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ISOMETRIC COPIES OF l^1 AND l^∞ IN ORLICZ SPACES EQUIPPED WITH THE ORLICZ NORM

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Abstract. Criteria in order that an Orlicz space equipped with the Orlicz norm contains a linearly isometric copy (or an order linearly isometric copy) of l^1 (or l^∞) are given.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ stand for the set of natural numbers, the set of reals and the set of nonnegative reals, respectively. Let (T, Σ, μ) be a σ-finite measure space that does not reduce to a finite number of atoms, where all atoms that appear in Σ have measure 1, and let $L^0 = L^0(T, \Sigma, \mu)$ be the space of all (equivalence classes of) Σ-measurable functions defined on T.

A function $\Phi : \mathbb{R} \to [0, +\infty]$ is said to be an Orlicz function if $\Phi(0) = 0$, $\Phi(u) \to \infty$ as $u \to \infty$, Φ is even and convex and $\lim_{u \to b(\Phi)_-} \Phi(u) = \Phi(b(\Phi))$, where $b(\Phi) = \sup\{u > 0 : \Psi(u) < \infty\} > 0$. Note that the case $\Phi(b(\Phi)) = \infty$ is not excluded. The function Ψ complementary to Φ in the sense of Young is defined by

$$\Psi(u) = \sup_{v>0}\{|u|v - \Phi(v)\}.$$

It is obvious that the Young inequality

$$uv \leq \Phi(u) + \Psi(v)$$

holds for all $u, v \in \mathbb{R}$ and that in the case when $b(\Phi) = \infty$ or $\Phi(b(\Phi)) < \infty$ and $\Phi'_-(b(\Phi)) < \infty$, we have the equality

$$uv = \Phi(u) + \Psi(v)$$

for all $u \in [0, b(\Phi))$ (resp. $u \in [0, b(\Phi)]$) and $v \in [\Phi'_-(u), \Phi'_+(u)]$, where $\Phi'_-(u)$ and $\Phi'_+(u)$ denote the left (resp. the right) derivative of Φ at the point u. For an example of an Orlicz function Φ with $0 < b(\Phi) < \infty$ and $\Phi'_-(b(\Phi)) = \infty$ such that the equality

$$b(\Phi)v = \Phi(b(\Phi)) + \Psi(v)$$

holds for no $v \in \mathbb{R}_+$ we refer to [6].

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Let us define for an Orlicz function Φ another important parameter,

$$a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\}.$$

Given an Orlicz function Φ we define on L^0 the convex modular

$$I_\Phi(x) = \int_T \Phi(x(t))d\mu$$

and the Orlicz space

$$L^\Phi = L^\Phi(T, \Sigma, \mu) := \{x \in L^0 : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

The most important norms in L^Φ are the following two:

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\},$$

called the Luxemburg norm (see [1] and [7] - [12]), and

$$\|x\|^0_\Phi = \sup\{|\int_T x(t)y(t)d\mu| : I_\Psi(y) \leq 1,\ y \in L^0\},$$

called the Orlicz norm (see [1] and [7] - [12]). The Amemiya formula

$$\|x\|^0_\Phi = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

for the Orlicz norm is very useful because it does not use the function Ψ complementary to Φ. For Orlicz functions Φ that are N-functions at infinity, that is, $(\Phi(u)/u) \to \infty$ as $u \to \infty$, this formula was well known from the beginning of the theory of Orlicz spaces (see [8]) and for arbitrary Orlicz functions it was proved in [6]. For any $x \in L^\Phi$, we define supp $x = \{t \in T : x(t) \neq 0\}$.

It is well known that for any Orlicz function the quotient $\Phi(u)/u$ is nondecreasing on $\mathbb{R}_+$. So the limit (finite or infinite) $A(\Phi) = \lim_{u\to\infty}(\Phi(u)/u)$ always exists. Let us define the function $R_\Phi(u) = A(\Phi)|u| - \Phi(u)$. As we will see below, this function will be of great importance.

We say that an Orlicz function Φ satisfies the $\triangle_2$-condition at zero (at infinity) [on $\mathbb{R}_+$] if there are positive constants $K \geq 2$ and u_0 with $0 < \Phi(u_0) < \infty$ such that the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in [0, u_0]$ ($u \in [u_0, \infty)$) [$u \in \mathbb{R}_+$]. We then write $\Phi \in \triangle_2(0)$ ($\Phi \in \triangle_2(\infty)$) [$\Phi \in \triangle_2$] for short. It is obvious that $\Phi \in \triangle_2$ $\iff$ $\Phi \in \triangle_2(0)$ and $\Phi \in \triangle_2(\infty)$. Moreover, $b(\Phi) = \infty$ whenever $\Phi \in \triangle_2(\infty)$ and $a(\Phi) = 0$ whenever $\Phi \in \triangle_2(0)$.

An Orlicz space L^Φequipped with the Orlicz norm $\|\cdot\|^0_\Phi$ will be denoted by L^Φ_0. The unit ball and the unit sphere of L^Φ_0 will be denoted by $B(L^\Phi_0)$ and $S(L^\Phi_0)$, respectively. For any $x \in L^\Phi_0 \setminus \{0\}$ we denote by $K(x)$ the set of these $k > 0$ such that $\|x\|^0_\Phi = \frac{1}{k}(1 + I_\Phi(kx))$. In the case when $\|x\|^0_\Phi = \lim_{k\to\infty} \frac{1}{k}(1 + I_\Phi(kx))$, we write $\infty \in K(x)$. If $A(\Phi) < \infty$, then it can happen that $K(x) = \emptyset$.

Since in the case when $a(\Phi) = 0$ the Orlicz space L^Φ_0 is strictly monotone (it does not matter if Φ satisfies or not the suitable $\triangle_2$-condition; see [5]), L^Φ_0 cannot contain an order isometric copy of l^∞ (in contrast to the case when the Luxemburg norm is considered). However, as we will see below in the case when $a(\Phi) = 0$, L^Φ_0 can contain an order isometric copy of l^∞.

In this paper we present criteria for the existence in Orlicz spaces L^Φ_0 equipped with the Orlicz norm a linearly isometric copy or an order linearly isometric copy of X, where X is equal to l^∞ or l^1. Such criteria are important when we are looking for criteria of other important topological and geometrical properties of L^Φ_0. Our results on l^∞-copies do not follow from [4] although they are connected

with those results. In the special case of Orlicz spaces our results are more precise. The results of [13] are also connected with our Theorems 1 and 2. A class of Orlicz spaces isomorphic or isomorphically isometric to l^∞ is distinguished in [13].

2. Results

We start with criteria for the existence of an order linearly isometric copy of l^∞ in L_0^Φ.

Theorem 1. *Let Φ be an Orlicz function with $b(\Phi) = \infty$. Then L_0^Φ contains an order linearly isometric copy of l^∞ if and only if μ is infinite and $a(\Phi) > 0$.*

Proof. Sufficiency. Assume that $b(\Phi) = \infty$, $a(\Phi) > 0$ and μ is infinite. Divide T into a sequence $(T_n)_{n=1}^\infty$ of pairwise disjoint sets such that $\mu(T_n) = \infty$ for any $n \in \mathbb{N}$. Define $x_n = a(\Phi)\chi_{T_n}$ $(n \in \mathbb{N})$. Then

$$\begin{aligned} 1 + I_\Phi(x_n) &= 1, \\ \frac{1}{k}(1 + I_\Phi(kx_n)) &= \frac{1}{k} > 1 \ \ (\forall k \in (0,1), \ \ n \in \mathbb{N}), \\ \frac{1}{k}(1 + I_\Phi(kx_n)) &= \infty \ \ (\forall k > 1, \ \ n \in \mathbb{N}). \end{aligned}$$

Therefore, $\|x_n\|_\Phi^0 = 1$ for any $n \in \mathbb{N}$. Moreover, in the same way we can prove that $\| \sum_{n=1}^\infty x_n\|_\Phi^0 = 1$. Hence it follows that the operator

$$Py = \sum_{n=1}^\infty y_n x_n \ \ (\forall y = (y_n) \in l^\infty),$$

which is obviously linear and positive, is an order isometry of l^∞ onto the closed subspace $P(l^\infty)$ of L_0^Φ (cf [4]).

Necessity. Note that the inequality $a(\Phi) > 0$ is necessary, since if $a(\Phi) = 0$, then L_0^Φ is strictly monotone (see [5]). Since strict monotonicity is preserved by linear order isometries and l^∞ is not strictly monotone, L_0^Φ cannot contain an order linearly isometric copy of l^∞ if $a(\Phi) = 0$. Therefore, we may assume in the remaining part of the proof of necessity that $a(\Phi) > 0$.

In order to prove the necessity of the condition $\mu(T) = \infty$, assume to the contrary that $\mu(T) < \infty$, $a(\Phi) > 0$ and L_0^Φ contains an order linearly isometric copy of l^∞. Since any order linear isometry preserves the orthogonality of elements (see [4]), there is in L_0^Φ a sequence $(x_n)_{n=1}^\infty$ such that $x_n \geq 0$, $\|x_n\|_\Phi^0 = 1$ for all $n \in \mathbb{N}$ and $\| \sum_{n=1}^\infty x_n\|_\Phi^0 = 1$. It is enough to take $e_n = (0, ..., 0, 1, 0, ...) \in l^\infty$ and $x_n = Pe_n$, where P is a linear order isometry of l^∞ onto the closed subspace $P(l^\infty)$ of L_0^Φ. Then the equality $\|x_n\|_\Phi^0 = 1$ for any $n \in \mathbb{N}$ is obvious. To prove the equality

$$\| \sum_{n=1}^\infty x_n\|_\Phi^0 = 1,$$

note first that

$$\| \sum_{n=1}^k x_n\|_\Phi^0 = \| \sum_{n=1}^k Pe_n\|_\Phi^0 = \|P(\sum_{n=1}^k e_n)\|_\Phi^0 = \| \sum_{n=1}^k e_n\|_\infty = 1$$

for any $k \in \mathbb{N}$.

Since L_0^Φ has the Fatou property (see [6]), we get $\sum_{n=1}^\infty x_n \in L_0^\Phi$ and

$$\|\sum_{n=1}^\infty x_n\|_\Phi^0 = \lim_{k\to\infty} \|\sum_{n=1}^k x_n\|_\Phi^0 = 1.$$

Since $\mu(T) < \infty$ and supp $x_n \cap$ supp $x_m = \emptyset$ for $n \neq m$, we get $\mu(\text{supp } x_n) \to 0$ as $n \to \infty$. By the assumption that all atoms that appear in T have measure 1 and Σ does not reduce to a finite number of atoms, there is no loss of generality in assuming that T is nonatomic. Therefore, we may assume that $\Phi \notin \triangle_2(\infty)$, because otherwise L_0^Φ is order continuous and so L_0^Φ cannot even contain an order linearly isomorphic copy of l^∞. In consequence we get that $A := A(\Phi) = \infty$. This implies that $K(x) \neq \emptyset$ for any $x \in L_0^\Phi \setminus \{0\}$ (see [3]). We claim that

(1) for any $c > 0$ and $m \in \mathbb{N}$ there is $n \in \mathbb{N}$

$$\text{such that } n > m \text{ and } \mu\{t \in T : x_n(t) > c\} > 0.$$

Otherwise, there are $c > 0$ and $m \in \mathbb{N}$ such that $x_n(t) \leq c$ μ-a.e. in T for all $n \in \mathbb{N}$, $n > m$. Since $\mu(T) < \infty$, $b(\Phi) = \infty$ and $\mu(\text{supp } x_n) \to 0$ as $n \to \infty$, we conclude that $I_\Phi(\lambda x_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$. Consequently, for any $k > 1$ taking $n \to \infty$, we get

$$1 = \|x_n\|_\Phi^0 \leq \frac{1}{k}(1 + I_\Phi(kx_n)) \to \frac{1}{k} < 1,$$

a contradiction. So, the proof of the claim is finished.

Let $n \in \mathbb{N}$ be such that

$$\mu\{t \in T : x_n(t) > a(\Phi)\} > 0$$

and choose an arbitrary $k \in K(x_1 + x_n)$. Since $\|x_1 + x_n\|_\Phi^0 = 1$, we have $k \geq 1$, whence $\mu(A) > 0$ for $A = \{t \in T : kx_n(t) > a(\Phi)\}$. Therefore,

$$\begin{aligned} \|x_1 + x_n\|_\Phi^0 &= \frac{1}{k}(1 + I_\Phi(k(x_1 + x_n))) \\ &> \frac{1}{k}(1 + I_\Phi(kx_1)) \geq \|x_1\|_\Phi^0 \\ &= 1, \end{aligned}$$

a contradiction, which finishes the proof. □

Theorem 2. *Let Φ be an Orlicz function with $b(\Phi) < \infty$. Then L_0^Φ contains an order linearly isomertric copy of l^∞ if and only if $a(\Phi) > 0$ and either* (a) $\mu(T) = \infty$ *or* (b) $\mu(T) < \infty$ *and $a(\Phi) = b(\Phi)$.*

Proof. If $a(\Phi) > 0$, $b(\Phi) < \infty$ and $\mu(T) = \infty$, we can repeat the appropriate part of the proof of Theorem 1 to prove the sufficiency.

Assume now that $b(\Phi) < \infty$, $a(\Phi) > 0$, $a(\Phi) = b(\Phi)$ and $\mu(T) < \infty$. Then $L_0^\Phi = L^\infty$ and $\|x\|_\Phi^0 = \frac{1}{a(\Phi)}\|x\|_\infty$ for any $x \in L_0^\Phi$, where $\|x\|_\infty := \text{esssup}_{t\in T} |x(t)|$.

Therefore, L_0^Φ is order linearly isometric to $(L^\infty, \| \|_\infty)$, so it contains an order linearly isometric copy of l^∞. □

Necessity. If $a(\Phi) = 0$, then L_0^Φ is strictly monotone; so it cannot contain an order linearly isometric copy of l^∞. So, under the assumptions that $\mu(T) < \infty$, $a(\Phi) > 0$ and $b(\Phi) < \infty$, we need to prove the necessity of the condition $a(\Phi) = b(\Phi)$. Assume to the contrary that $a(\Phi) < b(\Phi)$ and L_0^Φ contains an order linearly isometric copy of l^∞. Then there is a sequence $(x_n)_{n=1}^\infty$ in $S(L_0^\Phi)$ of disjointly supported elements

such that $\|\sum_{n=1}^{\infty} x_n\|_\Phi^0 = 1$ (see the proof of the necessity part of Theorem 1). Therefore, we have $\mu(\text{supp } x_n) \to 0$ as $n \to \infty$ and $K(x) \neq \emptyset$ for any $x \in L_0^\Phi \setminus \{0\}$. Let $m, n \in \mathbb{N}$, $m \neq m$, and $k_{m,n} \in K(x_m + x_n)$. Then

$$1 = \frac{1}{k_{m,n}}(1 + I_\Phi(k_{m,n}x_m)) + \frac{1}{k_{m,n}} I_\Phi(k_{m,n}x_n). \tag{2}$$

Since $\frac{1}{k_{m,n}}(1 + I_\Phi(k_{m,n}x_m)) \geq 1$, equality (1) yields

$$\frac{1}{k_{m,n}}(1 + I_\Phi(k_{m,n}x_m)) = 1 \text{ and } I_\Phi(k_{m,n}x_n) = 0. \tag{3}$$

In the same way we can get

$$\frac{1}{k_{m,n}}(1 + I_\Phi(k_{m,n}x_n)) = 1 \text{ and } I_\Phi(k_{m,n}x_m) = 0. \tag{4}$$

By (2), (3) and (4), we get $k_{m,n} = 1$ for all $m, n \in \mathbb{N}$, $m \neq n$, and $|x_n| \leq a(\Phi)$ μ-a.e. in T for any $n \in \mathbb{N}$.

For any $k > 1$ satisfying $ka(\Phi) < b(\Phi)$, since $\mu(\text{supp } x_n) \to 0$, we get $I_\Phi(kx_n) \to 0$, and consequently

$$1 = \|x_n\|_\Phi^0 \leq \frac{1}{k}(1 + I_\Phi(kx_n)) \to \frac{1}{k} < 1 \qquad \text{as} \quad n \to \infty,$$

a contradiction. This proves the necessity of the equality $a(\Phi) = b(\Phi)$ whenever $\mu(T) < \infty$. □

Theorems 1 and 2 can be summarized into the following result.

Theorem 3. *Let Φ be an Orlicz function. Then we have:*

(i) *if $\mu(T) = \infty$, then L_0^Φ contains an order linearly isometric copy of l^∞ if and only if $a(\Phi) > 0$;*

(ii) *if $\mu(T) < \infty$, then L_0^Φ contains an order linearly isometric copy of l^∞ if and only if $a(\Phi) > 0$, $b(\Phi) < \infty$ and $a(\Phi) = b(\Phi)$, that is, L_0^∞ is order linearly isometric to $(L^\infty, \| \|_\infty)$.*

In the next theorem we will use the function $R(u) = A|u| - \Phi(u)$, where $A = A(\Phi) = \lim_{u\to\infty}(\Phi(u)/u)$.

Theorem 4. *For any Orlicz function Φ with $a(\Phi) = 0$ and $b(\Phi) = +\infty$ and any nonatomic σ-finite measure space (T, Σ, μ), the following assertions are equivalent:*

(1) *L_0^Φ has a subspace order linearly isometric to l^1;*
(2) *L_0^Φ has a subspace linearly isometric to l^1;*
(3) *There exists a nonzero $x \in L_0^\Phi$ such that $K(x) = \emptyset$;*
(4) *The function $R(u)$ is upper bounded.*

Proof. The implication **1** ⇒ **2** is obvious. Let us now prove the implication **2** ⇒ **3**. If $K(x) \neq \emptyset$ for all nonzero x in L_0^Φ, then L_0^Φ is non-square by the proof of Theorem 3.26 in [1]. Therefore, assertion **2** is not true if assertion **3** is not true, which finishes the proof of the implication **2** ⇒ **3**. Assume that assertion **3** holds and $R(u)$ is not upper bounded. Let

$$f(k) := \frac{1}{k}(1 + \int_T \Phi(kx(t))d\mu) = A\int_T |x(t)|d\mu + \frac{1}{k}(1 - \int_T R(kx(t))d\mu)$$

for $k > 0$. Assertion **3** implies that $\|x\|_\Phi^0 = A\int_T |x(t)|d\mu$ (see [1] and [2]). Since R is not upper bounded, we conclude that

$$\frac{1}{k}(1 - \int_T R(kx(t))d\mu) < 0 \tag{5}$$

for $k > 0$ large enough. Therefore (5) implies that

$$\|x\|_\Phi^0 \le f(k) < A\int_T |x(t)|d\mu = \|x\|_\Phi^0$$

for $k > 0$ large enough, a contradiction.

4 $\Rightarrow$ **1**. Let $\sup\{R(u) : u \in \mathbb{R}_+\} =: c < \infty$. Pick pairwise disjoint subsets E_n $(n = 1, 2, ...)$ of T such that $\mu(E_n) > 0$ and $c\sum_{n=1}^\infty \mu(E_n) < 1$. Choose $a_n > 0$ such that $\|a_n\chi_{E_n}\|_\Phi^0 = 1$. Define $x_n = a_n\chi_{E_n}$. Then for any $(b_n)_{n=1}^\infty \in l^1$ and any $\varepsilon > 0$, pick $k > 0$ such that

$$\|\sum_{n=1}^\infty b_n x_n\|_\Phi^0 > \frac{1}{k}(1 + I_\Phi(k\sum_{n=1}^\infty b_n x_n)) - \varepsilon.$$

Then

$$\begin{aligned}
\sum_{n=1}^\infty |b_n| &\ge \|\sum_{n=1}^\infty b_n x_n\|_\Phi^0 \\
&> \frac{1}{k}(1 + \sum_{n=1}^\infty \Phi(kb_n a_n)\mu(E_n)) - \varepsilon \\
&= A\sum_{n=1}^\infty |b_n a_n|\mu(E_n) + \frac{1}{k}(1 - \sum_{n=1}^\infty R(k|b_n|a_n)\mu(E_n)) - \varepsilon \\
&\ge A\sum_{n=1}^\infty |b_n a_n|\mu(E_n) + \frac{1}{k}(1 - c\sum_{n=1}^\infty \mu(E_n)) - \varepsilon \\
&\ge A\sum_{n=1}^\infty |b_n a_n|\mu(E_n) - \varepsilon.
\end{aligned}$$

Combining this with the fact that

$$\begin{aligned}
\|x_n\|_\Phi^0 &\le \lim_{k\to\infty}\frac{1}{k}(1 + I_\Phi(kx_n)) \\
&= \lim_{k\to\infty}\frac{1}{k}(1 + \Phi(ka_n)\mu(E_n)) \\
&= Aa_n\mu(E_n),
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{n=1}^\infty |b_n| &\ge A\sum_{n=1}^\infty |b_n||a_n|\mu(E_n) \\
&\ge \sum_{n=1}^\infty |b_n|\|x_n\|_\Phi^0 = \sum_{n=1}^\infty |b_n|.
\end{aligned}$$

This shows that the map $P : l^1 \to L_0^\Phi$ defined by $P((b_n)_{n=1}^\infty) = \sum_{n=1}^\infty b_n x_n$ is a linear isometry. Since the map P is positive, it is an order isometry.

Theorem 5. *Let Φ be an Orlicz function and the measure space be nonatomic and σ-finite. Then the following are equivalent:*

(1) *The space L_0^Φ contains an order isometric copy of l^1;*
(2) *There exists a nonzero $x \in L_0^\Phi$ such that $K(x) = \emptyset$;*
(3) *The function $R(u)$ is upper bounded.*

Proof. The implications **2** $\Rightarrow$ **3** $\Rightarrow$ **1** can be proved in the same way as in Theorem 4.

1 $\Rightarrow$ **2**. Assume that **2** does not hold. Then $K(x) \neq \emptyset$ for any $x \in L_0^\Phi \setminus \{0\}$. Take any $x, y \in S(L_0^\Phi)$ with supp $x \cap$ supp $y = \emptyset$. Let $k \in K(x)$, $h \in K(y)$. We may assume, without loss of generality, that $k \leq h$. Then

$$\begin{aligned} 2 &= \|x\|_\Phi^0 + \|y\|_\Phi^0 = \frac{1}{k}(1 + I_\Phi(kx)) + \frac{1}{h}(1 + I_\Phi(hy)) \\ &\geq \frac{1}{k}(1 + I_\Phi(kx)) + \frac{1}{h} + \frac{1}{k} I_\Phi(ky) \\ &\geq \frac{1}{k}(1 + I_\Phi(k(x+y))) + \frac{1}{h} \geq \|x+y\|_\Phi^0 + \frac{1}{h}, \end{aligned}$$

whence $\|x+y\|_\Phi^0 \leq 2 - \frac{1}{h} < 2$. This yields that **1** cannot hold. Indeed, otherwise, by the fact that order isometry preserves disjointness of supports of functions (up to a set of measure zero), see [4], taking $e_1 = (1, 0, 0, ...)$, $e_2 = (0, 1, 0, ...)$ in l^1 and an order linear isometry P of l^1 onto a closed subspace of L_0^Φ, we get $\mu(\operatorname{supp} Px \cap \operatorname{supp} Py) = 0$ and $Px, Py \in S(L_0^\Phi)$. Therefore, we have by the above

$$2 = \|e_1 + e_2\|_{l^1} = \|Px + Py\|_\Phi^0 < 2,$$

a contradiction which finishes the proof of the implication **1** $\Rightarrow$ **2** as well as the proof of the theorem. □

Remark 1. The condition $\lim_{u\to\infty}(\Phi(u)/u) = A < \infty$ need not imply that L_0^Φ contains a linearly isometric copy of l^1, i.e. $A = A(\Phi) < \infty$ need not imply that $R(u)$ is bounded.

To show this, define for example the function $\Phi(u) = A(u+c) - K(u+c)^\alpha$ for $u \geq 0$ and $\Phi(-u) = \Phi(u)$, where $A > 0$, $K > 0$, $0 < \alpha < 1$, and $c = (K/A)^{1/(1-\alpha)}$. It is obvious that $\lim_{u\to\infty}(\Phi(u)/u) = A$. Moreover,

$$R(u) = Au - \Phi(u) = K(u+c)^\alpha - Ac,$$

whence we get $\sup\{R(u) : u \geq 0\} = \infty$.

Remark 2. It should be worth noting that the function $\Phi(u) = |u| - 1 + e^{-|u|}$ is strictly convex but, by Theorem 5, L_0^Φ contains an order linearly isometric copy of l^1. This contrasts nicely with what happens if Φ were an N-function, where the strict convexity of L_0^Φ and Φ go hand-in-hand for nonatomic measure spaces (see Milnes theorem, that is, Theorem 6 on page 274 in [12]).

Remark 3. It is well known that l^∞ contains a linearly isometric copy of any separable Banach space X. Namely if $(x_n)_{n=1}^\infty$ is the sequence which is dense in X and $(x_n^\star)_{n=1}^\infty \subset S(X^\star)$ is such that $x_n^\star(x_n) = \|x_n\|$ for any $n \in \mathbb{N}$, then the isometry $P : X \to l^\infty$ is defined by $Px = (x_n^\star(x))_{n=1}^\infty \in l^\infty$ for any $x \in X$.

From Theorems 3 and 4 and Remark 3, we get the following.

Corollary 1. *There are Orlicz spaces L_0^Φ containing a linearly isometric copy of l^1 but not containing an order linearly isometric copy of l^1.*

Indeed, if $a(\Phi) = b(\Phi)$, then by Theorem 3 and Remark 3, L_0^Φ contains a linearly isometric copy of l^1. However, the equality $a(\Phi) = b(\Phi)$ yields $a(\Phi) > 0$ and $b(\Phi) < \infty$, whence $K(x) \neq \emptyset$ for any $x \in L_0^\Phi \setminus \{0\}$. Consequently, by Theorem 5, L_0^Φ does not contain an order linearly isometric copy of L_0^Φ. This phenomena is possible only in Banach lattices X which are not strictly monotone. Otherwise (see [14]), if X contains a linearly isometric copy of l^1, then it also contains an order linearly isometric copy of l^1.

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