \( x^*(x) = 0 \). Thus, \( C_\perp = O(Y, X) \). 

**Definition 3.16.** (a) [16] Let \( Y \) be a subspace of \( X \). Let
\[
A(Y) = \{ x^* \in B(X^*) : x^*|_Y \text{ is an extreme point of } B(Y^*) \}.
\]
\( Y \) is a weakly separating subspace of \( X \) if \( Y \) separates points of \( A(Y) \).

(b) [9] A subspace \( Y \subseteq X \) is said to be an \( M \)-ideal if there exists a subspace \( N \subseteq X^* \) such that \( X^* = Y_\perp \oplus_1 N \).

**Proposition 3.17.** In each of the following cases, \( O(Y, X) \) is a closed subspace of \( X \), in or (a) \( Y \) is an \( AC \)-subspace, then \( Y \) is constrained by a unique norm 1 projection.

- (a) \( Y \) is a weakly separating subspace of \( X \).
- (b) \( Y \) is an \( M \)-ideal in \( X \).
- (c) \( Y \) is a subspace of \( X = C(K) \) containing the constants and separating points of \( K \).

**Proof.** (a) A careful examination of the proof of [16] Lemma 1] actually shows that \( A(Y) \subseteq C \). It is easy to see that \( A(Y) \) is a norming set for \( Y \). The result follows from Proposition 3.15.

(b) [9] Theorem I.1.12] observes that an \( M \)-ideal is a \( U \)-subspace.

(c) As observed in [16], such a \( Y \) is weakly separating.

**Remark 3.18.** (a) In [16], it is shown that for a weakly separating subspace in \( C(K) \), if there is a norm 1 projection, it must be unique. Clearly, our conclusion is stronger.

(b) In [13], it is shown that an \( M \)-ideal with the \( IP_{f,\infty} \) is an \( M \)-summand. An argument similar to [2] Proposition 2.8] shows that an \( M \)-ideal \( Y \) in \( X \) with the
$IP_{f_{\infty}}$ is an $AC$-subspace of $X$. Thus, Proposition 3.17(b) improves the result in [13].

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References


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