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REFINING THE CONSTANT IN A MAXIMUM PRINCIPLE FOR THE BERGMAN SPACE

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Abstract. Let $A^2(\mathbb{D})$ be the Bergman space over the open unit disk $\mathbb{D}$ in the complex plane. Korenblum conjectured that there is an absolute constant c, $0 < c < 1$, such that whenever $|f(z)| \leq |g(z)|$ $(f, g \in A^2(\mathbb{D}))$ in the annulus $c < |z| < 1$, then $\|f\| \leq \|g\|$. In this note we give an example to show that $c < 0.69472$.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Bergman space $A^2(\mathbb{D})$ consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\| = \left[\int_{\mathbb{D}} |f(z)|^2 dA(z)\right]^{\frac{1}{2}} < +\infty,$$

where

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi} r\, dr\, d\theta,\ z = x + iy = re^{i\theta}$$

is the normalized Lebesgue area measure on $\mathbb{D}$. Korenblum [1] conjectured that there is an absolute constant c, $0 < c < 1$, such that whenever $|f(z)| \leq |g(z)|$ in the annulus $c < |z| < 1$ $(f, g \in A^2(\mathbb{D}))$, then $\|f\| \leq \|g\|$.

W. K. Hayman [2] proved Korenblum's conjecture for $c = 0.04$. Hinkkanen [3] improved Hayman's result that $c = 0.157\cdots$.

On the other hand, the example of $f(z) = \frac{1}{\sqrt{2}}$, $g(z) = z$ shows that $c \leq \frac{1}{\sqrt{2}}$. However, Martin (see [1]) gave the following example to show that $c = \frac{1}{\sqrt{2}}$ is not sharp.

Example. Let

$$f(z) = \frac{1 + (\sqrt{2} - 1)z^{20}}{1 + (\sqrt{2} - 1)2^{-10}}, \qquad g(z) = \sqrt{2}z.$$

Then $|f(z)| \leq |g(z)|$ for $\frac{1}{\sqrt{2}} < |z| < 1$ but $\|f\| > \|g\| = 1$.

In fact, an upper bound on c can be found from Martin's example. Namely, if f and g are as in Martin's example, consider instead the pair h and g, where $h = \frac{1}{\|f\|} f$. Then $\|h\| = \|g\| = 1$ and $|h(z)| \leq |g(z)|$ in an annulus $c' < |z| < 1$. Using *Mathematica* and Lemma 1 below, we can easily obtain that $c' = 0.70450 \cdots < \frac{1}{\sqrt{2}}$.

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Lemma 1 (see [4]). *If* $f(z) = \sum_{k=0}^{+\infty} a_k z^k \in A^2(\mathbb{D})$, *then*

$$\|f\| = \left(\sum_{k=0}^{+\infty} \frac{|a_k|^2}{k+1} \right)^{\frac{1}{2}}.$$

Before stating our example, we recall that the singular inner functions are defined as

$$S_a(z) = \exp\left(-a \frac{1+z}{1-z} \right),$$

which play an important role in Bergman spaces [5], where a is any positive constant. Our main result is the following.

Theorem. *Let*

$$\begin{aligned} f(z) &= e^{-a} S_a(z^n) = e^{-\frac{2a}{1-z^n}}, \\ g(z) &= \frac{e^{-\frac{2a}{1+c^n}}}{c} z, \end{aligned}$$

where $0 < c < 1,\ a = -\frac{1+c^n}{1-c^n} \log c > 0,\ n \in \mathbb{N}$. *Then* $|f(z)| \leq |g(z)|$ *in* $c < |z| < 1$. *Moreover, when* $n = 14$ *and* $c = 0.69472$, *we have* $\|f\| > \|g\|$.

Proof. It is easy to see that

$$\varphi(r) = \max_{|z|=r} \left| \frac{f(z)}{g(z)} \right| = \frac{\max_{|z|=r} |f(z)|}{\frac{e^{-\frac{2a}{1+c^n}}}{c} r} = \frac{e^{-\frac{2a}{1+r^n}}}{e^{-a} r}.$$

Hence, we have

$$\varphi(c) = 1,\ \varphi(1) = \lim_{r \to 1^-} \varphi(r) = 1.$$

Since $\frac{f(z)}{g(z)}$ is analytic in $c \leq |z| < 1$, the maximum modulus theorem implies that $|f(z)| \leq |g(z)|$ in $c < |z| < 1$.

A direct calculation shows that the Taylor expansion of $f(z)$ at 0 is

$$f(z) = e^{-2a} \left[1 - 2az^n + 2(a^2 - a)z^{2n} - \frac{4a^3 - 12a^2 + 6a}{3} z^{3n} + \cdots \right].$$

It follows from Lemma 1 that

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)|^2 dA(z) - \int_{\mathbb{D}} |g(z)|^2 dA(z) \\ > \ & e^{-4a} \left[1 + \frac{4a^2}{n+1} + \frac{4(a^2-a)^2}{2n+1} + \frac{(4a^3 - 12a^2 + 6a)^2}{9(3n+1)} - \frac{e^{2a}}{2} \right] \\ \triangleq \ & I(a). \end{aligned}$$

Using *Mathematica*, we obtain that when $n = 14$ and $c = 0.69472$,

$$e^{4a} I(a) = 0.0000214904 > 0.$$

So we have $\|f\| > \|g\|$. $\square$

Remark. It is likely that for all functions $f(z)$ and $g(z)$ (which depend on n and $a > 0$) defined in the theorem, $c = 0.6947116 \cdots$ is the best one.

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