

TIGHT WAVELET FRAMES GENERATED BY THREE SYMMETRIC B -SPLINE FUNCTIONS WITH HIGH VANISHING MOMENTS

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ABSTRACT. In this note, we show that one can derive from any B -spline function of order m ($m \in \mathbb{N}$) an MRA tight wavelet frame in $L_2(\mathbb{R})$ that is generated by the dyadic dilates and integer shifts of three compactly supported real-valued symmetric wavelet functions with vanishing moments of the highest possible order m .

1. INTRODUCTION

B -spline functions and tight wavelet frames are of great interest in many applications. The B -spline function of order m ($m \in \mathbb{N}$), denoted by B_m throughout this note, can be obtained via the following recursive formula: $B_1 = \chi_{[0,1]}$, the characteristic function of the interval $[0, 1]$, and

$$(1.1) \quad B_m(x) := \int_0^1 B_{m-1}(x-t) dt, \quad x \in \mathbb{R}, \quad m = 2, 3, \dots$$

The B -spline function $B_m \in C^{m-2}(\mathbb{R})$ is a function of piecewise polynomials of degree less than m , vanishes outside the interval $[0, m]$ and is symmetric about the point $x = m/2$ (that is, $B_m(m-x) = B_m(x)$ for all $x \in \mathbb{R}$). It is well known that the B -spline function B_m is a refinable function satisfying the refinement equation

$$(1.2) \quad \widehat{B}_m(2\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^m \widehat{B}_m(\xi), \quad \xi \in \mathbb{R},$$

where the Fourier transform of $f \in L_1(\mathbb{R})$ is defined to be $\widehat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-it\xi} dt$, $\xi \in \mathbb{R}$.

Tight wavelet frames can be constructed from refinable functions. We say that a set $\{\psi^1, \dots, \psi^r\}$ of functions in $L_2(\mathbb{R})$ generates a (normalized) **tight wavelet frame** in $L_2(\mathbb{R})$ if

$$(1.3) \quad \sum_{\ell=1}^r \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle f, \psi_{j,k}^\ell \rangle|^2 = \|f\|^2 \quad \forall f \in L_2(\mathbb{R}),$$

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where $\langle f, g \rangle := \int_{\mathbb{R}} f(t)\overline{g(t)} dt$, $\|f\|^2 := \langle f, f \rangle$ and $\psi_{j,k}^\ell := 2^{j/2}\psi^\ell(2^j \cdot -k)$. The set $\{\psi^1, \dots, \psi^r\}$ is called a set of generators for the tight frame $\{\psi_{j,k}^\ell : \ell = 1, \dots, r; j, k \in \mathbb{Z}\}$. Throughout this note, we assume that the generators for any tight wavelet frame discussed in this note are real-valued and compactly supported functions in $L_2(\mathbb{R})$.

One of the most important properties of a tight wavelet frame is its vanishing moments. A set $\{\psi^1, \dots, \psi^r\}$ of compactly supported functions in $L_2(\mathbb{R})$ is said to have **vanishing moments** of order n if

$$\int_{\mathbb{R}} \psi^\ell(t)t^k dt = 0, \quad \ell = 1, \dots, r; k = 0, \dots, n-1.$$

Recently, there is a growing interest in constructing MRA tight wavelet frames derived from refinable functions, in particular, derived from the B -spline functions B_m ($m \in \mathbb{N}$) (see [1], [2], [4], [5], [8], [9]). For example, Ron and Shen in [9] demonstrated that a tight wavelet frame with m symmetric generators with short support can be derived from the B -spline function B_m ($m \in \mathbb{N}$). Recently, Chui and He in [1] further showed that a tight wavelet frame with only 3 symmetric generators (or 2 generators without symmetry) of compact support can be derived from the B -spline function B_m ($m \in \mathbb{N}$). Though the constructions in [1], [9] are very interesting, such tight wavelet frames derived from the B -spline functions in [1], [9] can have vanishing moments of order no more than one ([2], [4]). It is known that in certain applications the order of vanishing moments of a tight wavelet frame is very important. In order to achieve high order of vanishing moments, very recently, Chui, He and Stöckler [2] and Daubechies, Han, Ron and Shen [4] proposed a very interesting method to derive MRA wavelet tight frames from refinable functions. In particular, it was shown in [2], [4] that a tight wavelet frame with 2 generators of compact support can be derived from any B -spline function B_m ($m \in \mathbb{N}$) such that it can have vanishing moments of the highest possible order m . Unfortunately, except for very few cases, the tight wavelet frames with 2 generators derived from B -spline functions in [2], [4] are not symmetric. It is desirable to have symmetric tight wavelet frames with high vanishing moments in applications. On the other hand, it was shown in Petukhov [8] that it is not possible to construct tight wavelet frames with 2 symmetric generators of compact support from the B -spline functions $B_m(1 \leq m \leq 100, m \neq 1, 2, 3, 7)$ by using the unitary extension principle in [9].

All the above known results naturally lead to the following interesting question:

Question. Can one obtain a tight wavelet frame with 3 symmetric generators of compact support from any given B -spline function B_m ($m \in \mathbb{N}$) of order m such that the symmetric tight wavelet frame can have the vanishing moments of the highest possible order m ?

By numerical computation, it was possible to verify in [4] (also see [2]) that the above question is true for $m = 1, 2, 4$, and 6. It is the purpose of this note to completely settle the above question. The following is the main result of this note.

Theorem 1. *Let m be a positive integer and let B_m denote the B -spline function of order m which is defined in (1.1). Then there exist three finitely supported sequences b^1, b^2, b^3 on \mathbb{Z} , which can be easily constructed by a simple procedure, such that by defining*

$$\psi^\ell := \sum_{k \in \mathbb{Z}} b^\ell(k)B_m(2 \cdot -k), \quad \ell = 1, 2, 3,$$

one has

- (1) $\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$ and has the vanishing moments of order m ;
- (2) ψ^1, ψ^2, ψ^3 are real-valued, symmetric and compactly supported functions such that

$$\psi^1(1-t) = (-1)^m \psi^1(t), \quad \psi^2(m-t) = \psi^2(t), \quad \psi^3(m-t) = -\psi^3(t), \quad t \in \mathbb{R}.$$

The following is an outline of the note. In Section 2, we shall establish some auxiliary results. In Section 3, we shall prove Theorem 1. Also in Section 3, we shall give a step-by-step simple procedure for deriving the sequences b^1, b^2, b^3 in Theorem 1.

2. AUXILIARY RESULTS

In order to prove Theorem 1, let us introduce some auxiliary results in this section which are of interest in their own right.

The following lemma can be easily verified.

Lemma 2. *If $a_j \geq 0$ and $b_j > 0$ for all $j = 1, \dots, n$ such that $\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \dots \geq \frac{a_n}{b_n}$, then*

$$\frac{a_1}{b_1} \geq \frac{a_1 + a_2}{b_1 + b_2} \geq \dots \geq \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n}.$$

For any positive integer m , throughout this note, we define

$$(2.1) \quad c_0 := 1 \quad \text{and} \quad c_j := \frac{(2j-1)!!}{(2j)!!(2j+1)} = \frac{1}{2j+1} \prod_{k=1}^j \left(1 - \frac{1}{2k}\right), \quad j \in \mathbb{N}.$$

Note that

$$(2.2) \quad \frac{\xi/2}{\sin(\xi/2)} = \frac{\arcsin(\sin(\xi/2))}{\sin(\xi/2)} = \sum_{j=0}^{\infty} c_j \sin^{2j} \frac{\xi}{2}, \quad \xi \in [-\pi, \pi].$$

Clearly, we have

$$(2.3) \quad \sum_{j=0}^{\infty} c_j = \frac{\pi}{2}.$$

The following estimate will be needed later.

Lemma 3. *Let c_j ($j \in \mathbb{N} \cup \{0\}$) be defined in (2.1). For any positive integer m , define*

$$(2.4) \quad f_m(x) := 4m(1+x)^{2m} \sum_{j=0}^{\infty} c_{m+j} (1-x^2)^j, \quad x \in [0, 1].$$

Then for $m \geq 3$, f_m is an increasing function on the interval $[0, 1]$ and

$$f_m(x) \geq \frac{\pi}{1 - 2^{1/2-m}} \quad \forall x \in [0, 1].$$

Proof. By (2.3), it is easy to see that f_m is a continuous function on $[0, 1]$. For $x \in (0, 1)$, by computation, we have

$$f'_m(x) = 8m^2(1+x)^{2m-1} \sum_{j=0}^{\infty} c_{m+j} (1-x^2)^j - 8m(1+x)^{2m} x \sum_{j=0}^{\infty} j c_{m+j} (1-x^2)^{j-1}.$$

Consequently,

$$\frac{f'_m(x)}{8m(1+x)^{2m}x} = m \frac{x^{-1} - 1}{1-x^2} \sum_{j=0}^{\infty} c_{m+j}(1-x^2)^j - \sum_{j=0}^{\infty} j c_{m+j}(1-x^2)^{j-1}, \quad x \in (0, 1).$$

Denote $y := 1 - x^2$. Then

$$\frac{1}{x} = \frac{1}{\sqrt{1-y}} = (1-y)^{-1/2} = 1 + \sum_{j=1}^{\infty} (2j+1)c_j y^j, \quad x \in (0, 1).$$

Therefore, for $x \in (0, 1)$, we have

$$\begin{aligned} \frac{f'_m(x)}{8m(1+x)^{2m}x} &= m \left(\sum_{j=0}^{\infty} (2j+3)c_{j+1}y^j \right) \left(\sum_{j=0}^{\infty} c_{m+j}y^j \right) - \sum_{j=0}^{\infty} (j+1)c_{m+j+1}y^j \\ &= \sum_{j=0}^{\infty} g_{m,j}c_{m+j+1}y^j, \end{aligned}$$

where $y = 1 - x^2$ and the numbers $g_{m,j}$ are defined by

$$(2.5) \quad g_{m,j} := m \sum_{k=0}^j (2k+3)c_{k+1} \frac{c_{m+j-k}}{c_{m+j+1}} - (j+1), \quad j \in \mathbb{N} \cup \{0\}.$$

By the definition of the numbers c_j ($j \in \mathbb{N} \cup \{0\}$) in (2.1), we have

$$(2.6) \quad (2k+3)c_{k+1} \frac{c_{m+j-k}}{c_{m+j+1}} = \frac{2m+2j+3}{2m+2j-2k+1} \frac{\prod_{\ell=1}^{k+1} (1 - \frac{1}{2\ell})}{\prod_{\ell=m+j-k+1}^{m+j+1} (1 - \frac{1}{2\ell})}.$$

Note that for any nonnegative integer k , we have

$$\begin{aligned} \prod_{\ell=1}^{k+1} \left(1 - \frac{1}{2\ell}\right) &= \frac{1}{2} \prod_{\ell=2}^{k+1} \left(1 - \frac{1}{2\ell}\right) \geq \frac{1}{2} \prod_{\ell=2}^{k+1} \sqrt{1 - \frac{1}{2\ell}} \sqrt{1 - \frac{1}{2\ell-1}} \\ &= \frac{1}{2} \prod_{\ell=2}^{k+1} \sqrt{\frac{\ell-1}{\ell}} = \frac{1}{2\sqrt{k+1}} \end{aligned}$$

and for $0 \leq k \leq j$,

$$\begin{aligned} \prod_{\ell=m+j-k+1}^{m+j+1} \left(1 - \frac{1}{2\ell}\right) &\leq \prod_{\ell=m+j-k+1}^{m+j+1} \sqrt{1 - \frac{1}{2\ell}} \sqrt{1 - \frac{1}{2\ell+1}} \\ &= \prod_{\ell=m+j-k+1}^{m+j+1} \frac{\sqrt{2\ell-1}}{\sqrt{2\ell+1}} = \frac{\sqrt{2m+2j-2k+1}}{\sqrt{2m+2j+3}}. \end{aligned}$$

It follows from (2.6) and the above two inequalities that

$$\begin{aligned} (2k+3)c_{k+1} \frac{c_{m+j-k}}{c_{m+j+1}} &\geq \frac{2m+2j+3}{2m+2j-2k+1} \cdot \frac{1}{2\sqrt{k+1}} \cdot \frac{\sqrt{2m+2j+3}}{\sqrt{2m+2j-2k+1}} \\ &= \frac{1}{2\sqrt{k+1}} \frac{(m+j+3/2)^{3/2}}{(m+j+1/2-k)^{3/2}}. \end{aligned}$$

Hence,

$$\begin{aligned}
g_{m,j} + (j+1) &= m \sum_{k=0}^j (2k+3) c_{k+1} \frac{c_{m+j-k}}{c_{m+j+1}} \\
&\geq m(m+j+3/2)^{3/2} \sum_{k=0}^j \frac{1}{2\sqrt{k+1}(m+j+1/2-k)^{3/2}} \\
&= m(m+j+3/2)^{3/2} \sum_{k=1}^{j+1} \frac{1}{2\sqrt{k}(m+j+3/2-k)^{3/2}}.
\end{aligned}$$

Let $t_{m,j,k}^1 := \sqrt{(k+1)(m+j+5/2-k)}$ and $t_{m,j,k}^2 := \sqrt{k(m+j+5/2-(k+1))}$. Then $t_{m,j,k}^1 \geq \sqrt{k}\sqrt{m+j+3/2-k} = t_{m,j,k}^2$. We have

$$\begin{aligned}
&\frac{\sqrt{k+1}}{\sqrt{m+j+5/2-(k+1)}} - \frac{\sqrt{k}}{\sqrt{m+j+5/2-k}} \\
&= \frac{t_{m,j,k}^1 - t_{m,j,k}^2}{\sqrt{m+j+5/2-(k+1)}\sqrt{m+j+5/2-k}} \\
&= \frac{(k+1)(m+j+5/2-k) - k(m+j+5/2-(k+1))}{\sqrt{m+j+3/2-k}\sqrt{m+j+5/2-k}(t_{m,j,k}^1 + t_{m,j,k}^2)} \\
&= \frac{m+j+5/2}{\sqrt{m+j+3/2-k}\sqrt{m+j+5/2-k}(t_{m,j,k}^1 + t_{m,j,k}^2)} \\
&\leq \frac{m+j+5/2}{2\sqrt{k}(m+j+3/2-k)^{3/2}}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&g_{m,j} + (j+1) \\
&\geq \frac{m(m+j+3/2)^{3/2}}{m+j+5/2} \sum_{k=1}^{j+1} \left[\frac{\sqrt{k+1}}{\sqrt{m+j+5/2-(k+1)}} - \frac{\sqrt{k}}{\sqrt{m+j+5/2-k}} \right] \\
&= \frac{m(m+j+3/2)^{3/2}}{m+j+5/2} \left[\frac{\sqrt{j+2}}{\sqrt{m+1/2}} - \frac{\sqrt{1}}{\sqrt{m+j+3/2}} \right] \\
&= \frac{m(m+j+3/2)^{3/2}}{m+j+5/2} \\
&\quad \times \frac{(j+2)(m+j+3/2) - (m+1/2)}{\sqrt{m+1/2}\sqrt{m+j+3/2}(\sqrt{(j+2)(m+j+3/2)} + \sqrt{m+1/2})} \\
&= \frac{m(m+j+3/2)(j+1)}{\sqrt{(m+1/2)(j+2)(m+j+3/2)} + (m+1/2)}.
\end{aligned}$$

By computation, for $m \geq 2$ and $j \geq 0$, we have

$$\begin{aligned} & [m(m+j+3/2) - m - 1/2]^2 - (m+1/2)(j+2)(m+j+3/2) \\ &= [m(m-1) - 1/2]j^2 + [m(2m^2 - 5) - 7/4]j \\ & \quad + m(m+1)(m^2 - 4) + \frac{5}{4}m(m-1) + \frac{3}{4}(m - \frac{5}{3}) > 0. \end{aligned}$$

Consequently, we have

$$m(m+j+3/2) > \sqrt{(m+1/2)(j+2)(m+j+3/2)} + (m+1/2) \quad \forall m \geq 2, j \geq 0.$$

Hence, $g_{m,j} + (j+1) > j+1$ for all $m \geq 2$ and $j \geq 0$. That is,

$$g_{m,j} > 0 \quad \forall m \geq 2 \quad \text{and} \quad j \geq 0,$$

which implies that $f'_m(x) > 0$ for all $x \in (0, 1)$ and $m \geq 2$.

So, for $m \geq 2$, f_m is an increasing function on the interval $[0, 1]$ and

$$f_m(x) \geq f_m(0) \quad \forall x \in [0, 1].$$

Note that $1 - \frac{1}{2k} = \frac{2k-1}{2k} \geq \sqrt{\frac{k-1}{k}}$ for all $k \in \mathbb{N}$. We deduce that

$$\begin{aligned} (2.7) \quad c_j &= \frac{1}{2j+1} \prod_{k=1}^j \left(1 - \frac{1}{2k}\right) \geq \frac{1}{2(2j+1)} \prod_{k=2}^j \sqrt{\frac{k-1}{k}} \\ &= \frac{1}{(2j+1)\sqrt{4j}} \geq \frac{1}{2\sqrt{j}} - \frac{1}{2\sqrt{j+1}} \end{aligned}$$

for all $j \in \mathbb{N}$. When $m \geq 4$, by (2.7), we have

$$f_m(0) = 4m \sum_{j=m}^{\infty} c_j \geq 4m \sum_{j=m}^{\infty} \left(\frac{1}{2\sqrt{j}} - \frac{1}{2\sqrt{j+1}} \right) = 2\sqrt{m} \geq \frac{\pi}{1 - 2^{1/2-m}}$$

since $h(t) := 2\sqrt{t}(1 - 2^{1/2-t})$, $t \geq 1$ is an increasing function and $h(4) = 4 - 2^{-3/2} > \pi$.

Note that $c_0 = 1$, $c_1 = 1/6$ and $c_2 = 3/40$. When $m = 3$, by (2.3), we have

$$f_3(0) = 12 \sum_{j=0}^{\infty} c_{3+j} = 12 \left(\frac{\pi}{2} - c_0 - c_1 - c_2 \right) = 6\pi - \frac{149}{10} \geq \frac{\pi}{1 - 2^{1/2-3}},$$

which completes the proof. \square

Now the main result in this section, which plays a critical role in our proof of Theorem 1, is as follows:

Theorem 4. *For any positive integer m ,*

$$(2.8) \quad \frac{\sum_{j=0}^{m-1} c_j \sin^{2j} \frac{\xi}{2}}{\sum_{j=0}^{m-1} c_j \sin^{2j} \xi} \geq \left(\cos^{2m} \frac{\xi}{2} + \sin^{2m} \frac{\xi}{2} \right)^{\frac{1}{2m}} \quad \forall \xi \in \mathbb{R},$$

where the numbers c_j ($j \in \mathbb{N} \cup \{0\}$) are defined in (2.1).

Proof. It is easy to see that in order to show the inequality (2.8), it suffices to prove it for $\xi \in [0, \pi/2]$ since $\sin^2 \frac{\pi-\xi}{2} = \cos^2 \frac{\xi}{2} \geq \sin^2 \frac{\xi}{2}$ for all $\xi \in [0, \pi/2]$. Let $x = \cos(\xi)$. By Lemma 3, for $m \geq 3$, we have the following estimate:

$$(2.9) \quad 4m \left(2 \cos^2 \frac{\xi}{2} \right)^{2m} \sum_{j=0}^{\infty} c_{m+j} \sin^{2j} \xi \geq \frac{\pi}{1 - 2^{1/2-m}} \quad \forall \xi \in [0, \pi/2].$$

Define

$$A(\xi) := \sum_{j=0}^{m-1} c_j \sin^{2j} \frac{\xi}{2} \quad \text{and} \quad B(\xi) := \frac{\xi/2}{\sin(\xi/2)} - A(\xi).$$

By (2.2), $B(\xi) = \sum_{j=m}^{\infty} c_j \sin^{2j} \frac{\xi}{2}$. It follows from (2.9) that for $\xi \in [0, \pi/2]$,

$$\begin{aligned} B(2\xi) &= \sum_{j=m}^{\infty} c_j \sin^{2j} \xi = (\sin \xi)^{2m} \sum_{j=0}^{\infty} c_{m+j} \sin^{2j} \xi \\ &= \frac{1}{4m} \cdot \frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)} \cdot 4m \left(2 \cos^2 \frac{\xi}{2}\right)^{2m} \sum_{j=0}^{\infty} c_{m+j} \sin^{2j} \xi \\ &\geq \frac{\frac{\pi}{2} \cdot \frac{1}{2m} \cdot \frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}}{1 - 2^{1/2-m}} \\ &\geq \frac{\xi}{\sin \xi} \cdot \frac{\frac{1}{2m} \cdot \frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}}{1 - \frac{1}{[4 \cos^2(\xi/2)]^m} \cdot \frac{1}{\cos(\xi/2)}}, \end{aligned}$$

where we have used the fact that $\cos(\xi/2) \geq 2^{-1/2}$ for all $\xi \in [0, \pi/2]$. Observe that $(1+x)^{\frac{1}{2m}} - 1 \leq \frac{x}{2m}$ for all $x \geq 0$ and $m \in \mathbb{N}$. It follows that

$$\left(1 + \frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}\right)^{\frac{1}{2m}} - 1 \leq \frac{1}{2m} \cdot \frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}, \quad \xi \in [0, \pi/2].$$

Since $A(2\xi) + B(2\xi) = \frac{\xi}{\sin \xi}$, $\xi \in [0, \pi/2]$, from the above two inequalities we have

$$\begin{aligned} \frac{B(2\xi)}{A(2\xi) + B(2\xi)} &= \frac{B(2\xi)}{\frac{\xi}{\sin \xi}} \geq \frac{\left(1 + \frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}\right)^{\frac{1}{2m}} - 1}{\left(1 + \frac{\sin^{2m}(\xi/2)}{\cos^{2m}(\xi/2)}\right)^{\frac{1}{2m}} - \frac{1}{[4 \cos^2(\xi/2)]^m} \cdot \frac{1}{\cos(\xi/2)}} \\ &= \frac{\left(\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2)\right)^{\frac{1}{2m}} - \cos(\xi/2)}{\left(\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2)\right)^{\frac{1}{2m}} - \frac{1}{[4 \cos^2(\xi/2)]^m}}. \end{aligned}$$

The above inequality is equivalent to

$$(2.10) \quad \frac{\cos \frac{\xi}{2} - \frac{B(2\xi)}{A(2\xi)+B(2\xi)} \cdot \frac{1}{[4 \cos^2(\xi/2)]^m}}{1 - \frac{B(2\xi)}{A(2\xi)+B(2\xi)}} \geq \left(\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2)\right)^{\frac{1}{2m}}.$$

Since $2 \cos(\xi/2) > 1$ for all $\xi \in [0, \pi/2]$ and

$$\frac{c_j \sin^{2j} \frac{\xi}{2}}{c_j \sin^{2j} \xi} = \frac{1}{\left(2 \cos \frac{\xi}{2}\right)^{2j}} \geq \frac{1}{\left(2 \cos \frac{\xi}{2}\right)^{2j+2}} = \frac{c_{j+1} \sin^{2j+2} \frac{\xi}{2}}{c_{j+1} \sin^{2j+2} \xi}, \quad \xi \in [0, \pi/2],$$

by Lemma 2 we have

$$\frac{B(\xi)}{B(2\xi)} = \frac{\sum_{j=m}^{\infty} c_j \sin^{2j} \frac{\xi}{2}}{\sum_{j=m}^{\infty} c_j \sin^{2j} \xi} \leq \frac{\sin^{2m} \frac{\xi}{2}}{\sin^{2m} \xi} = \frac{1}{[4 \cos^2(\xi/2)]^m}, \quad \xi \in [0, \pi/2].$$

Hence, $B(\xi) \leq \frac{B(2\xi)}{[4 \cos^2(\xi/2)]^m}$ for all $\xi \in [0, \pi/2]$ and

$$\begin{aligned} \frac{A(\xi)}{A(2\xi)} &= \frac{[A(\xi) + B(\xi)] - B(\xi)}{[A(2\xi) + B(2\xi)] - B(2\xi)} \geq \frac{[A(\xi) + B(\xi)] - \frac{B(2\xi)}{[4 \cos^2(\xi/2)]^m}}{[A(2\xi) + B(2\xi)] - B(2\xi)} \\ &= \frac{\frac{A(\xi)+B(\xi)}{A(2\xi)+B(2\xi)} - \frac{B(2\xi)}{A(2\xi)+B(2\xi)} \cdot \frac{1}{[4 \cos^2(\xi/2)]^m}}{1 - \frac{B(2\xi)}{A(2\xi)+B(2\xi)}} \\ &= \frac{\cos \frac{\xi}{2} - \frac{B(2\xi)}{A(2\xi)+B(2\xi)} \cdot \frac{1}{[4 \cos^2(\xi/2)]^m}}{1 - \frac{B(2\xi)}{A(2\xi)+B(2\xi)}}, \end{aligned}$$

since $A(\xi) + B(\xi) = \frac{\xi/2}{\sin(\xi/2)}$ for $\xi \in [-\pi, \pi]$. It follows from the above inequality and (2.10) that for $\xi \in [0, \pi/2]$,

$$\frac{A(\xi)}{A(2\xi)} \geq \frac{\cos \frac{\xi}{2} - \frac{B(2\xi)}{A(2\xi)+B(2\xi)} \cdot \frac{1}{[4 \cos^2(\xi/2)]^m}}{1 - \frac{B(2\xi)}{A(2\xi)+B(2\xi)}} \geq \left(\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2) \right)^{\frac{1}{2m}}.$$

Therefore, (2.8) holds for $m \geq 3$. It is obvious that (2.8) holds for $m = 1$. In the following, let us check the case $m = 2$. Let $x = \sin^2 \frac{\xi}{2}$. Then $\sin^2 \xi = 4x(1-x)$. When $m = 2$, to prove (2.8) it suffices to prove

$$(2.11) \quad \frac{1 + \frac{x}{6}}{1 + \frac{2}{3}x(1-x)} \geq [1 - 2x(1-x)]^{1/4} \quad \forall x \in [0, 1].$$

By computation, for $x \in [0, 1]$, we have

$$\begin{aligned} \left[1 + \frac{2}{3}x(1-x)\right] [1 - 2x(1-x)]^{1/4} &\leq \left[1 + \frac{2}{3}x(1-x)\right] \left[1 - \frac{1}{4}2x(1-x)\right] \\ &= 1 + \left(\frac{2}{3} - \frac{1}{2}\right)x(1-x) - \frac{1}{3}x^2(1-x)^2 \\ &\leq 1 + \frac{1}{6}x(1-x) \\ &\leq 1 + \frac{1}{6}x, \end{aligned}$$

which verifies (2.11). Therefore, the proof is completed. \square

3. PROOF OF THE MAIN RESULT

In this section, using the auxiliary results in Section 2, we shall prove Theorem 1. In particular, we shall give a step-by-step procedure for constructing the sequences b^1, b^2, b^3 in Theorem 1.

The **symbol** of a sequence a on \mathbb{Z} is defined to be

$$(3.1) \quad \widehat{a}(\xi) := \sum_{k \in \mathbb{Z}} a(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}.$$

Proof of Theorem 1: Let B_m be the B -spline function of order m and let $\widehat{a}(\xi) := \left(\frac{1+e^{-i\xi}}{2}\right)^m$. Then it is known that $\widehat{B}_m(2\xi) = \widehat{a}(\xi)\widehat{B}_m(\xi)$. Let c_j ($j \in \mathbb{N} \cup \{0\}$) be the numbers that are defined in (2.1). We define the numbers $d_{m,j}$ ($m \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}$),

which are uniquely determined by the following identity:

$$(3.2) \quad \left(\sum_{j=0}^{m-1} c_j x^j \right)^m = \sum_{j=0}^{\infty} d_{m,j} x^j, \quad x \in \mathbb{R}.$$

Clearly, $d_{m,0} = [c_0]^m = 1$ and $d_{m,j} = 0$ for all $j > m(m-1)$. Define two 2π -periodic trigonometric polynomials θ_1 and θ as follows:

$$(3.3) \quad \theta_1(\xi) := 1 + \sum_{j=1}^{m-1} d_{m,j} \sin^{2j} \frac{\xi}{2} \quad \text{and} \quad \theta(\xi) := |\theta_1(\xi)|^2 = [\theta_1(\xi)]^2.$$

Since $\frac{\sin^2(\xi/2)}{\sin^2 \xi} = \frac{1}{4 \cos^2(\xi/2)} \leq \frac{1}{2}$ for $\xi \in [0, \pi/2]$, we have $\left(\frac{\sin^2(\xi/2)}{\sin^2 \xi} \right)^{j-1} \geq \left(\frac{\sin^2(\xi/2)}{\sin^2 \xi} \right)^j$ for all $j \in \mathbb{N}$ and for all $\xi \in [0, \pi/2]$. By Lemma 2, for $\xi \in [0, \pi/2]$, we have

$$\begin{aligned} \frac{\theta_1(\xi)}{\theta_1(2\xi)} &= \frac{\sum_{j=0}^{m-1} d_{m,j} \sin^{2j}(\xi/2)}{\sum_{j=0}^{m-1} d_{m,j} \sin^{2j} \xi} \geq \frac{\sum_{j=0}^{m(m-1)} d_{m,j} \sin^{2j}(\xi/2)}{\sum_{j=0}^{m(m-1)} d_{m,j} \sin^{2j} \xi} \\ &= \left(\frac{\sum_{j=0}^{m-1} c_j \sin^{2j}(\xi/2)}{\sum_{j=0}^{m-1} c_j \sin^{2j} \xi} \right)^m. \end{aligned}$$

By Theorem 4 in Section 2, for $\xi \in [0, \pi/2]$ we have

$$\begin{aligned} \frac{\theta_1(\xi)}{\theta_1(2\xi)} &\geq \left(\frac{\sum_{j=0}^{m-1} c_j \sin^{2j}(\xi/2)}{\sum_{j=0}^{m-1} c_j \sin^{2j} \xi} \right)^m \geq \left(\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2) \right)^{1/2} \\ &= \sqrt{|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + \pi)|^2}. \end{aligned}$$

In other words, $\theta(\xi) - \theta(2\xi)(|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + \pi)|^2) \geq 0$ for all $\xi \in [0, \pi/2]$. By the definition of θ , we have $\theta(-\xi) = \theta(\xi)$ and $\theta(\xi) \leq \theta(\pi - \xi)$ for all $\xi \in [0, \pi/2]$ since $d_{m,j} \geq 0$ and $\sin^2 \frac{\pi - \xi}{2} = \cos^2 \frac{\xi}{2} \geq \frac{1}{2} \geq \sin^2 \frac{\xi}{2}$ for all $\xi \in [0, \pi/2]$. Consequently, we have

$$(3.4) \quad \theta(\xi) - \theta(2\xi)(|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + \pi)|^2) \geq 0 \quad \forall \xi \in [-\pi, \pi].$$

By the Fejér-Riesz lemma, there exists a 2π -periodic trigonometric polynomial θ_2 such that

$$(3.5) \quad |\theta_2(\xi)|^2 = \theta(\xi) - \theta(2\xi)(|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + \pi)|^2).$$

Now define

$$(3.6) \quad \begin{aligned} \widehat{b}^1(\xi) &:= \overline{\widehat{a}(\xi + \pi)} e^{-i\xi} \theta_1(2\xi) = \left(\frac{1 - e^{i\xi}}{2} \right)^m e^{-i\xi} \theta_1(2\xi), \\ \widehat{b}^2(\xi) &:= \widehat{a}(\xi) [\theta_2(2\xi) + \overline{\theta_2(2\xi)}] / 2 = \left(\frac{1 + e^{-i\xi}}{2} \right)^m [\theta_2(2\xi) + \overline{\theta_2(2\xi)}] / 2, \\ \widehat{b}^3(\xi) &:= \widehat{a}(\xi) [\theta_2(2\xi) - \overline{\theta_2(2\xi)}] / 2 = \left(\frac{1 + e^{-i\xi}}{2} \right)^m [\theta_2(2\xi) - \overline{\theta_2(2\xi)}] / 2. \end{aligned}$$

It is evident that b^1, b^2 , and b^3 are real-valued finitely supported sequences on \mathbb{Z} such that

$$(3.7) \quad \overline{\widehat{b}^1(\xi)} = (-1)^m e^{i(2-m)\xi} \widehat{b}^1(\xi), \quad \overline{\widehat{b}^2(\xi)} = e^{im\xi} \widehat{b}^2(\xi), \quad \overline{\widehat{b}^3(\xi)} = -e^{im\xi} \widehat{b}^3(\xi).$$

Let

$$(3.8) \quad \Theta(\xi) := \theta(2\xi)(|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + \pi)|^2).$$

Note that $|\theta_2(\xi) + \overline{\theta_2(\xi)}|^2 + |\theta_2(\xi) - \overline{\theta_2(\xi)}|^2 = 4|\theta_2(\xi)|^2$. By calculation, we have

$$\begin{aligned} & |\widehat{a}(\xi)|^2 \Theta(2\xi) + |\widehat{b}^1(\xi)|^2 + |\widehat{b}^2(\xi)|^2 + |\widehat{b}^3(\xi)|^2 \\ &= |\widehat{a}(\xi)|^2 \Theta(2\xi) + |\widehat{a}(\xi + \pi)|^2 |\theta_1(2\xi)|^2 + |\widehat{a}(\xi)|^2 |\theta_2(2\xi)|^2 \\ &= |\widehat{a}(\xi)|^2 \theta(4\xi) (|\widehat{a}(2\xi)|^2 + |\widehat{a}(2\xi + \pi)|^2) + |\widehat{a}(\xi + \pi)|^2 \theta(2\xi) \\ &\quad + |\widehat{a}(\xi)|^2 [\theta(2\xi) - \theta(4\xi) (|\widehat{a}(2\xi)|^2 + |\widehat{a}(2\xi + \pi)|^2)] \\ &= \theta(2\xi) (|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + \pi)|^2) \\ &= \Theta(\xi) \end{aligned}$$

and

$$\begin{aligned} & \widehat{a}(\xi) \overline{\widehat{a}(\xi + \pi)} \Theta(2\xi) + \widehat{b}^1(\xi) \overline{\widehat{b}^1(\xi + \pi)} + \widehat{b}^2(\xi) \overline{\widehat{b}^2(\xi + \pi)} + \widehat{b}^3(\xi) \overline{\widehat{b}^3(\xi + \pi)} \\ &= \widehat{a}(\xi) \overline{\widehat{a}(\xi + \pi)} \Theta(2\xi) - \widehat{a}(\xi) \overline{\widehat{a}(\xi + \pi)} |\theta_1(2\xi)|^2 + \widehat{a}(\xi) \overline{\widehat{a}(\xi + \pi)} |\theta_2(2\xi)|^2 \\ &= \widehat{a}(\xi) \overline{\widehat{a}(\xi + \pi)} \left[\Theta(2\xi) - \theta(2\xi) + [\theta(2\xi) - \theta(4\xi) (|\widehat{a}(2\xi)|^2 + |\widehat{a}(2\xi + \pi)|^2)] \right] \\ &= 0. \end{aligned}$$

Define $\widehat{\psi}^\ell(2\cdot) = \widehat{b}^\ell \widehat{B}_m$, $\ell = 1, 2, 3$. Since $\Theta(0) = 1$, by [4, Proposition 1.11] or [2], $\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$.

It follows from (2.2) that $\theta(\xi) - \theta(2\xi) |\widehat{a}(\xi)|^2 = O(|\xi|^{2m})$, $\xi \rightarrow 0$ (also see [4]). Thus, we deduce that $\Theta(\xi) - \Theta(2\xi) |\widehat{a}(\xi)|^2 = O(|\xi|^{2m})$, $\xi \rightarrow 0$. Consequently, each wavelet function ψ^ℓ , $\ell = 1, 2, 3$ has the vanishing moments of order m . The symmetry of the wavelet functions ψ^1, ψ^2, ψ^3 follows directly from (3.7). \square

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