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THE LUSTERNIK-SCHNIRELMANN CATEGORY OF Sp(3)

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ABSTRACT. We show that the Lusternik-Schnirelmann category of the symplectic group Sp(3) is 5. This L-S category coincides with the cone length and the stable weak category.

1. Introduction

The Lusternik-Schnirelmann category of a CW complex X, $\mathrm{cat}(X)$, is the least integer n for which X can be covered by $n+1$ subcomplexes, each of which is contractible in X. This invariant was introduced by Lusternik and Schnirelmann in [13]; they proved that any smooth function on a compact manifold M has strictly more than $\mathrm{cat}(M)$ critical points. Fox showed that the Lusternik-Schnirelmann category is a homotopy invariant [4], which means that calculating the category of spaces is a problem of homotopy theory. For the analytic applications, it is crucial to compute the L-S category of well-known compact manifolds. Indeed, determining the L-S category of Lie groups is the first problem on Ganea's famous problem list [6].

The present status of this problem is as follows. Singhof determined the values $\mathrm{cat}(\mathrm{SU}(n)) = n-1$ and $\mathrm{cat}(\mathrm{U}(n)) = n$ in [19]. The values $\mathrm{cat}(\mathrm{SO}(n))$ for $n < 5$ are easily computed by classical methods. More recently, James and Singhof [10] calculated $\mathrm{cat}(\mathrm{SO}(5)) = 8$. The symplectic groups have offered the most resistance: the only known value is $\mathrm{cat}(\mathrm{Sp}(2)) = 3$, which was obtained by Schweitzer [18] using secondary cohomology operations. Singhof extended the method to operations of arbitrarily large order, and proved in [20] the lower bound $\mathrm{cat}(\mathrm{Sp}(n)) \geq n+1$.

Actually, Schweitzer proved a little bit more. He showed that the weak category of Sp(2), which is a lower bound for $\mathrm{cat}(\mathrm{Sp}(2))$, is bounded below by 3. Since this lower bound coincides with an upper bound, namely the cone length, he concluded that $\mathrm{cat}(\mathrm{Sp}(2)) = 3$. (See §2 for the definitions.)

Recent work of N. Iwase [7, 8], P. Lambrechts, D. Stanley and L. Vandembroucq [11] has resulted in the resolution of several problems from Ganea's list. In this paper we make use of the new techniques developed in [3, 21] to make further progress on Problem 1. Our main result is the following.

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Theorem 1. *The symplectic group* Sp(3) *has Lusternik-Schnirelmann category* 5.

It follows from the proof of Theorem 1 that the 5-fold reduced diagonal $\overline{\Delta}_5$ can be factored

$$\begin{tikzcd} \mathrm{Sp}(3) \arrow[r, "\overline{\Delta}_5"] \arrow[d] & \mathrm{Sp}(3)^{\wedge 5} \\ S^{21} \arrow[r, "\nu^2"] & S^{15} \arrow[u, "j"'] \end{tikzcd}$$

where $\nu^2 \in \pi_{21}(S^{15}) \cong \mathbb{Z}/2$ is the unique nonzero element.

The proof of Theorem 1 has the same general outline as Schweitzer's argument for Sp(2). That is, we show that the cone length of Sp(3) is at most 5 and the weak category of Sp(3) is at least 5, and conclude that $\mathrm{cat}(\mathrm{Sp}(3)) = 5$. Our methods are quite different, however. In place of higher order operations, we use decompositions of the diagonal map as in [21] together with the Hopf-Ganea invariants studied in [3].

Theorem 1 gives another proof of the fact, first proved in [25], that Sp(3) satisfies Ganea's conjecture. To see this, simply apply Theorem 8 in [22], or observe that, in the terminology of [16], the stable class of the 5-fold reduced diagonal $\overline{\Delta}_5$ is a detecting class.

We now consider the analytic consequences of the main result. Denote by $\mathrm{Crit}(M)$ the minimum number of critical points for any smooth function on the compact manifold M. Lusternik and Schnirelmann showed that $\mathrm{Crit}(M) \geq \mathrm{cat}(M) + 1$. In view of Theorem 1, we see that $\mathrm{Crit}(M) \geq 6$. Since $\mathrm{Crit}(\mathrm{Sp}(3)) \leq 7$ can be deduced from a result of F. Takens [23], the value of $\mathrm{Crit}(\mathrm{Sp}(3))$ is either 6 or 7.

The minimal number of critical points on a compact manifold M for smooth functions which are quadratic at infinity, denoted $\widetilde{\mathrm{Crit}}(M)$, is also of interest to analysts. This invariant was introduced by F. Laudenbach and J.-C. Sikhorav in [12]. Our proof shows that the category of the space Sp(3) coincides with the cone length of Sp(3) and with a stabilized version of the category, denoted $\mathrm{Qcat}(\mathrm{Sp}(3))$; see [17, 25]. From the main theorem of P.-M. Moyaux and L. Vandembroucq in [15] we know that $\widetilde{\mathrm{Crit}}(\mathrm{Sp}(3)) - 1$ is less than the cone length and is bounded below by Qcat. Theorem 1 now implies that $\widetilde{\mathrm{Crit}}(\mathrm{Sp}(3)) = 6$.

The rest of the paper is devoted to the proof of Theorem 1. In §2 we establish the basic definitions, notation and results that we will use. Section 3 is devoted to an overview of the proof and its reduction to some lemmas. Their proofs are the content of the last section.

2. Background

All spaces are pointed; the basepoint of a space and the trivial map between two spaces are both denoted $*$. The (reduced) cone on a space A will be denoted $\mathcal{C}(A)$. For a CW complex X, we write X_n to denote the n-skeleton of X. The fat wedge in the n-fold cartesian product X^n of X with itself is the subcomplex

$$T^n(X) = \{(x_1, \dots, x_n) \,|\, \text{at least one } x_i = *\} \subseteq X^n.$$

The cofiber of $T^n(X) \hookrightarrow X^n$ is the n-fold smash product of X with itself, denoted $X^{\wedge n}$; the smash product of two spaces is denoted $X \wedge Y$, as usual. The diagonal map $\Delta_n : X \longrightarrow X^n$ is the map defined by $\Delta_n(x) = (x, \dots, x)$. The reduced diagonal $\overline{\Delta}_n : X \longrightarrow X^{\wedge n}$ is the composite of Δ_n with the quotient map $X^n \longrightarrow X^{\wedge n}$.

According to Whitehead [26], the **category** of X is the least integer n for which the map $\Delta_{n+1} : X \longrightarrow X^{n+1}$ lifts through $T^{n+1}(X)$, up to homotopy. The **weak category** of X, $\mathrm{wcat}(X)$, is the least n such that $\overline{\Delta}_{n+1} \simeq *$; clearly $\mathrm{wcat}(X) \leq \mathrm{cat}(X)$. See [9] for a survey of the Lusternik-Schnirelmann category and related invariants. The **strong category**, or **cone length**, of X is the least n for which there is a sequence of cofibrations $A_k \longrightarrow X_{k-1} \longrightarrow X_k$, $1 \leq k \leq n$, with $X_0 \simeq *$, $X_n \simeq X$, and $A_k = \Sigma B_k$ for $k > 1$ [2, 5].

We write $(f, g) : X \vee Y \longrightarrow Z$ for the map with components $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$. We write $[f, g] : X \longrightarrow Y \vee Z$ for the map with components $f : X \longrightarrow Y$ and $g : X \longrightarrow Z$; we use this notation only in the stable range, so there will be no ambiguity.

We refer to Toda [24] for information about the homotopy groups of spheres, and we use his notation here. In particular, $\omega' = \nu' + \alpha_1(3) \in \pi_6(S^3) = \mathbb{Z}_4 \oplus \mathbb{Z}_3$ denotes the Blakers-Massey element; ω' is the attaching map of the 7-cell in a cellular decomposition $S^3 \cup D^7 \cup D^{10}$ of Sp(2). Also, for $n > 3$, ν_n is the generator of the 2-primary component of $\pi_{n+3}(S^n)$; we know that $E^{n-3}\nu' = 2\nu_n$ for $n > 4$. Write $C = S^3 \cup_{\nu'} D^7$ and $C_n = S^n \cup_{\nu_n} D^{n+4}$ for $n > 4$.

We now recall a result from [1] that will be needed in the sequel.

Theorem A. *If X is a compact n-manifold and an H-space, then in any CW decomposition $X = X_{n-1} \cup_\alpha D^n$, the map α is stably trivial.*

We will frequently use this fact in the case $X = \mathrm{Sp}(n)$, $n = 2, 3$. It implies that if the connectivity of K is strictly greater than $\dim(X)/2$, then a composite $X \to S^{\dim X} \to K$ is essential if, and only if, $S^{\dim X} \to K$ is essential.

The key to our calculation is the following extension of a result from [3].

Proposition 2. *The space* Sp(3) *has a cone decomposition of the form*

$$\mathrm{Sp}(3) = S^3 \cup \mathcal{C}(C_6) \cup \mathcal{C}(C_9) \cup D^{18} \cup D^{21}.$$

Proof. It is proved in [3] that Sp(3) has a cellular decomposition of the form

$$\mathrm{Sp}(3) = S^3 \cup \mathcal{C}(C_6) \cup D^{10} \cup D^{14} \cup D^{18} \cup D^{21}.$$

It remains to show that the 10-cell and the 14-cell can be attached at the same time as the cone on a map $C_9 \longrightarrow S^6 \cup (C_6)$. Consider the locally trivial bundle $S^3 \longrightarrow \mathrm{Sp}(3) \longrightarrow V_{3,2}$, where $V_{3,2}$ is the homogeneous space $\mathrm{Sp}(3)/S^3$. This Stiefel manifold admits a cell decomposition $V_{3,2} = C_7 \cup D^{18}$, so $V_{3,2}$ has cone length 2 and hence $\mathrm{cat}(V_{3,2}) = 2$. We now apply [14, Theorem 6.2] to complete the proof. □

It follows that Sp(3) has cone length at most 5. Therefore Theorem 1 will be proved once we show that $\mathrm{wcat}(\mathrm{Sp}(3)) \geq 5$.

3. Proof of Theorem 1

As the problem is reduced to the non-triviality of a map (the 5-reduced diagonal of Sp(3)) we simplify the situation by proving that the 2-localization of $\overline{\Delta}_5$ is not trivial (see also the remark at the end of the paper) which is obviously sufficient for our purpose.

Moreover, we may also observe that Sp(3) is of 21-dimensional and $\mathrm{Sp}(3)^{\wedge 5}$ is 14-connected so the map $\overline{\Delta}_5$ is trivial if, and only if, it is stably trivial.

Therefore, for the end of the paper, all spaces and maps are supposed to be localized at 2 *and we are working stably. Let* X *denote* $\mathrm{Sp}(3)$ *with the CW decomposition given by Proposition 2.*

Since the subcomplex $A = S^3 \cup \mathcal{C}(C_6) \subseteq X$ has weak category 2, we may decompose the 5-fold reduced diagonal $\overline{\Delta}_5$ using the *main diagram:*

$$\begin{tikzcd}
X \arrow[r, "\overline{\Delta}_3"] \arrow[d, equal] & X \wedge X \wedge X \arrow[r, "1\wedge 1\wedge\overline{\Delta}_3"] \arrow[d] & X^{\wedge 5} \arrow[d, equal] \\
X \arrow[r] \arrow[d, equal] & X \wedge X \wedge (X/A) \arrow[r] & X^{\wedge 5} \\
X \arrow[r] \arrow[d] & C \wedge C \wedge C_{10} \arrow[r] \arrow[u, hook] \arrow[d, equal] & C^{\wedge 5} \arrow[u, hook] \arrow[d, equal] \\
S^{18} \vee S^{21} \arrow[r, "(q_1,q_2)"] \arrow[rr, bend right, "(k,h)"'] & C \wedge C \wedge C_{10} \arrow[r, "1\wedge 1\wedge\gamma"] & C^{\wedge 5}
\end{tikzcd}$$

The factorization in the second line is possible because $\mathrm{wcat}(A) = 2 < 3$, and so $\overline{\Delta}_3$ factors through a map $X/A \longrightarrow X^{\wedge 3}$. Neither this map, nor its restriction γ to C_{10}, is uniquely determined; however, the restriction to $S^{10} \subseteq C_{10}$ factors uniquely through the map $S^{10} \longrightarrow S^9$ induced by $\overline{\Delta}_3 : \mathrm{Sp}(2) \longrightarrow \mathrm{Sp}(2)^{\wedge 3}$, which is known to be η [18]. The third line is a cellular approximation. The fourth line is obtained by collapsing the 14-skeleton of X and (h, k) denotes the composition $(1 \wedge 1 \wedge \gamma) \circ (q_1, q_2)$.

This diagram expresses the reduced diagonal $\overline{\Delta}_5$ as a composition of several maps, and our proof amounts to explicitly determining each of them. The details are contained in the proofs of the following lemmas.

We first show that the sphere S^{18} does not play a role in the map $\overline{\Delta}_5$.

Lemma 3. *In the main diagram, the map* $k : S^{18} \longrightarrow C^{\wedge 5}$ *is trivial.*

Our next result focuses on the map γ.

Lemma 4. *The composite* $C_{10} \xrightarrow{\gamma} C^{\wedge 3} \longrightarrow C^{\wedge 3}/(S^3 \wedge S^3 \wedge S^3)$ *is trivial.*

Our two last results determine the behaviour of $q_2 \colon S^{21} \longrightarrow C \wedge C \wedge C_{10}$.

Lemma 5. *The map* $q_2 \colon S^{21} \longrightarrow C \wedge C \wedge C_{10}$ *admits a lifting* $\overline{q}_2$ *through the canonical injection:*

$$\begin{tikzcd}
 & S^3 \wedge S^3 \wedge C_{10} = C_{16} \arrow[d, hook] \\
S^{21} \arrow[ur, "\overline{q}_2"] \arrow[r, "q_2"] & C \wedge C \wedge C_{10}
\end{tikzcd}$$

Lemma 6. *The composite* $S^{21} \xrightarrow{\overline{q}_2} C_{16} \longrightarrow S^{20}$ *of* $\overline{q}_2$ *with the pinch map is equal to* η.

We now use these results to prove Theorem 1. The proofs of Lemmas 3, 4, 5 and 6 are given in the last section.

Proof of Theorem 1. From Lemma 4 and Lemma 5, the 5-fold reduced diagonal has a factorization of the form

$$\begin{tikzcd}
X \arrow[r, "\overline{\Delta}_5"] \arrow[d] & X^{\wedge 5} \\
S^{21} \arrow[r, "h"] & S^{15} \arrow[u, hook, "j"']
\end{tikzcd}$$

Since X is an H-space and a manifold, the attaching map for the 21-cell is stably trivial by Theorem A, and it follows that $j \circ h$ is the *unique* map $S^{21} \longrightarrow X^{\wedge 5}$ making the diagram commute.

We begin by showing that $h = \nu^2$. It follows from Lemma 5 that the map $h\colon S^{21} \longrightarrow C^{\wedge 5}$ factors through a map $C_{16} \longrightarrow S^{15}$. Examination of the third row of the main diagram reveals that the restriction of $C_{16} \longrightarrow S^{15}$ to S^{16} is the sixth suspension of the map $S^{10} \longrightarrow S^9$ induced by $\overline{\Delta}_3$. This map has been determined to be η in [18] (see also [3, 21]). Therefore the map $S^{21} \longrightarrow S^{15}$ that lifts h is the composition on the third row of the diagram

$$\begin{tikzcd}
S^{20} \arrow[r, "\eta"] \arrow[d] & S^{19} \arrow[d, "\nu_{16}"] & \\
* \arrow[r] \arrow[d] & S^{16} \arrow[d] \arrow[dr, "\eta"] & \\
S^{21} \arrow[r] \arrow[dr, "\eta"'] & C_{16} \arrow[r] \arrow[d] & S^{15} \\
& S^{20} &
\end{tikzcd}$$

In other words, the map $S^{21} \longrightarrow S^{15}$ is an element of the Toda secondary composition $\{\eta, \nu, \eta\}_0$, which is the singleton set $\{\nu^2\}$ by [24, Lemma 5.12].

It remains to show that $j \circ \nu^2 \neq 0$. Observe that the 22-skeleton of $X^{\wedge 5}$ has the homotopy type of $\mathrm{Sp}(2)^{\wedge 5}$. The attaching map for the 10-cell of Sp(2) being stably trivial by Theorem A, we have $(X^{\wedge 5})_{22} \simeq (S^{15} \cup_{2\nu} D^{19}) \vee \bigvee_{i=1}^{4} S^{19} \vee \bigvee_{i=1}^{5} S^{22}$.

Thus, to prove $j \circ \nu^2 \neq 0$ it is thus sufficient to establish that $r \circ \nu^2 \neq 0$, where r is defined by the cofiber sequence

$$S^{18} \xrightarrow{2\nu} S^{15} \xrightarrow{r} S^{15} \cup_{2\nu} D^{19} .$$

This previous sequence gives an exact sequence of homotopy groups

$$\pi_{21}(S^{18}) \longrightarrow \pi_{21}(S^{15}) \xrightarrow{r_*} \pi_{21}(S^{15} \cup_{2\nu} D^{19}) .$$

Since $2\nu \circ \nu = 0$, it follows that r_* is injective and so $r \circ \nu^2 \neq 0$. Consequently $\overline{\Delta}_5 \not\simeq *$ and hence $\mathrm{cat}(\mathrm{Sp}(3)) = \mathrm{wcat}(\mathrm{Sp}(3)) = 5$. $\square$

4. Proof of the lemmas

Proof of Lemma 3. Since $\mathrm{wcat}(X_{18}) \leq 4$ by Proposition 2, the map $\overline{\Delta}_5 : X_{18} \longrightarrow X^{\wedge 5}$ is trivial. Also, the inclusion $X_{18} \hookrightarrow X$ induces the inclusion of the first summand $S^{18} \hookrightarrow S^{18} \vee S^{21}$ after collapsing 14-skeleta. Thus we obtain the commutative diagrams:

$$\begin{tikzcd} X_{18} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X^{\wedge 5} \arrow[d, equal] \\ S^{18} \arrow[r, hook] & S^{18}\vee S^{21} \arrow[r, "(k,h)"'] & X^{\wedge 5} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_{18} \arrow[r] \arrow[d] & X^{\wedge 5} \arrow[d, equal] \\ S^{18} \arrow[r, "*"'] & X^{\wedge 5} \end{tikzcd}$$

We will show that there is a *unique* map $S^{18} \longrightarrow X^{\wedge 5}$ making the second diagram commute and conclude that $k = 0$.

From the cofiber sequence $X_{18} \longrightarrow S^{18} \longrightarrow \Sigma X_{14} \longrightarrow \Sigma X_{18}$, we see that it is enough to show that every map $\Sigma X_{14} \longrightarrow X^{\wedge 5}$ can be extended to ΣX_{18}. For this, we consider the commutative diagram

$$\begin{tikzcd} \Sigma X_{14} \arrow[r] \arrow[d] & S^{15} \arrow[r] \arrow[d] & X^{\wedge 5} \\ \Sigma X_{18} \arrow[r] & S^{15}\cup_x D^{19} \arrow[ur, dashed] & \end{tikzcd}$$

in which the attaching map x is some multiple of ν_{15} (we do not have to consider the element α_1 because the entire situation is 2-localized). Since the homotopy type of the 19-skeleton of $X^{\wedge 5}$ is given by

$$(X^{\wedge 5})_{19} \simeq (S^{15} \cup_{2\nu} D^{19}) \vee \left(\bigvee_{i=1}^{4} S^{19} \right),$$

the extension will be possible if x is an even multiple of ν. If x were an odd multiple of ν, then the operation $\mathrm{Sq}^4 : H^{14}(S^{14} \cup_x D^{18}) \longrightarrow H^{18}(S^{14} \cup_x D^{18})$ would be an isomorphism. But a simple calculation in $H^*(\mathrm{Sp}(3); \mathbb{Z}/2)$ shows that this operation is zero. $\square$

Proof of Lemma 4. We apply Theorem 2.2 of [3] to the CW-structure of $\mathrm{Sp}(3)_{14}$ given by Proposition 2 to get a factorization of the 3-reduced diagonal of $\mathrm{Sp}(3)_{14}$ as

$$\begin{tikzcd} \mathrm{Sp}(3)_{14} \arrow[rr] \arrow[dd] & & \mathrm{Sp}(3)_{14}^{\wedge 3} \\ & & C^{\wedge 3} \arrow[u] \\ C_{10} \arrow[r] & S^3 \wedge (S^7 \cup D^{11}) \arrow[r] & S^3 \wedge S^3 \wedge S^3 \arrow[u] \end{tikzcd}$$

Therefore the composite $C_{10} \xrightarrow{\gamma} C^{\wedge 3} \longrightarrow C^{\wedge 3}/(S^3 \wedge S^3 \wedge S^3)$ is trivial. $\square$

Proof of Lemma 5. The cone length of the space $X/A = (S^{10} \cup D^{14} \cup D^{18}) \cup D^{21}$ is at most 2 since $S^{10} \cup D^{14} \cup D^{18}$ is a suspension. Let $\varphi\colon S^{20} \to S^{10} \cup D^{14} \cup D^{18}$ denote the attaching map of the 21-cell. We deduce from Theorem A that the composite of φ with the projection $S^{10} \cup D^{14} \cup D^{18} \to S^{14} \cup D^{18}$ is stably trivial, therefore trivial. As a consequence, the map φ lifts to S^{10}. The suspension homomorphism $E\colon \pi_{19}(S^9) \to \pi_{20}(S^{10})$ is surjective [24, Theorem 7.3]. Therefore X/A is a suspension and its 2-reduced diagonal is trivial.

From the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\overline{\Delta}_2"] \arrow[d] & X \wedge X \arrow[d] \\ X/A \arrow[r] & (X/A) \wedge (X/A) \end{tikzcd}$$

we deduce the triviality of the composite map $X \xrightarrow{\overline{\Delta}_2} X \wedge X \longrightarrow X/A \wedge X/A$.

Recall now from Theorem 2.2 of [3] that the 2-reduced diagonal of A can be factored as $A \longrightarrow \Sigma C_6 \longrightarrow S^3 \wedge S^3 \longrightarrow A \wedge A$. We now consider a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \wedge X \arrow[d] \\ X/A \arrow[r] & (X \wedge X)/(S^3 \wedge S^3) \end{tikzcd}$$

in which the bottom map is the map induced by $A \to S^3 \wedge S^3$ between the cofibers of the inclusion $A \hookrightarrow X$ and $S^3 \wedge S^3 \hookrightarrow X \wedge X$. We now consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\overline{\Delta}_2"] & X \wedge X \arrow[r, "\overline{\Delta}_2 \wedge 1"] \arrow[d] & X \wedge X \wedge X \arrow[d] \\ & X \wedge X/A \arrow[r] \arrow[d] & X \wedge X \wedge X/A \arrow[d] \\ & X/A \wedge X/A \arrow[r] & ((X \wedge X)/(S^3 \wedge S^3)) \wedge X/A \end{tikzcd}$$

The second line is obtained by collapsing A in the last right factor. The third line is the previous factorization on the first factor. From the first step of this proof, we deduce the triviality of the composite map

$$X \xrightarrow{\overline{\Delta}_2} X \wedge X \xrightarrow{\overline{\Delta}_2 \wedge 1} X \wedge X \wedge X \longrightarrow ((X \wedge X)/(S^3 \wedge S^3)) \wedge X/A$$

which implies the triviality of

$$S^{21} \xrightarrow{q_2} C \wedge C \wedge C_{10} \xrightarrow{\rho} C \wedge C \wedge C_{10}/(S^3 \wedge S^3 \wedge C_{10}).$$

Observe that C_{16} is the 22-skeleton of the homotopy fiber of ρ. The existence of the lifting now comes from the associated long exact sequence of homotopy. $\square$

Proof of Lemma 6. We decompose the 3-reduced diagonal as follows:

$$\begin{tikzcd} X \arrow[r, "\overline{\Delta}_2"] \arrow[d] & X \wedge X \arrow[r, "\overline{\Delta}_2 \wedge 1"] \arrow[d] & X \wedge X \wedge X \arrow[d] \\ S^{21} \arrow[r] \arrow[d, equal] & X/S^3 \wedge (X/X_{11}) \arrow[r] & X \wedge X \wedge (X/X_{11}) \\ S^{21} \arrow[r, "\simeq"] & S^7 \wedge S^{14} \arrow[r, "\eta \wedge 1"] \arrow[u, hook] & S^3 \wedge S^3 \wedge S^{14} \arrow[u, hook] \end{tikzcd}$$

The second line is obtained by collapsing the last factor in $X^{\wedge 2}$, $X^{\wedge 3}$ by X_{11} and using the fact that the two maps $S^3 \to X \wedge X$, $X_{18} \to (X/S^3) \wedge (X/X_{11})$ are trivial. The third line is a cellular approximation.

The cup product structure of Sp(3) shows that the map $S^{21} \longrightarrow S^7 \wedge S^{14}$ is a homotopy equivalence and the map $S^7 \wedge S^{14} \longrightarrow S^6 \wedge S^{14}$ is evidently the 14-fold suspension of the map $S^7 \longrightarrow S^6$ induced by $\overline{\Delta}_2$. This latter map was shown in [18] to be η. □

Remark. We may observe that the only obstruction to the non-triviality of the 5-reduced diagonal of Sp(3) is 2-local. To see this, recall from [3] that the 4-fold reduced diagonal of $\mathrm{Sp}(3)_{18}$ factors through a Hopf-Ganea invariant which lies in the group $\pi_{18}(C_{10} \wedge \mathrm{Sp}(3)_{14})$. The 19-skeleton of $C_{10} \wedge \mathrm{Sp}(3)_{14}$ is the total space of a fibration over $S^{17} \vee S^{17}$ and whose fiber has S^{13} as 19-skeleton. As a consequence, the homotopy group $\pi_{18}(C_{10} \wedge \mathrm{Sp}(3)_{14})$ consists entirely of 2-torsion. Thus, if p is an odd prime, then the p-localization of $\mathrm{Sp}(3)_{18}$ has weak category ≤ 3 which implies that the p-localization of Sp(3) has weak category ≤ 4.

References

1. W. Browder and E. Spanier, *H-Spaces and duality*, Pacific J. Math. **12** (1962), 411-414. MR **26:**1891
2. O. Cornea, *There is just one rational cone-length*, Trans. Amer. Math. Soc. **344** (1994), 835–848. MR **95b:**55009
3. L. Fernández-Suárez, A. Gómez-Tato and D. Tanré, *Hopf-Ganea invariants and weak LS category.* Topology Appl. **115** (2001), 305–316. MR **2002g:**55023
4. R. Fox, *On the Lusternik-Schnirelmann category*, Ann. of Math. **42** (1941), 333–370. MR **2:**320f
5. T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois J. Math. **11** (1967), 417–427. MR **37:**4814
6. T. Ganea, *Some problems on numerical homotopy invariants*, Symp. Algebraic Topology, Lecture Notes in Math., no. 249, Springer-Verlag, Berlin, 1971, pp. 23–30. MR **49:**3910
7. N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. London Math. Soc. **30** (1998), 623-634. MR **99j:**55003
8. N. Iwase, *A_∞-method in Lusternik-Schnirelmann category*, Topology **41** (2002), 695–723.
9. I. M. James, *On category, in the sense of Lusternik and Schnirelmann*, Topology **17** (1978), 331–348. MR **80i:**55001
10. I. M. James and W. Singhof, *On the category of fiber bundles, Lie groups, and Frobenius maps*, Contemporary Math. **227** (1999), 177–189. MR **2000a:**55007
11. P. Lambrechts, D. Stanley and L. Vandembroucq, *Embeddings up to homotopy of two-cones in Euclidean space*, Trans. Amer. Math. Soc. **354** (2002), 3973–4013.
12. F. Laudenbach and J.-C. Sikorav, *Persistance d'intersection avec la section nulle au cours d'une isotopie hamiltonienne dans un fibré cotangent*, Invent. Math. **82** (1985), 349–357. MR **87c:**58042
13. L. Lusternik and L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Hermann, Paris, 1934.
14. H. Marcum, *Fibrations over double mapping cylinders*, Illinois J. Math. **24** (1980), 344–358. MR **81j:**55011
15. P.-M. Moyaux and L. Vandembroucq, *Lagrangian intersections in cotangent bundles*, Preprint (2002).
16. Y. Rudyak, *On category weight and its applications*, Topology **38** (1999), 37–55. MR **99f:**55007
17. H. Scheerer, D. Stanley and D. Tanré, *Fibrewise construction applied to Lusternik-Schnirelmann category*, Israel J. of Math. **131** (2002), 333–359.
18. P. A. Schweitzer, *Secondary cohomology operations induced by the diagonal mapping*, Topology **3** (1965), 337–355. MR **32:**451

19. W. Singhof, *On the Lusternik-Schnirelmann category of Lie groups*, Math. Zeitschrift **145** (1975), 111–116. MR **52:**11897
20. W. Singhof, *On the Lusternik-Schnirelmann category of Lie groups II*, Math. Zeitschrift **151** (1976), 143–148. MR **54:**13902
21. J. Strom, *Decomposition of the diagonal map.* Topology **42** (2003), 349–364.
22. J. Strom, *Essential category weight and phantom maps*, In: Cohomological methods in homotopy theory, Progress in Mathematics **196**, Birkhäuser, Basel, Boston, Berlin, 2001. MR **2002f:**55018
23. F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik-Schnirelmann category*, Invent. Math. **6** (1968), 197–244. MR **38:**5235
24. H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Annals of Mathematics Studies **49**, Princeton Universtity Press, 1962. MR **26:**777
25. L. Vandembroucq, *Fiberwise suspension and Lusternik-Schnirelmann category*, Topology **41** (2002), 1239-1258.
26. G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics **61**, Springer-Verlag, 1978. MR **80b:**55001

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