ON THE INVARIANCE OF CLASSES $\Phi BV, ABV$ UNDER COMPOSITION

PAMELA B. PIERCE AND DANIEL WATERMAN

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Abstract. The necessary and sufficient condition for $g \circ f$ to be in the class $\Phi BV, ABV$ for every $f$ of that class whose range is in the domain of $g$ is that $g$ be in $\text{Lip 1}.$

Several different classes of functions arise naturally in the study of the convergence of Fourier series [W1]{[W4], [S], [Y]. We concern ourselves here with two such classes: $ABV$ and $\Phi BV.$ Waterman [W1] defined the class $ABV$ as follows: Suppose $\lambda = \{\lambda_n\}$ is an increasing sequence such that $\sum_{n=1}^{\infty} 1/\lambda_n = \infty.$ We say that $f \in ABV$ on $[a,b]$ if $\sum_{n=1}^{\infty} |f(I_n)|/\lambda_n < \infty$ for every set $\{I_n\}$ of nonoverlapping intervals in $[a,b].$ If $\{\lambda_n\} = \{1\},$ the resulting class is called $HBV,$ the functions of harmonic bounded variation.

To define the class $\Phi BV,$ we let $\varphi$ be a convex function with domain $[0, \infty)$ having the following three properties:

1. $\varphi(0) = 0,$ $\varphi(x) > 0$ for $x > 0;$
2. $\varphi(x) \to 0$ as $x \to 0;$
3. $\varphi(x) \to \infty$ as $x \to \infty.$

Another property which is sometimes assumed is:

4. there exists $a > 0$ and $\delta > 0$ such that
$$\frac{\varphi(2x)}{\varphi(x)} \leq \delta \quad \text{for} \quad x \in (0, a].$$

The last condition above is usually called the condition $\Delta_2$ (often called $\Delta_2$ for small values).

Let $P = \{a = x_0, x_1, \ldots, x_m = b\}$ be a partition of $[a, b].$ We use the notation $I_n = [x_{n-1}, x_n],$ and we write $f(I_n) = f(x_n) - f(x_{n-1}).$ Since $\varphi$ may be represented as an integral of a nondecreasing function, clearly $\varphi$ itself is a strictly increasing continuous function on $[0, \infty).$

We say that $f \in \Phi BV$ [MO] on an interval $[a, b]$ if there exists a constant $M$ such that whenever $P = \{I_n\}$ is an arbitrary partition of $[a, b],$ we have
$$\sum_{P} \phi(|f(I_n)|) < M.$$
Alternatively we may require the infinite sum \( \sum_{n=1}^{\infty} \phi(|f(I_n)|) \) to be finite whenever \( \{I_n\}_{n=1}^{\infty} \) is a collection of nonoverlapping intervals in \([a, b]\). These two definitions have been shown to be equivalent. We will henceforth omit all reference to the interval \([a, b]\) and simply write \( \Phi BV \) to denote this class. We see the importance of the condition \( \Delta_2 \) in the following theorem.

**Theorem** (Musielak and Orlicz). The class \( \Phi BV \) is linear if and only if \( \Delta_2 \) is satisfied.

In what follows we shall assume that \( \varphi \) satisfies \( \Delta_2 \).

\( GW \) represents the class of functions (necessarily having only simple discontinuities) that have a convergent Fourier series for every change of variable, and the class \( UGW \) is defined analogously with respect to uniform convergence. Chaika and Waterman \([CW]\) proved the following.

**Theorem** (Chaika and Waterman). \( g \circ f \) is in one of the classes \( GW, UGW \) or \( HBV \) for each \( f \) of that class whose range is in the domain of \( g \) if and only if \( g \in \text{Lip} \, 1 \).

Josephy \([J]\) had proved an analogous theorem for the class \( BV \).

We are interested now in necessary and sufficient conditions for a function \( f \) to be preserved as a member of the classes \( \Lambda BV, \Phi BV \) when it is composed with a function \( g \) on the left. We prove here the following extension of the theorem of Chaika and Waterman:

**Theorem.** \( g \circ f \) is in the class \( \Lambda BV \) or \( \Phi BV \) for each \( f \) of that class whose range is in the domain of \( g \) if and only if \( g \in \text{Lip} \, 1 \).

**Proof.** We observe that the continuity of \( g \) is necessary. Suppose \( g \) is discontinuous. We may suppose \( g(0) = 0, g(t_n) \geq 1 \), where \( t_n \searrow 0 \) and \( \sum_{n=1}^{\infty} t_n < 1 \). We will define a function \( f \in \text{BV} \) such that \( g \circ f \notin \Lambda BV \cup \Phi BV \). Let \( \{a_n\} \) be a sequence in \((0, 1)\) converging downward to 0. Let \( f \) be 0 at 0 and on \([a_1, 1]\). On each interval \([a_{n+1}, a_n]\) we define \( f \) to be a tent function such that \( f(a_{n+1}) = f(a_n) = 0 \) and \( f((a_{n+1} + a_n)/2) = t_n \). Note that \( f \in \text{BV} \) and therefore \( f \in \Lambda BV \) and \( f \in \Phi BV \). Let \( I_n = [(a_n + a_{n+1})/2, a_n] \). Then \( (g \circ f)(I_n) = g(t_n) \geq 1 \). Hence

\[
\sum_{n=1}^{\infty} \frac{|(g \circ f)(I_n)|}{\lambda_n} \geq \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty
\]

and

\[
\sum_{n=1}^{\infty} \varphi(|g \circ f(I_n)|) \geq \sum_{n=1}^{\infty} \varphi(1) = \infty.
\]

Let us now suppose \( g \in \text{Lip} \, 1 \) and \( I_n \) is a collection of nonoverlapping intervals in the domain of \( f \). Then there is a \( c > 0 \) such that for any interval \( I_n, \quad |g \circ f(I_n)| < c|f(I_n)| \). Thus if \( f \) is in \( \Lambda BV \) or \( \Phi BV \), then so is \( g \circ f \).

Now we assume that \( g \) is continuous but \( g \notin \text{Lip} \, 1 \), and the range of \( f \) is in the domain of \( g \). Without loss of generality, we may suppose that the domain of \( g \) contains \([0, a]\) for some positive \( a \) and that there is a sequence of disjoint intervals \( J_n = [p_n, q_n] \) in the domain of \( g \), with \( p_n, q_n \searrow 0 \), and a sequence of positive numbers \( c_n \to \infty, \quad c_1 > 1 \), such that

\[
\sum_{n=1}^{\infty} \frac{1}{c_n} < \infty, \quad |J_n| c_n \leq g(J_n),
\]
and
\[ \sum_{n=1}^{\infty} \varphi(q_n) < \infty. \]

Since \( g \) is continuous and \( |J_n| \to 0 \), we see that \( |g(J_n)| \to 0 \) implying \( c_n|J_n| \to 0 \). We may choose \( \{J_n\} \) so that \( \{c_n|J_n|\} \) is monotone.

We now construct a function \( f \in \Lambda BV \) such that \( g \circ f \notin \Lambda BV \). To do this, we define a function \( L : [0, \infty) \to [0, \infty) \) by setting
\[ L(q) = \sum_{k=1}^{q} \frac{1}{\lambda_k} \quad \text{for} \quad q \in Z^+, \quad L(0) = 0, \]
and extending \( L \) to be linear on each interval \([q - 1, q], q \in Z^+\).

We observe that \( L \) is strictly increasing on \([0, \infty)\), \( L(x) \to \infty \) as \( x \to \infty \), and hence \( L^{-1} \) exists. There is no loss of generality if we assume that \( \lambda_1 \geq 1 \). Let \( k_n = \lfloor L^{-1}(1/(c_n|J_n|) + 1) \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the greatest integer less than or equal to \( t \). Then
\[ k_n \to \infty, \quad L(k_n) > \frac{1}{c_n|J_n|} \quad \text{and} \quad g(J_n)L(k_n) > \frac{g(J_n)}{c_n|J_n|} \geq 1. \]

There is a \( C > 1 \) such that
\[ \frac{1}{c_n} \leq |J_n|L(k_n) < C \frac{1}{c_n}. \]

For this \( C \) we then have
\[ \sum_{n=1}^{\infty} |J_n|L(k_n) < C \sum_{n=1}^{\infty} \frac{1}{c_n} < \infty. \]

For each \( n = 1, 2, \ldots \) let \( I_{n,1}, I_{n,2}, \ldots, I_{n,k_n} \) be a collection of disjoint closed intervals in \((2^{-n}, 2^{-(n+1)})\) with \( I_{n,m} \) to the left of \( I_{n,m+1} \) for each \( m \).

Let \( f \) be defined on each \( I_{n,m} (m = 1, 2, \ldots, k_n; n = 1, 2, \ldots) \) to be the increasing linear map of \( I_{n,m} \) onto \( J_n \). Set \( f(0) = f(1) = 0 \). Define \( f \) to be linear on each of the component intervals of the remainder of \([0, 1]\) and continuous on \([0, 1] \). We now claim that \( f \in \Lambda BV \) while \( g \circ f \notin \Lambda BV \).

To see that \( g \circ f \notin \Lambda BV \) we observe that
\[ V_A \left( g \circ f, \left[ \frac{\pi}{2^n}, \frac{\pi}{2^{n-1}} \right] \right) \geq \sum_{m=1}^{k_n} \frac{(g \circ f)(I_{n,m})}{\lambda_m} = g(J_n)L(k_n) \geq 1. \]

If \( g \circ f \) were in \( \Lambda BV \), then the right continuity of \( g \circ f \) at 0 would imply \( V_A(g \circ f, [0, \epsilon]) \to 0 \) as \( \epsilon \to 0 \) (see [W2]). Hence \( g \circ f \notin \Lambda BV \).

To see that \( f \in \Lambda BV \) we first recall a definition of Banach:

**Definition.** If \( f \) is a continuous function, then \( \mathcal{N}(y) = \mathcal{N}(f; y) = \text{card}\{x | f(x) = y\} \). (This function \( \mathcal{N} \) has been called the Banach indicatrix of \( f \).)

This notion is easily extended to regulated functions, i.e., those functions with only simple discontinuities, by adjoining a vertical line connecting \( f(x+) \) and \( f(x-) \) at each point \( x \) of discontinuity (see [W1]). \( \mathcal{N}(y) \) is then defined to be the number of intersection points of this “extended” graph of \( f \) and the horizontal line of height
Theorem. If \( \inf f = A \) and \( \sup f = B \), \( f \) has only simple discontinuities, and \( L(x) \) is an increasing function such that
\[
L(n) \sim \sum_{k=1}^{n} \frac{1}{\lambda_k} \quad \text{as} \quad n \to \infty,
\]
then \( \int_{A}^{B} L(N(y)) \, dy < \infty \) implies that \( f \) is in the class \( \Lambda BV \).

The importance of the class of functions satisfying \( \int_{A}^{B} L(N(y)) \, dy < \infty \) was first observed by Garsia and Sawyer \( [GS] \) for continuous functions and \( L(x) = \log(x) \).

Now we need only observe
\[
\int L(N(f; y)) \, dy \leq q_1 L(2) + \sum_{n=1}^{\infty} |J_n| L(2k_n)
\]
\[
< q_1 L(2) + 2 \sum_{n=1}^{\infty} |J_n| L(k_n)
\]
\[
< \infty
\]
by (1). Hence \( f \in \Lambda BV \), and we have the portion of the desired result which is concerned with \( \Lambda BV \).

To complete the proof for \( \Phi BV \), we now define a function \( P \) by setting
\[
P(n) = \frac{1}{\varphi(c_n |J_n|)} \quad \text{for} \quad n \in \mathbb{Z}^+, \quad P(0) = 0,
\]
and extending \( P \) to be linear on each interval \([k-1, k], k \in \mathbb{Z}^+\). Then \( P \) is a one-to-one increasing mapping of \([0, \infty)\) onto \([0, \infty)\). Let \( k_n = [P(n) + 1] \). Thus \( k_n > P(n) = 1/(\varphi(c_n |J_n|)) \).

Let \( f \) be defined as in the construction for \( \Lambda BV \) but with this definition of \( k_n \). We now claim that \( f \in \Phi BV \) while \( g \circ f \notin \Phi BV \).

To see that \( g \circ f \notin \Phi BV \) we observe that
\[
\sum_{m=1}^{k_n} \varphi((g \circ f)(I_{n,m})) = \sum_{m=1}^{k_n} \varphi(g(J_n)) = k_n \varphi(g(J_n)) \geq k_n \varphi(c_n |J_n|) > \frac{\varphi(c_n |J_n|)}{\varphi(c_n |J_n|)} = 1.
\]
So for this collection \( \{I_{n,m}\} \) of intervals, we have
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{k_n} \varphi((g \circ f)(I_{n,m})) \geq \sum_{n=1}^{\infty} 1 = \infty,
\]
implying \( g \circ f \notin \Phi BV \).
We now show that \( f \in \Phi BV \). Let \( \{0 = x_0, x_1, \ldots, x_m = 1\} \) be an arbitrary partition of \([0,1]\). We will find an upper bound for the sum

\[
\sum_{i=1}^{m} \varphi(|f(x_i) - f(x_{i-1})|).
\]

Consider the contribution of an arbitrary term, \( \varphi(|f(x_i) - f(x_{i-1})|) \). For \( i = 1 \) we have

\[
\varphi(|f(x_1) - f(0)|) = \varphi(|f(x_1)|) \leq \varphi(q_1)
\]

and similarly for \( i = m \) we get \( \varphi(|0 - f(x_{m-1})|) \leq \varphi(q_1) \). For \( i \neq 1, m \) we have the following cases:

**Case 1.** There exists \( n \) such that \( f(x_i), f(x_{i-1}) \in J_n \). If \( I_{n,m} = [a_{n,m}, b_{n,m}] \), then \( x_i, x_{i-1} \in [a_{n,1}, b_{n,k_n}] \). Clearly,

\[
\varphi(|f(x_i) - f(x_{i-1})|) < \varphi(|J_n|).
\]

We observe that increasing the number of partition points along intervals of monotonicity does not increase the sum, and hence the contribution of all such terms is bounded by \( 2k_n \varphi(|J_n|) \).

**Case 2.** There does not exist an \( n \) such that \( f(x_i), f(x_{i-1}) \in J_n \). Then there is a smallest integer \( j \) satisfying \( f(x_i) \leq q_j \).

Suppose there are \( r \) such intervals having this same smallest integer \( j \) satisfying the above. If these intervals are

\[
[x_{i_0}, x_{i_0+1}], [x_{i_0+1}, x_{i_0+2}], \ldots, [x_{i_0+r-1}, x_{i_0+r}],
\]

we then have, since \( \varphi \) is convex,

\[
\sum_{l=1}^{r} \varphi(|f(x_{i_0+l}) - f(x_{i_0+l-1})|) \leq \varphi(|f(x_{i_0+r}) - f(x_{i_0})|) \leq \varphi(q_j).
\]

We observe that the above situation can arise at most once for each index \( j \), \( j = 1, \ldots, m \).

By combining all of the above information we show that an upper bound for \( \sum_{i=1}^{m} \varphi(|f(x_i) - f(x_{i-1})|) \) is

\[
M = 2\varphi(q_1) + \sum_{n=1}^{\infty} 2k_n \varphi(|J_n|) + \sum_{n=1}^{\infty} \varphi(q_n)
\]

\[
= I + II + III.
\]

Clearly I, II, and III are independent of the choice of the partition. Obviously I is finite and \( q_n \) was chosen so that III is finite. Since \( c_n \geq 1 \), it follows that
\[ \varphi(c_n|J_n|) > c_n \varphi(|J_n|) \]. Thus
\[
\sum_{n=1}^{\infty} k_n \varphi(|J_n|) = \sum_{n=1}^{\infty} [P(n) + 1] \varphi(|J_n|)
\]
\[
\leq \sum_{n=1}^{\infty} \left( \frac{1}{\varphi(c_n|J_n|)} + 1 \right) \varphi(|J_n|)
\]
\[
\leq \sum_{n=1}^{\infty} \left( \frac{\varphi(|J_n|)}{c_n \varphi(|J_n|)} + \varphi(|J_n|) \right)
\]
\[
= \sum_{n=1}^{\infty} \frac{1}{c_n} + \sum_{n=1}^{\infty} \varphi(|J_n|)
\]
\[
\leq \sum_{n=1}^{\infty} \frac{1}{c_n} + \sum_{n=1}^{\infty} \varphi(q_n) < \infty.
\]

Thus \( f \in \Phi BV \), and the proof is complete.

REFERENCES


DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, THE COLLEGE OF WOOSTER, WOOSTER, OHIO 44691
E-mail address: ppierce@acs.wooster.edu

DEPARTMENT OF MATHEMATICS, FLORIDA ATLANTIC UNIVERSITY, BOCA RATON, FLORIDA 33431-0991
E-mail address: fourier@earthlink.net