PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 3, Pages 841–845
S 0002-9939(03)07092-8
Article electronically published on August 19, 2003

THE CAUCHY PROBLEM FOR A CLASS OF KOVALEVSKIAN PSEUDO-DIFFERENTIAL OPERATORS

ROSSELLA AGLIARDI AND MASSIMO CICOGNANI

(Communicated by David S. Tartakoff)

ABSTRACT. We prove the H^∞ well-posedness of the forward Cauchy problem for a pseudo-differential operator P of order $m \geq 2$ with the Log-Lipschitz continuous symbol in the time variable. The characteristic roots λ_k of P are distinct and satisfy the necessary Lax-Mizohata condition $\mathrm{Im}\lambda_k \geq 0$. The Log-Lipschitz regularity has been tested as the optimal one for H^∞ well-posedness in the case of second-order hyperbolic operators. Our main aim is to present a simple proof which needs only a little of the basic calculus of standard pseudo-differential operators.

INTRODUCTION

Let us consider the Cauchy problem

$$\left\{ \begin{array}{l} Pu(t,x) = f(t,x), \quad 0 \leq t \leq T, \quad x \in \mathbf{R}^n, \\ \partial_t^j u(0,x) = g_j(x), \quad 0 \leq j \leq m-1 \end{array} \right. \tag{1.1}$$

for a pseudo-differential operator of Kovalevskian type

$$P = D_t^m + \sum_{j=0}^{m-1} A_j(t,x,D_x) D_t^j, \quad A_j(t) \in OPS^{m-j} , \tag{1.2}$$

of order $m \geq 2$ in $[0,T] \times \mathbf{R}^n$.

One says that problem (1.1) is well posed in the Sobolev space $H^\infty = H^\infty(\mathbf{R}^n) = \bigcap_\mu H^\mu(\mathbf{R}^n)$ if for every $f \in \mathcal{C}([0,T]; H^\infty)$ and $g_j \in H^\infty$, $0 \leq j \leq m$, there is a unique solution $u \in \mathcal{C}^m([0,T]; H^\infty)$.

In this paper we are concerned with the question of what kind of regularity in the time variable t one has to assume for the A_j's in (1.2) in order to obtain such a well-posedness. From [1] and [2] we know that for second-order strictly hyperbolic differential operators the sharp regularity is the Log-Lipschitz continuity: a function $a(t)$ is said to be Log-Lipschitz continuous, in short $a \in LL([0,T])$, if it satisfies

$$|a(t) - a(s)| \leq C|t-s| \big| \log|t-s| \big|, \quad 0 < |t-s| < \tfrac{1}{2} .$$

Received by the editors September 30, 2002 and, in revised form, November 5, 2002.
2000 *Mathematics Subject Classification.* Primary 35G10, 35L30.
Key words and phrases. Strictly hyperbolic operators, energy estimates, Log-Lipschitz continuity.

Here we prove that H^∞ well-posedness holds true for any operator of the type (1.2) with $A_j(t) \in LL([0,T]; OPS^{m-j})$, $j = 0, ..., m-1$, provided that the characteristic roots $\lambda_k(t,x,\xi)$ of P are distinct and satisfy the necessary Lax-Mizohata condition

$$\mathcal{I}m\lambda_k(t,x,\xi) \geq 0 \ \text{ for large } \ |\xi| \ , \ k = 1, ..., m \ .$$

Our main aim is to give a simple proof which needs only basic calculus of classical pseudo-differential operators.

Notation. Throughout this paper, $x, \xi \in \mathbf{R}^n, \langle\xi\rangle$ denotes $\sqrt{1+|\xi|^2}$ and $D_x = -i\nabla_x$. Since it concerns the notation for pseudo-differential operators, we follow [3], which we refer to also for all the results we need.

Therefore, S^N will denote the class of the symbols $a(x,\xi)$ such that

$$\sup_{x,\xi} |\partial_\xi^\alpha D_x^\beta a(x,\xi)| \cdot \langle\xi\rangle^{|\alpha|-N} < \infty \tag{2.1}$$

for every $\alpha, \beta \in \mathbf{N}^n$. Moreover, $S^N_{\ell og}$ will denote the class of symbols with the following property:

$$\sup_{x,\xi} |\partial_\xi^\alpha D_x^\beta a(x,\xi)| \cdot \langle\xi\rangle^{|\alpha|-N} / \log(1+\langle\xi\rangle) < \infty \tag{2.2}$$

for every $\alpha, \beta \in \mathbf{N}^n$.

For symbols depending also on a time variable, we introduce the following notation: $a(t,x,\xi) \in LL([0,T]; S^N)$, that is, $a(t,x,\xi)$ is Log-Lipschitz with respect to t, whenever for every $\alpha, \beta \in \mathbf{N}^n$ there exists $C_{\alpha,\beta} > 0$ such that

(2.3)
$$\sup_{x\in\mathbf{R}^n} |\partial_\xi^\alpha D_x^\beta \big(a(t,x,\xi) - a(s,x,\xi)\big)| \leq C_{\alpha,\beta}|t-s|\Big|\log|t-s|\Big|\langle\xi\rangle^{N-|\alpha|},$$

$0 < |t-s| < 1/2$.

The Cauchy problem. Let us consider an operator $P = P(t,x,D_t,D_x)$ in $[0,T] \times \mathbf{R}^n$ given by

$$P = D_t^m + \sum_{j=0}^{m-1} A_j(t,x,D_x) D_t^j \tag{3.1}$$

where $A_j \in \mathcal{C}([0,T]; S^{m-j})$. Let $P^0 = D_t^m + \sum_{j=0}^{m-1} A_j^0(t,x,D_x) D_t^j$ be such that

$$A_j - A_j^0 \in \mathcal{C}([0,T]; S^{m-1-j}) \ \ , \ \ j = 0, ..., m-1, \tag{3.2}$$

and

$$P^0(t,x,\tau,\xi) = \prod_{k=1}^m (\tau - \lambda_k(t,x,\xi)) \tag{3.3}$$

with

$$\lambda_k \in LL([0,T]; S^1), \ \ k = 1, ..., m. \tag{3.4}$$

We assume that the roots are distinct:

$$|\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \geq c|\xi| \ \ \text{for large } \ |\xi|, \ j \neq k, c > 0, \tag{3.5}$$

and that they fulfill the Lax-Mizohata condition:

$$\mathcal{I}m\lambda_k(t,x,\xi) \geq 0 \quad \text{for large } \ |\xi| \ . \tag{3.6}$$

Remark 3.1. If (3.2) holds for A_j^0 such that each function $A_j^0(t,x,\xi)$ is homogeneous of degree $m-j$ in ξ for large $|\xi|$, i.e. $A_j^0(t,x,\theta\xi) = \theta^{m-j}A_j^0(t,x,\xi)$, $\theta \geq 1$, $|\xi| \geq M$, and we assume

$$A_j^0 \in LL([0,T]; S^{m-j}) \ \ , \ \ j = 0, ..., m-1 \ , \tag{3.4$'$}$$

then, after a modification in a neighborhood of $\xi = 0$, we have (3.4), in view of (3.5).

Our main result is that the Cauchy problem

$$\begin{cases} Pu(t,x) = f(t,x), & 0 \leq t \leq T, \\ D_t^j u(0,x) = g_j(x), & 0 \leq j \leq m-1, \end{cases} \tag{3.7}$$

is well posed in H^∞ (with a loss of derivatives). In fact, we have the following.

Theorem 3.1. *Let P satisfy* (3.2), (3.4), (3.5) *and* (3.6). *Then there is $\delta > 0$ such that for every $\mu \in \mathbf{R}$, every $f \in \mathcal{C}([0,T]; H^\mu)$ and $g_j \in H^{\mu+m-j-1}$, $0 \leq j \leq m-1$, the Cauchy problem* (3.7) *has a unique solution $u \in \bigcap_{j=0}^{m-1} \mathcal{C}^j([0,T]; H^{\mu-\delta T+m-j-1})$. The solution satisfies the inequality*

(3.8)
$$\sum_{j=0}^{m-1} \|\partial_t^j u(t)\|^2_{\mu-\delta t+m-j-1} \leq C\Big\{\sum_{j=0}^{m-1} \|g_j\|^2_{\mu+m-j-1} + \int_0^t \|f(s)\|^2_{\mu-\delta s} ds\Big\}, \ 0 \leq t \leq T \ ,$$

for some $C = C_\mu > 0$.

Proof. The first step is to carry the factorization (3.3) to the operator level. Let us introduce the following regularization of λ_k with respect to the variable t :

$$\tilde{\lambda}_k(t,x,\xi) = \int \lambda_k(s,x,\xi)\rho((t-s)\langle\xi\rangle)\langle\xi\rangle ds, \tag{3.9}$$

where $\rho \in \mathcal{C}_0^\infty(\mathbf{R})$, $0 \leq \rho \leq 1$, $\displaystyle\int \rho = 1$ and we have set $\lambda_k(s,x,\xi) = \lambda_k(T,x,\xi)$ for $s > T$ and $\lambda_k(s,x,\xi) = \lambda_k(0,x,\xi)$ for $s < 0$. It is easy to see that

$$\begin{aligned} &\tilde{\lambda}_k \in \mathcal{C}([0,T]; S^1), \\ &\tilde{\lambda}_k - \lambda_k \in \mathcal{C}([0,T]; S^0_{\ell og}), \\ &\partial_t^h \tilde{\lambda}_k \in \mathcal{C}([0,T]; S^h_{\ell og}) \ \text{ for any } \ h \geq 1. \end{aligned} \tag{3.10}$$

Thus the operator P can be written in the form

(3.11)
$$P = (D_t - \tilde{\lambda}_m(t,x,D_x))...(D_t - \tilde{\lambda}_1(t,x,D_x)) + \sum_{j=0}^{m-1} R_j(t,x,D_x)D_t^j$$

with $R_j \in \mathcal{C}([0,T]; S^{m-j-1)}_{\ell og})$. Next we want to reduce the scalar equation $Pu = f$ to an $m \times m$ system $\mathcal{LU} = \mathcal{F}$. Let us define $\mathcal{U} = {}^t(u_0, ..., u_{m-1})$ by

$$
\begin{aligned}
&u_0 = \langle D_x \rangle^{m-1} u, \\
&u_1 = \langle D_x \rangle^{m-2}(D_t - \tilde{\lambda}_1(t, x, D_x))u, \\
&...... \\
&u_{m-1} = (D_t - \tilde{\lambda}_{m-1}(t, x, D_x))...(D_t - \tilde{\lambda}_1(t, x, D_x))u.
\end{aligned} \tag{3.12}
$$

Denoting ${}^t(\langle D_x \rangle^{m-1} u, \langle D_x \rangle^{m-2} D_t u, ..., D_t^{m-1} u)$ by $\mathcal{V}$, we immediately have

$$
\mathcal{U} = Q(t, x, D_x)\mathcal{V}, \quad \mathcal{V} = Q_0(t, x, D_x)\mathcal{U}, \tag{3.13}
$$

where the symbols of the entries of the $m \times m$ matrices Q and Q_0 belong to $\mathcal{C}([0,T]; S^0)$. Thus the equation $Pu = f$ is equivalent to an $m \times m$ system

$$
\mathcal{LU} = \mathcal{F} \ , \tag{3.14}
$$

where $\mathcal{F} ={}^t (0, ..., 0, if)$ and

$$
\mathcal{L} = \partial_t - i\Lambda(t, x, D_x) + B(t, x, D_x) \ , \tag{3.15}
$$

with

$$
\Lambda = \begin{bmatrix} \tilde{\lambda}_1 \langle D_x \rangle & ... & ... & 0 \\ 0 & ... & ... & 0 \\ 0 & ... & \tilde{\lambda}_{m-1} \langle D_x \rangle & 0 \\ 0 & ... & ... & \tilde{\lambda}_m \end{bmatrix}
$$

and the entries of $B(t, x, \xi)$ in $\mathcal{C}([0,T]; S^0_{\ell og})$. The operator Λ can be diagonalized by means of

$$
M = \begin{bmatrix} 1 & & d_{ij} \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \quad d_{ij}(t, x, \xi) = \langle \xi \rangle^{j-1} / \prod_{k=i}^{j-1} (\tilde{\lambda}_j - \tilde{\lambda}_k), \ i < j, \ \text{for large } |\xi| \ . \tag{3.16}
$$

From (3.10) we have $M, M^{-1} \in \mathcal{C}([0,T]; S^0)$, $\partial_t M, \partial_t M^{-1} \in \mathcal{C}([0,T]; S^0_{\ell og})$. Thus, we have

$$
\mathcal{L}_1 := M^{-1} \mathcal{L} M = \partial_t - i\Delta(t, x, D_x) + B_1(t, x, D_x), \tag{3.17}
$$

where Δ is the diagonal matrix of the $\tilde{\lambda}_k$'s and $B_1 \in \mathcal{C}([0,T]; S^0_{\ell og})$. Now, for $\mu \in \mathbf{R}$ and $\delta > 0$ let us define the operator $\mathcal{L}_2 = \langle D_x \rangle^{\mu - \delta t} \mathcal{L}_1 \langle D_x \rangle^{-\mu + \delta t}$. We have

$$
\mathcal{L}_2 = \partial_t - i\Delta(t, x, D_x) + B_1(t, x, D_x) + \delta \log \langle D_x \rangle I + B_2(t, x, D_x), \tag{3.18}
$$

where $B_2 \in \mathcal{C}([0,T]; S^0)$.

It remains to prove the following.

Proposition 3.2. *It is possible to fix $\delta > 0$ such that the Cauchy problem*

$$
\begin{cases} \mathcal{L}_2 \mathcal{U}(t, x) = F(t, x), & 0 \le t \le T, \\ \mathcal{U}(0, x) = G(x) \end{cases} \tag{3.19}
$$

has a unique solution $\mathcal{U} \in \mathcal{C}([0,T]; H^1) \cap \mathcal{C}^1([0,T]; H^0)$ *for any given* $F \in \mathcal{C}([0,T]; H^1)$ *and* $G \in H^1$. *The solution satisfies the energy inequality*

$$\|\mathcal{U}(t)\|_0^2 \leq C(\|\mathcal{U}(0)\|_0^2 + \int_0^t \|\mathcal{F}(s)\|_0^2 ds), \quad 0 \leq t \leq T, \tag{3.20}$$

for some $C = C_\mu > 0$.

Proof. We have only to prove that it is possible to fix $\delta > 0$ such that

$$\|\mathcal{U}(t)\|_0^2 \leq C(\|\mathcal{U}(0)\|_0^2 + \int_0^t \|\mathcal{L}_2\mathcal{U}(s)\|_0^2 ds), \quad 0 \leq t \leq T, \tag{3.21}$$

for every $\mathcal{U} \in \mathcal{C}([0,T]; H^1) \cap \mathcal{C}^1([0,T]; H^0)$, with a constant $C > 0$ depending only on the seminorms of the symbols of Δ, B_1, B_2 (hence depending on μ because of B_2). Then one can apply the usual energy method to solve (3.19), e.g. [3], pp. 236-240. To prove (3.21), we begin by choosing δ large enough in order to have positive operators $B_1(t,x,D_x) + \delta \log\langle D_x\rangle I$ in (3.18) for $t \in [0,T]$. This is possible from $B_1 \in \mathcal{C}([0,T]; S^0_{\ell og})$. So, we obtain

$$\begin{aligned} \frac{d}{dt}\|\mathcal{U}(t)\|_0^2 &\leq 2\mathcal{R}e(i\Delta\mathcal{U}(t) - B_2\mathcal{U}(t) + \mathcal{L}_2\mathcal{U}, \mathcal{U}(t)) \\ &\leq 2\mathcal{R}e(i\Delta\mathcal{U}(t), \mathcal{U}(t)) + C_\mu(\|\mathcal{U}(t)\|_0^2 + \|\mathcal{L}_2\mathcal{U}(t)\|_0^2) \ . \end{aligned} \tag{3.22}$$

Then from (3.6) we can apply the sharp Gårding inequality to the first-order operator $-i\Delta$:

$$2\mathcal{R}e(-i\Delta\mathcal{U}(t), \mathcal{U}(t)) \geq -C\|\mathcal{U}(t)\|_0^2, \quad C > 0 \ ,$$

which gives

$$\frac{d}{dt}\|\mathcal{U}(t)\|_0^2 \leq C'_\mu(\|\mathcal{U}(t)\|_0^2 + \|\mathcal{L}_2\mathcal{U}(t)\|_0^2) \ . \tag{3.23}$$

Gronwall's inequality yields (3.21), completing the proof.

References

[1] F. Colombini, E. De Giorgi, and S. Spagnolo, Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps, *Ann. Scuola Norm. Sup. Pisa* 6 (1979), 511–559. MR **81c:**35077

[2] F. Colombini and N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, *Duke Math. J.* 77 (1995), no. 3, 657–698. MR **96d:**35075

[3] H. Kumano-go, *Pseudodifferential operators*, The MIT Press, Cambridge, Massachusetts, and London, England, 1981. MR **84c:**35113

University of Ferrara, via Machiavelli 35, 44100 Ferrara, Italy
E-mail address: `agl@dns.unife.it`

University of Bologna, via Genova 181, 47023 Cesena, Italy
E-mail address: `cicognan@dm.unibo.it`