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BANACH SPACES EMBEDDING ISOMETRICALLY INTO L_p WHEN $0 < p < 1$

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ABSTRACT. For $0 < p < 1$ we give examples of Banach spaces isometrically embedding into L_p but not into any L_r with $p < r \le 1$.

1. INTRODUCTION

It is a consequence of the Maurey-Nikishin factorization theory that every Banach space that embeds isomorphically into $L_p(0,1)$ for some $0 < p < 1$ embeds into every $L_p(0,1)$ for $0 < p < 1$ (see [10], [11] and [15] pp. 257ff.). It is, however, an open problem whether every Banach space that embeds isomorphically into L_p for some $0 < p < 1$ must also embed isomorphically into L_1. This problem was formulated by Kwapien [8] in 1969; see [4] where it is shown that X embeds into L_1 if and only if $\ell_1(X)$ embeds into L_p for some $p < 1$. The isometric version of the problem asks: if X isometrically embeds into L_p for some $p < 1$ does it follow that X isometrically embeds into L_1? This problem was solved negatively by the second author in 1996 [6] who showed that there is a Banach space embedding into $L_{1/2}$ but not into L_1. The construction also yielded an example of a Banach space embedding into $L_{1/4}$ but not $L_{1/2}$. Later, J. Borwein and the Center for Computational Mathematics at Simon Fraser University (unpublished) showed by computer methods that this algorithm yields examples of Banach spaces embedding into $L_{a/64}$ but not into $L_{(a+1)/64}$ for $a = 1, 2, \cdots, 63$.

The purpose of this note is to show that for every $0 < p < 1$ we can find a (real) Banach space X embedding isometrically into L_p but not into any L_r for $p < r \le 1$. The example constructed in [6] is finite-dimensional and is obtained by a perturbation method. By contrast, our spaces are infinite-dimensional and we use probabilistic ideas to construct them. It is, of course, true that an infinite-dimensional space X embeds isometrically into L_p if and only if every finite-dimensional subspace does, and so our methods also imply the existence of finite-dimensional examples.

We start in Section 2 by discussing the Plotkin-Rudin Equimeasurability and Uniqueness Theorems, which we need for our applications. In Section 3 we construct a very basic example, which we denote by E_p. This is the subspace of $L_p(0,1)$

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spanned by a constant function and a sequence of symmetric 1-stable random variables. It turns out that this space is a Banach space that is an absolute direct sum of a one-dimensional space and an isometric copy of ℓ_1. The spaces E_p provide our first family of examples. We show this by establishing that they have a certain extremal property (see Proposition 3.5).

In Section 4 we provide a second family of examples that are renormings of Hilbert spaces. For each $0 < p < 1$ we construct an example of such a space X_p that embeds isometrically into L_p but not into any L_r for $r > p$. These spaces are absolute direct sums of two infinite-dimensional Hilbert spaces. We observe that these examples have the additional property that no subspace of finite codimension can be embedded into any L_r where $r > p$.

2. Remarks on the Plotkin-Rudin theorem

In this section we discuss some essentially known results based on the Plotkin-Rudin theorems on isometric embeddings ([12], [13], [14]). See [7] for a discussion of these results.

We will always work in the setting of a Polish space Ω equipped with a nonatomic Borel probability measure μ; we then say that (Ω, μ) is a standard probability space. All functions are assumed to be Borel; if $f_1, \cdots, f_n$ are real Borel functions, then their joint distribution is the Borel measure on $\mathbb{R}^n$ given by $\mu \circ (f_1, \cdots, f_n)^{-1}$, and this will be denoted by $\rho_{f_1,\cdots,f_n}$.

We say that if (Ω_1, μ_1) and (Ω_2, μ_2) are two standard probability spaces, then a Borel map $\sigma : \Omega_1 \to \Omega_2$ is a measure isomorphism if there is a Borel map $\tau : \Omega_2 \to \Omega_1$ (an essential inverse) such that

- $\tau\sigma(\omega_1) = \omega_1$, μ_1–a.e.;
- $\sigma\tau(\omega_2) = \omega_2$, μ_2–a.e.;
- $\mu_2 \circ \tau^{-1} = \mu_1$ and $\mu_1 = \mu_2 \circ \sigma^{-1}$.

If σ is a measure isomorphism, then it may be modified on a set of μ_1-measure zero to become a Borel isomorphism (i.e., an invertible Borel map). If (Ω, μ) is a standard probability space, then there is always a Borel isomorphism $\sigma : \Omega \to [0, 1]$ such that $\lambda = \mu \circ \sigma^{-1}$ where λ is Lebesgue measure.

We shall need the following fact.

Proposition 2.1. *Let (Ω, μ) be a standard probability space and suppose K is a Polish space. Suppose $\sigma : \Omega \to K$ is a Borel map and $\nu = \mu \circ \sigma^{-1}$. Suppose there exists a Borel function f on Ω such that $\rho_f = \mu \circ f^{-1}$ is nonatomic and f is independent of σ (i.e., f is independent of the σ-algebra of sets of the form $\sigma^{-1}B$ for B a Borel subset of K). Then there is a Borel map $\tau : \Omega \to [0, 1]$ so that $\sigma \times \tau$ is a measure isomorphism of Ω onto $(K \times [0, 1], \nu \times \lambda)$.*

Proof. This is surely well known, but we do not know an explicit reference. It follows, for example, from Proposition 2.2 of [3] once one observes that σ is anti-injective (i.e., if B is a Borel set such that σ is injective on B, then $\mu(B) = 0$)). It suffices by Lusin's theorem to consider the case when B is compact and σ is continuous on B; then σ is a Borel isomorphism of B onto $\sigma(B)$. To see this, suppose $C_1, \cdots, C_N$ form a partition of $\mathbb{R}$ so that $\rho_f(C_k) = N^{-1}$. Let $B_k = B \cap f^{-1}(C_k)$. Then $\sigma(B_k)$ is Borel and $\mu(f^{-1}(C_k) \cap \sigma^{-1}\sigma(B_k)) = N^{-1}\nu(\sigma(B_k))$. Hence $\mu(B) \le N^{-1} \sum_{k=1}^{N} \nu(\sigma(B_k)) \le N^{-1}$. □

Let X be a separable normed space, and $T : X \to L_p(\Omega, \mu)$ an isometric embedding. We say that T is *in canonical position* if it satisfies the following two conditions:

- There exists $x \in X$ so that Tx has full support, i.e., $\mu(Tx \neq 0) = 1$.
- There exists a function f with ρ_f nonatomic such that f is independent of the smallest σ-algebra Σ such that each Tx is Σ-measurable.

It is well known that if X embeds into L_p, then there is also an embedding in canonical position.

Let us say that two embeddings $S : X \to L_p(\Omega_1, \mu_1)$ and $T : X \to L_p(\Omega_2, \mu_2)$ are *equivalent* if

$$\rho_{Sx_1,\cdots,Sx_n} = \rho_{Tx_1,\cdots,Tx_n} \qquad x_1, \cdots, x_n \in X.$$

Theorem 2.2 ([12], [13], [14])**.** (1) *Suppose p is not an even integer and (Ω, μ_1) and (Ω_2, μ_2) are two standard probability spaces. If $S : X \to L_p(\Omega, \mu_1)$ and $T : X \to L_p(\Omega, \mu_2)$ are isometric embeddings such that for some x_0 we have $Sx_0 = \chi_{\Omega_1}$ and $Tx_0 = \chi_{\Omega_2}$, then S and T are equivalent.*

(2) *If, in addition, S and T are in canonical position, then there exists a measure isomorphism $\sigma : \Omega_1 \to \Omega_2$ such that $\mu_2 = \mu_1 \circ \sigma^{-1}$ and $Tx \circ \sigma = Sx$ for $x \in X$.*

Proof. (1) is the usual Plotkin-Rudin equimeasurability theorem [12], [13], [14], [7]. (2) is surely well known and follows directly from Proposition 2.1. Let us indicate one proof. Let (x_n) be any dense sequence in X and define, for $j = 1, 2$, $\tau_j : \Omega_j \to \mathbb{R}^{\mathbb{N}}$ by $\tau_1(\omega_1) = (Sx_n(\omega_1))$ and $\tau_2(\omega_2) = ((Tx_n)(\omega_2))$. Then by (1) $\mu_1 \circ \tau_1^{-1} = \mu_2 \circ \tau_2^{-1} = \nu$, say. By Proposition 2.1 we can define Borel maps $\kappa_j : \Omega_j \to [0, 1]$ so that $\tau_j \times \kappa_j$ is a measure isomorphism of (Ω_j, μ_j) onto $(\mathbb{R}^{\mathbb{N}} \times [0, 1], \nu \times \lambda)$. The map σ is then the composition $\alpha(\tau_1 \times \kappa_1)$ where α is the essential inverse of $\tau_2 \times \kappa_2$. □

If $T : X \to L_p(\Omega, \mu)$ is an isometric embedding, then we can always construct a new embedding by a change of density. If φ is a nonvanishing Borel function, and $\int |\varphi|^p d\mu = 1$, we define $d\nu = |\varphi|^p d\mu$ and $T'x = \varphi^{-1}Tx$; then $T' : X \to L_p(\Omega, \nu)$ is a new isometric embedding. We then say that T' is obtained from T by a change of density.

Theorem 2.3. *Suppose p is not an even integer and $S : X \to L_p(\Omega, \mu)$ is an isometric embedding of canonical type. Then, if $T : X \to L_p(\Omega_1, \mu_1)$ is any other isometric embedding, there exists a nonvanishing Borel function φ so that T' is equivalent to T where $T' : X \to L_p(\Omega, |\varphi|^p d\mu)$ is given by $T'x = \varphi^{-1}Sx$. (Thus T is obtained from S by a change of density.)*

Proof. We assume S is also of canonical type. Pick any x_0 with $\|x_0\| = 1$ so that $Sx_0 = f$ and $Tx_0 = g$ have full support. Consider $V_1x = f^{-1}Sx$ and $V_2x = g^{-1}Tx$. Then $V_1 : X \to L_p(\Omega, |f|^p d\mu)$ and $V_2 : X \to L_p(\Omega_1, |g|^p d\mu_1)$ are isometric embeddings with $V_1x_0 = \chi_\Omega$ and $V_2x_0 = \chi_{\Omega_1}$. It follows that there is a measure isomorphism $\sigma : \Omega \to \Omega_1$ so that $|g|^p\mu_1 = |f|^p\mu \circ \sigma^{-1}$ and $V_1x = V_2x \circ \sigma$. Now $Tx \circ \sigma = g \circ \sigma V_2 x \circ \sigma = g \circ \sigma f^{-1} Sx$, and if B is a Borel subset of $\mathbb{R}^n$ and $x_1, \cdots, x_n \in X$, then

$$\mu_1((Tx_1, \cdots, Tx_n) \in B) = \int |g \circ \sigma|^{-p} |f|^p \chi_{((Tx_1 \circ \sigma, \cdots, Tx_n \circ \sigma) \in B)} d\mu$$

and the conclusion follows with $\varphi = f(g \circ \sigma)^{-1}$. □

Corollary 2.4. *Let X be a (separable) Banach space that embeds into L_p where $p<1$. Let E be a subspace of X and suppose $T:E\to L_p(\Omega,\mu)$ is a given isometric embedding. Then there is an isometric embedding $S:X\to L_p(\Omega_1,\mu_1)$ such that the restriction of S to E is equivalent to T.*

Proof. Let $R:X\to L_p(\Omega,\mu)$ be any isometric embedding of canonical type. We note that R is also of canonical type when restricted to E. In fact, it is only necessary to note that for *every* $x\in X$, Rx has full support in Ω. Indeed, if Rx_0 has full support, then

$$\int |Rx+tRx_0|^p d\mu \geq \|x\|^p + |t|^p \int_{Rx=0} |Rx_0|^p d\mu,$$

which contradicts the convexity of the norm unless Rx has full support. It follows that we can make a change of density so that the new embedding S restricted to E is equivalent to T. ☐

A random variable f is called symmetric p-stable $0<p<2$ if the Fourier transform of ρ_f is of the form $e^{-c|t|^p}$ for some $c>0$. We recall that there is an isometric embedding T of $L_r(0,1)$ into $L_p(0,1)$ when $0<p<r<2$ so that each Tf has a symmetric r-stable distribution. (See the remarks on p. 213 of [9].) We will call this the r-stable embedding. A particular case is that ℓ_1 can be embedded into L_p for $p<1$ by mapping the basic vectors to a sequence of independent 1-stable random variables.

We will also need the following standard lemmas.

Lemma 2.5. *Suppose X is a Banach space and $T:X\to L_p(\Omega,\mu)$ is an isometric embedding where $0<p<1$. Then $\{|Tx|^p : \|x\|\leq 1\}$ is equi-integrable.*

Proof. This follows by contradiction: if $\{|Tx|^p : \|x\|\leq 1\}$ is not equi-integrable, then (see [15] p. 137) there exists $\delta>0$, a disjoint sequence of Borel sets (A_k) and x_k with $\|x_k\|\leq 1$ so that $\int_{A_k}|Tx_k|^p d\mu > \delta^p$. Then by an application of Khintchine's inequality we have for suitable $c>0$,

$$\begin{aligned} N^p &\geq \operatorname*{Ave}_{\epsilon_k=\pm1} \|\sum_{k=1}^N \epsilon_k x_k\|^p \\ &\geq c^p \int (\sum_{k=1}^N |Tx_k|^2)^{\frac{p}{2}} d\mu \\ &\geq c^p N\delta^p, \end{aligned}$$

and for large enough N this gives a contradiction. ☐

Lemma 2.6. *Let $F:\mathbb{R}^{m+1}\to\mathbb{R}$ be a continuous function. Suppose $g_1,\cdots,g_m$ are measurable functions on (Ω,μ) and that $(f_n)_{n=1}^\infty$ is any sequence of identically distributed independent random variables with common distribution $\rho=\rho_{f_n}$. If the functions $F(g_1,\cdots,g_m,f_n)$ are equi-integrable for $n=1,2,\cdots,$ then $F(g_1,\cdots,$ $g_m,f_0)$ is integrable and*

$$\lim_{n\to\infty}\int F(g_1,\cdots,g_m,f_n)d\mu = \int_\Omega\int_{\mathbb{R}} F(g_1,\cdots,g_m,t)d\rho(t)d\mu. \tag{2.1}$$

Proof. First, suppose that $F, g_1, \cdots, g_m, f_n$ are all bounded functions. Note that for $a_1, \cdots, a_m, b = 0, 1, 2, \cdots,$ we have

$$\lim_{n\to\infty} \int g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m} f_n^b d\mu = \left(\int g_1^{a_1} \cdots g_m^{a_m} d\mu\right)\left(\int t^b d\rho(t)\right)$$

since the f_n^b converge weakly in L_2 to the constant $\int f_n^b d\mu$. Hence for any polynomial P,

$$\lim_{n\to\infty} \int P(g_1, \cdots, g_m, f_n) d\mu = \int_\Omega \int_{\mathbb{R}} P(g_1, \cdots, g_m, t) d\rho(t) d\mu.$$

If $|f_n|, |g_1|, \cdots, |g_m| \le M$ and $\epsilon > 0$, we approximate F on the cube $[-M, M]^{m+1}$ by a polynomial P so that the range of

$$|P(x_1, \cdots, x_m, y) - F(x_1, \cdots, x_m, y)| \le \epsilon \qquad |x_j| \le M,\ 1 \le j \le m,\ |y| \le M.$$

Then it follows that we have

$$\left|\lim_{n\to\infty} \int F(g_1, \cdots, g_m, f_n) d\mu - \int_\Omega \int_{\mathbb{R}} F(g_1, \cdots, g_m, t) d\rho(t) d\mu\right| \le \epsilon.$$

Letting $\epsilon \to 0$ we obtain (2.1) under the assumption that $f, g_1, \cdots, g_m$ are bounded.

Next assume that $|F|$ is bounded by M, but allow f and g_j to be unbounded. For any $m \in \mathbb{N}$, let $f_{k,n} = f_n \chi_{|f_n| \le k}$, and $g_{k,j} = g\chi_{|g| \le k}$. Then for $n \ge 0$,

$$\begin{aligned} &\left|\int F(g_1, \cdots, g_m, f_n) d\mu - \int F(g_{k,1}, \cdots, g_{k,m}, f_{k,n}) d\mu\right| \\ &\le 2M\left(\mu(|f_0| > k) + \sum_{j=1}^m \mu(|g_j| > k)\right). \end{aligned}$$

Since we have (2.1) for bounded $f_n, g_1, \cdots, g_m$, we obtain the result in general for F bounded.

Now assume that $F(g_1, \cdots, g_m, f_n)$ is equi-integrable and let $F_k = \min(F, k)$ if $F \ge 0$ and $F_k = \max(F, -k)$ if $F \le 0$. Then

$$\lim_{n\to\infty} \int_\Omega |F_k(g_1, \cdots, g_m, f_n)| d\mu = \int_\Omega \int_{\mathbb{R}} |F_k(g_1, \cdots, g_m, t)| d\rho(t) d\mu,$$

and it follows that $F(g_1, \cdots, g_m, t)$ is integrable with respect to $\mu \times \rho$. We also have

$$\lim_{k\to\infty} \int F_k(g_1, \cdots, g_m, f_n) d\mu = \int F(g_1, \cdots, g_m, f_n) d\mu$$

uniformly in k, so that the general result follows by uniform convergence. □

3. The spaces E_p for $0 < p < 1$

Lemma 3.1. *Suppose* $0 < p < 1$. *Then for* $-\pi/2 < \theta \le \pi/2$,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x \cos\theta + \sin\theta|^p}{1 + x^2} dx = \frac{\cos p\theta}{\cos p\pi/2}. \tag{3.1}$$

Proof. We consider the case $\theta \ne 0$ of (3.1); the other cases are similar. We define $f(z)$ to be the branch of $(z\cos\theta + \sin\theta)^p$ defined in $\mathbb{C} \setminus \{-\tan\theta - it :\ t \ge 0\}$ such that $f(x)$ is real and positive if $x \ge -\tan\theta$. Now by a routine contour integration we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{1 + x^2} dx = e^{ip(\frac{\pi}{2} - \theta)}.$$

Taking imaginary parts gives

$$\frac{1}{\pi}\int_{-\infty}^{-\tan\theta}\frac{|x\cos\theta+\sin\theta|^p}{1+x^2}dx=\frac{\sin p(\frac{\pi}{2}-\theta)}{\sin p\pi}.$$

Taking real parts and substituting in, we have

$$\frac{1}{\pi}\int_{-\tan\theta}^{\infty}\frac{|x\cos\theta+\sin\theta|^p}{1+x^2}dx=\cos p(\frac{\pi}{2}-\theta)-\cot p\pi\sin p(\frac{\pi}{2}-\theta)=\frac{\sin p(\frac{\pi}{2}+\theta)}{\sin p\pi}.$$

Combining gives (3.1). $\square$

Lemma 3.2. *Let $M:\mathbb{C}\to[0,\infty)$ be a continuous nonnegative function. Suppose M is subharmonic and positively homogeneous (i.e., $M(az)=aM(z)$ for $a\geq 0$). Then M is convex.*

Proof. First, we assume that M is C^2 on $\mathbb{C}\setminus\{0\}$. Then for any $z=x+iy\neq 0$ the second derivative of M is given by a symmetric 2×2 matrix that has rank at most one. To see this, note that the equation $M(az)=aM(z)$ implies on differentiation by a, and then by setting $a=1$ that

$$x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}=M.$$

Differentiating again with respect to x and y gives

$$x\frac{\partial^2 M}{\partial x^2}+y\frac{\partial^2 M}{\partial x\partial y}=0,$$
$$x\frac{\partial^2 M}{\partial x\partial y}+y\frac{\partial^2 M}{\partial y^2}=0,$$

and hence the second derivative has determinant zero. Thus if $\nabla^2 M\geq 0$, the second derivative of M is nonnegative at z. This shows that M is convex.

If M is not C^2, then we may approximate it by functions of the form

$$\tilde{M}(z)=\int_0^{2\pi}\varphi(\theta)M(ze^{i\theta})d\theta$$

where φ is smooth and nonnegative. Each such function $\tilde{M}$ is convex and so M is convex. $\square$

Now, for $0<p<1$, let us define a function $N_p(x,y)$ on $\mathbb{R}^2$ by setting

$$N_p(x,y)=r\left(\frac{\cos p\theta}{\cos\frac{p\pi}{2}}\right)^{\frac{1}{p}},$$

whenever $x\geq 0$ and $x=r\cos\theta,\ y=r\sin\theta$ with $r\geq 0,\ -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$. Then extend N_p to be an even function, i.e., so that $N_p(x,y)=N_p(-x,-y)$ whenever $x\leq 0$. Note also that $N_p(0,1)=1$ but $N_p(1,0)=(\sec\frac{p\pi}{2})^{\frac{1}{p}}$.

Lemma 3.3. *If $0<p<1$, N_p is an absolute norm on $\mathbb{R}^2$; i.e., N_p is a norm so that $N_p(x,y)=N_p(|x|,|y|)$.*

Proof. Let $u(z)=r^p\cos p\theta$ when $z=re^{i\theta}$ with $-\pi<\theta\leq\pi$. Then u is subharmonic and $N_p(x,y)=(\sec\frac{p\pi}{2})^{\frac{1}{p}}(\max(u(z),u(-z)))^{\frac{1}{p}}$ where $z=x+iy$. Hence N_p is a norm by Lemma 3.2. The fact that N_p is absolute is trivial. $\square$

We now define a Banach space E_p for $0 < p < 1$. We define this to be the space $\ell_1 \oplus \mathbb{R}$ with the norm $\|(x, y)\|_{E_p} = N_p(\|x\|, |y|)$.

Let (f_n) be a sequence of independent 1-stable random variables on some probability space (Ω, μ) so that $\int e^{itf_n} d\mu = e^{-|t|}$. Then for any finitely nonzero sequence $(\xi_n)_{n=1}^{\infty}$ and any η we have

$$\|\sum_{n=1}^{\infty} \xi_n f_n + \eta\|_p = N_p(\sum_{n=1}^{\infty} |\xi_n|, |\eta|).$$

It follows that:

Proposition 3.4. *E_p is isometric to a closed subspace of L_p for $0 < p < 1$.*

Next, we show that E_p cannot be embedded into L_r for any $p < r < 1$. To do this we introduce the quantity

$$a_p = \lim_{t \to 0} \frac{N((\cos \frac{p\pi}{2})^{\frac{1}{p}} t, 1) - 1}{t} = (\cos \tfrac{p\pi}{2})^{\frac{1}{p}-1} \sin \tfrac{p\pi}{2}.$$

Proposition 3.5. *Suppose $0 < p < 1$ and that (g_n) is a sequence in $L_p(\Omega, \mu)$ that is 1-equivalent to the standard unit vector basis of ℓ_1. Suppose $h \in L_p$ and $\|h\|_p = 1$. Then*

$$\lim_{n \to \infty} \|h + tg_n\|_p \geq N_p((\cos \tfrac{p\pi}{2})^{\frac{1}{p}} t, 1) \geq 1 + a_p|t|.$$

Proof. It follows from Theorem 2.3 and Corollary 2.4 that it suffices to consider the case when $g_n = (\cos \frac{p\pi}{2})^{\frac{1}{p}} f_n$ where (f_n) is a sequence of independent 1-stable random variables with $\int e^{itf_n} d\mu = e^{-|t|}$. We now apply Lemma 2.6:

$$\begin{aligned} \lim_{n \to \infty} \int |h + \tau f_n|^p d\mu &= \frac{1}{\pi} \int_{\Omega} \int_{-\infty}^{\infty} \frac{|h(\omega) + \tau x|^p}{1 + x^2} dx\, d\mu(\omega) \\ &= \int N_p(\tau, h(\omega))^p d\mu(\omega). \end{aligned}$$

Now since N_p is an absolute norm,

$$\begin{aligned} \int N_p(\tau, 1)^{1-p} N_p(\tau, h(\omega))^p d\mu &\geq \int N_p(\tau, |h(\omega)|^p) d\mu \\ &\geq N_p(\tau, 1) \end{aligned}$$

and hence

$$\int N_p(\tau, h(\omega))^p d\mu(\omega) \geq N_p(\tau, 1)^p.$$

This gives us the first inequality.

For the second part observe that

$$\lim_{t \to 0+} \frac{N_p((\cos \frac{p\pi}{2})^{\frac{1}{p}} t, 1) - 1}{t} = a_p$$

and use the fact that N_p is a norm. □

Theorem 3.6. *For $0 < p < 1$ the space E_p is a Banach space isometric to a subspace of L_p, which is not isometric to a subspace of any L_r for $r > p$.*

Proof. This is immediate from Proposition 3.5 once we show that the function $p \to a_p$ is strictly increasing on $(0,1)$. Since L_r embeds into L_p when $p < r$ and E_r embeds into L_r, it is clear from Proposition 3.5 that $p \to a_p$ is increasing. This function is non-constant since $\lim_{p\to 1} a_p = 1$ and $a_{1/2} = \frac{1}{2}$. Since it is a real-analytic function, it must therefore be strictly increasing. □

Remark. It would be interesting to estimate the smallest integer $n = n(r,p)$ so that the n-dimensional subspace of E_p spanned by the constant function and $f_1, \cdots, f_{n-1}$ fails to embed into L_r. We also mention that the span of the constant function and the sequence $|f_n|$ is isomorphic to the Ribe space [2]; for similar examples involving p-stable random variables see [1].

4. Perturbed Hilbert spaces

In this section we give an alternative construction of examples that are isomorphic but not isometric to Hilbert spaces.

Lemma 4.1. *Suppose $0 < p < 1$. Then there exists $\epsilon(p) > 0$ so that if $0 < a < \epsilon(p)$, the following equation defines an absolute norm on $\mathbb{R}^2$:*

$$N(x,y)^p = \frac{1}{2}(x^2 + (1+a)^{\frac{2}{p}} y^2)^{\frac{p}{2}} + (x^2 + (1-a)^{\frac{2}{p}} y^2)^{\frac{p}{2}}. \tag{4.1}$$

Proof. This follows easily from Lemma 3.2 since, if a is small enough, $(x^2 + (1+a)^{\frac{2}{p}} y^2)^{\frac{p}{2}}$ and $(x^2 + (1-a)^{\frac{2}{p}} y^2)^{\frac{p}{2}}$ are both subharmonic. □

Theorem 4.2. *Suppose $0 < p < 1$ and N is given by (4.1). Then the space $X = \ell_2 \oplus_N \ell_2$ embeds into L_p but does not embed into any space L_r where $r > p$.*

Proof. We first establish an embedding of X into $L_p(\Omega, \mu)$. Let (e_n) and (e_n') be the canonical orthonormal bases of the two factors of X. Let $(f_n), (g_n)$ be two mutually independent sequences of independent normalized Gaussians; we denote by γ their common distribution so that $d\gamma(t) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{t^2}{2})dt$. Let E be a Borel set independent of (f_n, g_n) with $\mu E = \frac{1}{2}$. Let $h = (1+a)^{\frac{1}{p}}\chi_E + (1-a)^{\frac{1}{p}}\chi_{\tilde{E}}$. We define our embedding by

$$Te_n = b_1 f_n,$$
$$Te_n' = b_1 h g_n$$

where $b_1^{-p} = \|f_n\|_p^p = \int |t|^p d\gamma(t)$. We can and do assume that T is of canonical type. Suppose $(\xi_n), (\eta_n)$ are two finitely nonzero sequences of reals. Then

$$\begin{aligned}\int_\Omega |\sum_{n=1}^\infty \xi_n Te_n + \sum_{n=1}^\infty \eta_n Te_n'|^p d\mu &= b_1^p \int_\Omega |\sum_{n=1}^\infty \xi_n f_n + h \sum_{n=1}^\infty \eta_n g_n|^p d\mu \\ &= \int_\Omega (\sum_{n=1}^\infty |\xi|^2 + h^2 \sum_{n=1}^\infty \eta_n^2)^{\frac{p}{2}} d\mu \\ &= N((\sum_{n=1}^\infty \xi_n^2)^{\frac{1}{2}}, (\sum_{n=1}^\infty \eta_n^2)^{\frac{1}{2}})^p.\end{aligned}$$

Now assume X also embeds isometrically into L_r for some $p < r < 2$. Then X can also be embedded into L_p by an r-stable embedding S. In view of Theorem 2.3, it may be assumed that S is obtained from T by a change of density, i.e., there exists a nonvanishing Borel function φ with $\|\varphi\|_p = 1$ such that $S : X \to L_p(\Omega, |\varphi|^p d\mu)$ is given by $Sx = \varphi^{-1}Tx$. Fix any $0 < q < p$. It follows for an appropriate choice of b_2 that the map $S'x = b_2 Sx$ embeds X into $L_q(\Omega, |\varphi|^p d\mu)$. Now we make a further change of density. Let $b_3^q = \int_\Omega |\varphi|^{p-q} d\mu$ and define $\psi = b_3^{-1}\varphi^{-1}$. Let $R : X \to L_q(\Omega, |\psi|^q |\varphi|^p d\mu)$ by $Rx = \psi^{-1}S'x$. Then $Rx = b_3 b_2 Tx$. Let $b_0 = b_3 b_2 b_1$.

We now use Lemma 2.5 and Lemma 2.6. Suppose $x, y \in \mathbb{R}$.

$$\begin{aligned} N(x,y)^q &= b_0^q \lim_{m\to\infty} \lim_{n\to\infty} \int_\Omega |xf_m + yhg_n|^q |\varphi|^p |\psi|^q d\mu \\ &= b_0^q \lim_{m\to\infty} \int_\Omega \int_{\mathbb{R}} |xf_m + yth|^q d\gamma(t) |\varphi|^p |\psi|^q d\mu \\ &= b_0^q \int_\Omega \int_{\mathbb{R}} \int_{\mathbb{R}} |xs + yth|^q d\gamma(s) d\gamma(t) |\varphi|^p |\psi|^q d\mu \\ &= b_0^q \int_{\mathbb{R}} |t|^q d\gamma(t) \int_\Omega (x^2 + y^2h^2)^{\frac{q}{2}} |\varphi|^p |\psi|^q d\mu. \end{aligned}$$

Since h takes only the values $(1 \pm a)^{\frac{1}{p}}$, this implies that we can find positive constants c_1, c_2 so that for all x, y,

$$N(x,y)^q = c_1(x^2 + (1-a)^{\frac{2}{p}} y^2)^{\frac{q}{2}} + c_2(x^2 + (1+a)^{\frac{2}{p}} y^2)^{\frac{q}{2}}.$$

Since $N(1,0) = N(0,1) = 1$, this requires

$$\begin{aligned} c_1 + c_2 &= 1, \\ c_1(1-a)^{\frac{q}{p}} + c_2(1+a)^{\frac{q}{p}} &= 1. \end{aligned}$$

Note also that

$$\begin{aligned} \lim_{t\to 0} \frac{N(1,t)^2 - 1}{t^2} &= \frac{1}{2}((1+a)^{\frac{2}{p}} + (1-a)^{\frac{2}{p}}) \\ &= c_1(1-a)^{\frac{2}{p}} + c_2(1+a)^{\frac{2}{p}}. \end{aligned}$$

It is clearly impossible to satisfy these three conditions. This contradiction shows that we cannot embed X into L_r for any $r > p$. $\square$

Remark. It is worth remarking in this context that it is unknown if there is an infinite-dimensional space X that embeds isometrically into L_p and L_r where $p < 2 < r$ and is isomorphic but not isometric to a Hilbert space (see [5]).

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