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BOUNDEDNESS AND OSCILLATION FOR NONLINEAR DYNAMIC EQUATIONS ON A TIME SCALE

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ABSTRACT. We obtain some boundedness and oscillation criteria for solutions to the nonlinear dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)(f \circ x^{\sigma}) = 0,$$

on time scales. In particular, no explicit sign assumptions are made with respect to the coefficient $q(t)$. We illustrate the results by several examples, including a nonlinear Emden–Fowler dynamic equation.

1. INTRODUCTION

Consider the second order nonlinear dynamic equation

$$(p(t)x^{\Delta})^{\Delta} + q(t)(f \circ x^{\sigma}) = 0, \tag{1.1}$$

where p and q are real–valued, right–dense continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$. We also assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$f'(x) \geq \frac{f(x)}{x} > 0 \quad \text{for} \quad x \neq 0. \tag{1.2}$$

Although we shall assume p is a positive function we do not make any explicit sign assumptions on q in contrast to most results on nonlinear oscillations.

For completeness, we recall the following concepts related to the notion of time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and, since boundedness and oscillation of solutions is our primary concern, we make the blanket assumption that $\sup \mathbb{T} = \infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The *forward* and *backward jump operators* are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \; s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$; here $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be *left–dense* if $\rho(t) = t$, *right–dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left–scattered* if $\rho(t) < t$ and *right–scattered* if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *right–dense continuous* (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The *graininess*

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function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

The assumption (1.2) allows f to be of superlinear growth, say

$$f(x) = x^{2n+1}, \quad n \geq 1. \tag{1.3}$$

In several papers ([4], [12]), (1.1) has been studied with $q > 0$ and assuming the nonlinearity has the property

$$\left|\frac{f(x)}{x}\right| \geq K \quad \text{for} \quad x \neq 0. \tag{1.4}$$

This essentially says that the equation is, in some sense, not too far from being linear. We shall see that one may relate oscillation and boundedness of solutions of the nonlinear equation (1.1) to the linear equation

$$\left(p(t)x^\Delta\right)^\Delta + \lambda q(t)x^\sigma = 0, \tag{1.5}$$

where $\lambda > 0$, for which many oscillation criteria are known (see, e.g., [1], [2], [3], [4], [5], [7], [9], and [11]). In particular, we will obtain the time scale analogues of the results due to Erbe [6] for the continuous case $\mathbb{T} = \mathbb{R}$. We shall restrict attention to solutions of (1.1) which exist on some interval of the form $[T_x, \infty)$, where $T_x \in \mathbb{T}$ may depend on the particular solution. This paper is organized as follows. In Section 2 we present some preliminary results on the chain rule, integration by parts, and an auxiliary lemma. Section 3 contains the main results on oscillation and boundedness and several examples are given in Section 4.

2. Preliminary results

On an arbitrary time scale $\mathbb{T}$, the usual chain rule from calculus is no longer valid (see Bohner and Peterson [3], p. 31). One form of the extended chain rule, due to S. Keller [13] and generalized to measure chains by C. Pötzsche [14], is as follows. (See also Bohner and Peterson [3], p. 32.)

Lemma 2.1. *Assume $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. Assume further that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and satisfies*

$$(f \circ g)^\Delta(t) = \left\{\int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh\right\} g^\Delta(t). \tag{2.1}$$

We shall also need the following integration by parts formula (cf. [3]), which is a simple consequence of the product rule and which we formulate as follows:

Lemma 2.2. *Let $a, b \in \mathbb{T}$ and assume $f^\Delta, g^\Delta \in C_{rd}$. Then*

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t. \tag{2.2}$$

Before stating the next result, we recall that a solution of equation (1.1) is said to be oscillatory on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. Since $p(t) > 0$ we shall consider both cases

$$\int_a^\infty \frac{1}{p(t)}\Delta t = \infty \tag{2.3}$$

and

$$\int_a^\infty \frac{1}{p(t)}\Delta t < \infty. \tag{2.4}$$

We also introduce the following condition:

$$\liminf_{t\to\infty} \int_T^t q(s)\Delta s \geq 0 \quad \text{and } \not\equiv 0 \tag{2.5}$$

for all large T. It can be shown that (2.5) implies either $\int_a^\infty q(s)\Delta s = +\infty$ or that

$$\int_T^\infty q(s)\Delta s = \lim_{t\to\infty}\int_T^t q(s)\Delta s$$

exists and satisfies $\int_T^\infty q(s)\Delta s \geq 0$ for all large T.

We have the following lemma which describes the behavior of a nonoscillatory solution of (1.1) for the case when (2.3) and (2.5) hold.

Lemma 2.3. *Let x be a nonoscillatory solution of* (1.1) *and assume conditions* (2.3) *and* (2.5) *hold. Then there exists $T_1 \geq T$ such that*

$$x(t)x^\Delta(t) > 0 \quad \textit{for} \quad t \geq T_1.$$

Proof. Suppose that x is a nonoscillatory solution of (1.1) and without loss of generality, assume $x(t) > 0$ for $t \geq T_0$. Because of (2.5), we may assume that $T_1 \geq T_0$ is sufficiently large such that

$$\int_{T_1}^t q(s)\Delta s \geq 0 \quad \text{for all} \quad t \geq T_1. \tag{2.6}$$

Indeed, if no such $T_1 \geq T_0$ exists, then for any $T > T_0$ fixed but arbitrary, we define

$$T_1 = T_1(T) := \sup\{t > T : \int_T^t q(s)\Delta s < 0\}.$$

If $T_1 = \infty$, then choosing $t_n \to \infty$ such that $\int_T^{t_n} q(s)\Delta s < 0$ for all n, we obtain a contradiction to (2.5). Hence, we must have $T_1 < \infty$ which implies $\int_{T_1}^t q(s)\Delta s \geq 0$ for all $t \geq T_1$. Now assume that $x^\Delta(t)$ is not strictly positive for all large t. First consider the case when $x^\Delta(t) < 0$ for all large t. Then, without loss of generality, $x^\Delta(t) < 0$ for all $t \geq T_1 \geq T_0$. An integration of (1.1) for $t > T_1$ gives

$$p(t)x^\Delta(t) + \int_{T_1}^t q(s)f(x^\sigma(s))\Delta s = p(T_1)x^\Delta(T_1) < 0. \tag{2.7}$$

Now by the integration by parts formula (2.2) we have

$$\begin{aligned} \int_{T_1}^t q(s)f(x^\sigma(s))\Delta s &= f(x(t))\int_{T_1}^t q(s)\Delta s \\ &\quad - \int_{T_1}^t (f(x(s)))^\Delta \int_{T_1}^s q(r)\Delta r \Delta s. \end{aligned} \tag{2.8}$$

By (2.1) we have (with $g(t) = x(t)$)

$$(f(x(t)))^\Delta = \left\{\int_0^1 f'(x(t) + h\mu(t)x^\Delta(t))dh\right\} x^\Delta(t) \leq 0,$$

since $f'(u) > 0$ for all $u \neq 0$ and $x^\Delta(t) < 0$. Hence, it follows that

$$\int_{T_1}^{t} (f(x(s)))^\Delta \int_{T_1}^{s} q(r)\Delta r \Delta s \leq 0 \tag{2.9}$$

and so from (2.8) we have

$$\int_{T_1}^{t} q(s) f(x^\sigma(s))\Delta s \geq f(x(t)) \int_{T_1}^{t} q(s)\Delta s \geq 0. \tag{2.10}$$

Consequently, from (2.7) we have

$$p(t)x^\Delta(t) \leq p(T_1)x^\Delta(T_1) < 0, \quad t \geq T_1. \tag{2.11}$$

Now dividing by $p(t)$ and integrating (2.11) yields

$$x(t) \leq x(T_1) + p(T_1)x^\Delta(T_1) \int_{T_1}^{t} \frac{1}{p(s)}\Delta s \to -\infty, \tag{2.12}$$

which is a contradiction. Hence $x^\Delta(t)$ is not negative for all large t and since we are assuming $x^\Delta(t)$ is not positive for all large t, it follows that $x^\Delta(t)$ must change sign infinitely often.

Make the "Riccati–like" substitution

$$w(t) := -\frac{p(t)x^\Delta(t)}{f(x(t))}, \quad t \geq T_0. \tag{2.13}$$

We may suppose that $T_1 > T_0$ is sufficiently large so that (2.5) holds with $T = T_1$ and is such that $w(T_1) > 0$ (i.e., $x^\Delta(T_1) < 0$).

Differentiating w gives

$$\begin{aligned} w^\Delta(t) &= q(t) + w^2(t)\frac{f(x(t))}{p(t)f(x^\sigma(t))}\left\{\int_0^1 f'(x(t) + h\mu(t)x^\Delta(t))dh\right\} \\ &\geq q(t), \quad t \geq T_1, \end{aligned}$$

and this yields

$$w(t) \geq w(T_1) + \int_{T_1}^{t} q(s)\Delta s, \quad t \geq T_1. \tag{2.14}$$

Now taking the lim inf of both sides of (2.14) we have by (2.5) that

$$\liminf_{t\to\infty} w(t) \geq w(T_1) > 0,$$

which implies that $x^\Delta(t) < 0$ for all large t, which is a contradiction to the assumption that $x^\Delta(t)$ changes sign infinitely often.

□

3. Main results

The first result is a boundedness result for (1.1).

Theorem 3.1. *Let $\lambda > 0$ and assume that equation* (1.5) *is oscillatory. Assume that* (1.2) *holds and let x be a nonoscillatory solution of* (1.1) *with $x(t)x^\Delta(t) > 0$ for all $t \geq T_0$. Then*

$$\lim_{t\to\infty} \frac{f(x(t))}{x(t)} := \gamma \leq \lambda. \tag{3.1}$$

Proof. We consider only the case $x(t) > 0$, $x^\Delta(t) > 0$, for $t \geq T_0$, since the other case is similar. We define

$$g(x) = \frac{f(x)}{x}, \quad x \neq 0.$$

Since $g'(x) \geq 0$ by (1.2), it follows that $\lim_{t\to\infty} g(x(t))$ exists in the extended reals. If (3.1) does not hold, then we can assume there exists $T_1 \geq T_0$ such that

$$g(x(t)) \geq \lambda, \quad t \geq T_1. \tag{3.2}$$

Let z be the solution of (1.5) with $z(T_1) = 0$ and $p(T_1)z^\Delta(T_1) = 1$. Since (1.5) is assumed to be oscillatory there exists a $T_2 > T_1$ such that

$$p(t)z^\Delta(t) > 0, \quad \text{on} \quad [T_1, T_2), \tag{3.3}$$

so that $z(t) > 0$ on $(T_1, T_2]$ and

$$p(T_2)z^\Delta(T_2) \leq 0. \tag{3.4}$$

We have from (3.2) that for $t \geq T_1$,

$$p(t)(z^\Delta(t))^2(g(x(t)) - \lambda) \geq 0$$

so that an integration by parts gives

$$\begin{aligned}
0 &\leq \int_{T_1}^{T_2} (g(x(t)) - \lambda)\, p(t)(z^\Delta(t))^2 \Delta t = \left[z(t)p(t)z^\Delta(t)\,(g(x(t)) - \lambda)\right]_{T_1}^{T_2} \\
&- \int_{T_1}^{T_2} z^\sigma(t)\left[p(t)z^\Delta(t)(g(x(t)) - \lambda)\right]^\Delta \Delta t \\
&= z(T_2)p(T_2)z^\Delta(T_2)(g(x(T_2)) - \lambda) - \int_{T_1}^{T_2} z^\sigma(t)(p(t)z^\Delta(t))^\Delta(g(x^\sigma(t)) - \lambda)\Delta t \\
&- \int_{T_1}^{T_2} z^\sigma(t)p(t)z^\Delta(t)(g(x(t)))^\Delta \Delta t
\end{aligned}$$

$$\leq \lambda \int_{T_1}^{T_2} q(t)(z^\sigma(t))^2(g(x^\sigma(t)) - \lambda)\Delta t - \int_{T_1}^{T_2} p(t)z^\Delta(t)z^\sigma(t)\left(\frac{f(x(t))}{x(t)}\right)^\Delta \Delta t. \tag{3.5}$$

We note that

$$\begin{aligned}
\left(\frac{f(x(t))}{x(t)}\right)^\Delta &= \frac{x(t)(f(x(t)))^\Delta - f(x(t))x^\Delta(t)}{x(t)x^\sigma(t)} \\
&= \frac{x(t)\left\{\int_0^1 f'(x(t) + h\mu(t)x^\Delta(t))dh\right\}x^\Delta(t) - f(x(t))x^\Delta(t)}{x(t)x^\sigma(t)}.
\end{aligned} \tag{3.6}$$

If we denote $y_h(t) = x(t) + h\mu(t)x^\Delta(t)$, then since $x^\Delta(t) > 0$, it follows that $y_h(t) \geq x(t)$ for $0 \leq h \leq 1$ and so by (1.2) we have

$$f'(y_h(t)) \geq \frac{f(y_h(t))}{y_h(t)} \geq \frac{f(x(t))}{x(t)},$$

since $x(t) > 0$. Consequently, we have from (3.6)

$$\left(\frac{f(x(t))}{x(t)}\right)^\Delta \geq \frac{x(t)\left\{\int_0^1 \frac{f(x(t))}{x(t)}dh\right\}x^\Delta(t) - f(x(t))x^\Delta(t)}{x(t)x^\sigma(t)} = 0.$$

Using this in (3.5), since $\lambda > 0$ now yields

$$0 \le \int_{T_1}^{T_2} q(t)(z^\sigma(t))^2(g(x^\sigma(t)) - \lambda)\Delta t \tag{3.7}$$

(since $z^\Delta(t) > 0$, $z^\sigma(t) = z(t) + \mu(t)z^\Delta(t) > 0$ on $(T_1, T_2]$). From (3.7) we now have

$$\begin{aligned} 0 &\le \int_{T_1}^{T_2} q(t)(z^\sigma(t))^2(g(x^\sigma(t)) - \lambda)\Delta t \\ &= \int_{T_1}^{T_2} \frac{z^\sigma(t)}{x^\sigma(t)} \left(q(t)z^\sigma(t)f(x^\sigma(t)) - \lambda q(t)z^\sigma(t)x^\sigma(t)\right)\Delta t. \end{aligned} \tag{3.8}$$

Notice that

$$\begin{aligned} &\left(p(t)z^\Delta(t)x(t) - p(t)x^\Delta(t)z(t)\right)^\Delta \\ &= \left(p(t)z^\Delta(t)\right)^\Delta x^\sigma(t) + p(t)z^\Delta(t)x^\Delta(t) - (p(t)x^\Delta(t))^\Delta z^\sigma(t) - p(t)x^\Delta(t)z^\Delta(t) \\ &= (p(t)z^\Delta(t))^\Delta x^\sigma(t) - (p(t)x^\Delta(t))^\Delta z^\sigma(t) \\ &= -\lambda q(t)z^\sigma(t)x^\sigma(t) + q(t)f(x^\sigma(t))z^\sigma(t). \end{aligned}$$

Hence, (3.8) now gives

$$\begin{aligned} 0 &\le \int_{T_1}^{T_2} \frac{z^\sigma(t)}{x^\sigma(t)} \left(p(t)z^\Delta(t)x(t) - p(t)x^\Delta(t)z(t)\right)^\Delta \Delta t \\ &= \left[\frac{z(t)}{x(t)}(p(t)z^\Delta(t)x(t) - p(t)x^\Delta(t)z(t))\right]_{T_1}^{T_2} \\ &\quad - \int_{T_1}^{T_2} \left(\frac{z(t)}{x(t)}\right)^\Delta \left(p(t)z^\Delta(t)x(t) - p(t)x^\Delta(t)z(t)\right)\Delta t \\ &= \frac{z(T_2)}{x(T_2)} \left(p(T_2)z^\Delta(T_2)x(T_2) - p(T_2)x^\Delta(T_2)z(T_2)\right) \\ &\quad - \int_{T_1}^{T_2} \frac{p(t)(z^\Delta(t)x(t) - x^\Delta(t)z(t))^2}{x(t)x^\sigma(t)}\Delta t < 0, \end{aligned}$$

which is a contradiction. Hence it follows that (3.1) holds and this completes the proof. □

Corollary 3.2. *Let $\lambda > 0$ and assume that equation (1.5) is oscillatory. Suppose that x is a nonoscillatory solution of the generalized Emden–Fowler equation*

$$\left(p(t)x^\Delta\right)^\Delta + q(t)(x^\sigma)^{2n+1} = 0. \tag{3.9}$$

Then

$$\lim_{t\to\infty} |x(t)| = \gamma \le (\lambda)^{\frac{1}{2n}}.$$

Theorem 3.3. *Assume that equation (1.5) is oscillatory for all $\lambda > 0$ and suppose that (2.3), (2.5), and (1.2) hold. Then all solutions of (1.1) oscillate.*

Proof. If not, let x be a nonoscillatory solution. Then Lemma 2.3 and Theorem 3.1 imply that $x(t)x^\Delta(t) > 0$ for all large t and $\lim_{t\to\infty} g(x(t)) = 0$. But, since $(g(x(t)))^\Delta \ge 0$, this is a contradiction. □

The next theorem deals with the case when (2.4) holds.

Theorem 3.4. *Assume that equation* (1.5) *is oscillatory for all* $\lambda > 0$ *and suppose that* (1.2), (2.4), *and* (2.5) *hold. In addition, assume that*

$$\int_T^\infty \frac{1}{p(s)} \int_T^s q(\eta)\Delta\eta\Delta s = \infty. \tag{3.10}$$

Then every solution of (1.1) *is either oscillatory or converges to zero on* $[a, \infty)$.

Proof. Let x be a nonoscillatory solution of (1.1) and suppose that $x(t) > 0$ for $t \geq T$. We claim that $x^\Delta(t) < 0$ for all large t, say for $t \geq T_1$. For if $x^\Delta(t) > 0$ for all large t, then as in the proof of Theorem 3.1, we conclude that (3.1) holds for all $\lambda > 0$, which is a contradiction. Also, if $x^\Delta(t)$ changes sign infinitely often, then as in the proof of Lemma 2.3, we again obtain a contradiction. Therefore, it follows that $x^\Delta(t) < 0$, for $t \geq T_1$, and so

$$\lim_{t\to\infty} x(t) = b \geq 0.$$

We claim that $b = 0$. If not, then we have that

$$x(t) \geq x^\sigma(t) \geq b > 0$$

for $t \geq T_1$. Integrating (1.1) gives

$$p(t)x^\Delta(t) + \int_{T_1}^t q(s)f(x^\sigma(s))\Delta s = p(T_1)x^\Delta(T_1) < 0. \tag{3.11}$$

We may suppose, without loss of generality, that $\int_{T_1}^t q(s)\Delta s \geq 0$ for all $t \geq T_1$. Therefore, as in the proof of Lemma 2.3, we obtain (2.10) and so we have

$$\int_{T_1}^t q(s)f(x^\sigma(s))\Delta s \geq f(b)\int_{T_1}^t q(s)\Delta s. \tag{3.12}$$

From (3.11) we obtain

$$p(t)x^\Delta(t) < -f(b)\int_{T_1}^t q(s)\Delta s \tag{3.13}$$

and so dividing by $p(t)$ and integrating gives

$$x(t) < x(T_1) - f(b)\int_{T_1}^t \frac{1}{p(s)}\int_{T_1}^s q(\eta)\Delta\eta\Delta s \to -\infty,$$

as $t \to \infty$, a contradiction. This shows that $b = 0$ and completes the proof. $\square$

4. Examples

Clearly, equation (1.5) is oscillatory iff

$$\left(\frac{1}{\lambda}p(t)x^\Delta\right)^\Delta + q(t)x^\sigma = 0 \tag{4.1}$$

is oscillatory. It was shown in Erbe [6, Corollary 7] (see also Bohner and Peterson [3]) that

$$(p(t)x^\Delta)^\Delta + q(t)x^\sigma = 0 \tag{4.2}$$

is oscillatory if there exists a sequence $\{t_k\} \subset \mathbb{T}$ with $\lim_{k\to\infty} t_k = \infty$ and $\mu(t_k) > 0$ such that

$$\limsup_{k\to\infty}\left(Q(t_k) - \frac{p(t_k)}{\mu(t_k)}\right) = \infty, \tag{4.3}$$

where $Q(t) := \int_\tau^t q(s)\Delta s$. We can therefore conclude that all solutions of (1.1) oscillate in case (1.2), (2.3), and (2.5) hold along with

$$\limsup_{k\to\infty}\left(Q(t_k) - \frac{p(t_k)}{\lambda\mu(t_k)}\right) = \infty, \tag{4.4}$$

for all $\lambda > 0$. We note that there is no assumption on the boundedness of p and μ. If (1.2), (2.4), and (2.5) hold along with (4.4), then every solution oscillates or converges to zero. One may also apply averaging techniques or the telescoping principle to give some more sophisticated results (see Erbe, Kong, and Kong [8] and Erbe [7]). We leave this to the interested reader.

As a second example, suppose that $\mathbb{T}$ is such that there exists a sequence of points $t_k \in \mathbb{T}$ with $t_k \to \infty$ and positive numbers M, K such that $p(t_k) \le M$ and $\mu(t_k) \ge K$. Then if (1.2) and (2.5) hold and $\sum_1^\infty \mu(t_k)q(t_k) = \infty$, it follows from results of Erbe, Kong, and Kong [8, Corollary 4.1] that all solutions of (1.5) are oscillatory for all $\lambda > 0$. Consequently, all solutions of (1.1) are oscillatory.

As a third example, we would like to consider a particular example for the case when $\mathbb{T} = \mathbb{Z}$. If f has the form of (1.3) (i.e., $f(x) = x^{2n+1}$), $p(t) \equiv 1$, and $q(t) = \frac{\beta}{t\sigma(t)}$, then it is known that equation (4.2) is oscillatory if $\beta > \frac{1}{4}$, and is nonoscillatory if $\beta \le \frac{1}{4}$. Since in this case (2.5) holds trivially, it follows from Theorem 3.1 that all nonoscillatory solutions of (1.1) satisfy $\lim_{t:\to\infty} |x(t)| \le (\frac{1}{4})^{\frac{1}{2n}}$.

Remark 4.1. From Theorem 4.64 in [3] (Leighton–Wintner Theorem) it follows that equation (1.5) is oscillatory for all $\lambda > 0$ if

$$\int_a^\infty \frac{1}{p(t)}\Delta t = \int_a^\infty q(t)\Delta t = +\infty. \tag{4.5}$$

Since the second condition in (4.5) implies that (2.5) holds, Theorem 3.3 implies that all solutions of the Emden–Fowler equation (3.9) are oscillatory. That is, the Leighton–Wintner Theorem is valid for (3.9) and more generally for (1.1) if (1.2) holds. We note again that there are no explicit sign conditions on $q(t)$. For the special case when $\mathbb{T} = \mathbb{Z}$ and (1.1) is

$$\Delta^2 x_n + q_n (x_{n+1})^{2m+1} = 0, \tag{4.6}$$

where $m \in \mathbb{N}$, it follows that (4.6) is oscillatory if

$$\sum_{n=1}^\infty q_n = +\infty. \tag{4.7}$$

That is (4.7) implies that the linear equation

$$\Delta^2 x_n + \lambda q_n x_{n+1} = 0 \tag{4.8}$$

is oscillatory for all $\lambda > 0$ and so oscillation of (4.6) is a consequence of Theorem 3.3. If we consider equation (4.8) with $\lambda = 1$, then Theorem 4.51 of [3] (see also [10]) implies that (4.8) is oscillatory if for any $k \ge 1$ there exists $k_1 \ge k$ such that

$$\lim_{n\to\infty} \sum_{j=k_1}^n q_j \ge 1.$$

Consequently, by Corollary 3.2, all nonoscillatory solutions of (4.6) satisfy

$$\lim_{n\to\infty} |x_n| \le 1.$$

As a final example consider the Euler-Cauchy dynamic equation

$$x^{\Delta\Delta} + \frac{\lambda}{t\sigma(t)} x = 0 \tag{4.9}$$

on the time scale $\mathbb{T} = r^{\mathbb{N}_0} = \{1, r, r^2, \cdots\}$, where $r > 1$. Assume $\lambda > \frac{1}{4}$ is fixed and let

$$z_0 := \frac{1}{2} + i\frac{\sqrt{4\lambda - 1}}{2}.$$

Then (see [3, Section 3.7])

$$\begin{aligned} x(t) &= e_{\frac{z_0}{t}}(t,1) \\ &= \exp\left[\int_1^t \frac{\mathrm{Log}[1+\mu(s)\frac{z_0}{s}]}{\mu(s)}\Delta s\right] \\ &= \exp\left[\int_1^t \frac{\mathrm{Log}[1+(r-1)z_0]}{(r-1)s}\Delta s\right] \\ &= \exp\left[\frac{1}{(r-1)}\mathrm{Log}[1+(r-1)z_0]\int_1^t \frac{1}{s}\Delta s\right] \end{aligned}$$

is a complex valued solution, where Log denotes the principal logarithm and Arg denotes the principal argument. Using $\int_1^t \frac{1}{s}\Delta s = \frac{(r-1)\ln t}{\ln r}$ and simplifying we get

$$x(t) = (|1+(r-1)z_0|)^{\frac{\ln t}{\ln r}} e^{i\frac{\mathrm{Arg}(1+(r-1)z_0)\ln t}{\ln r}}.$$

This implies that

$$x_0(t) = (|1+(r-1)z_0|)^{\frac{\ln t}{\ln r}} \cos\left(\frac{\mathrm{Arg}(1+(r-1)z_0)\ln t}{\ln r}\right)$$

is a real valued solution. It follows from this that (4.9) is oscillatory for $\lambda > \frac{1}{4}$. It is much easier to show that (4.9) is nonoscillatory for $0 \le \lambda \le \frac{1}{4}$.

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