A NEGATIVE ANSWER TO NEVANLINNA’S TYPE QUESTION
AND A PARABOLIC SURFACE
WITH A LOT OF NEGATIVE CURVATURE

ITAI BENJAMINI, SERGEI MERENKOV, AND ODED SCHRAMM

(Communicated by Jozef Dodziuk)

In memory of Bob Brooks

Abstract. Consider a simply-connected Riemann surface represented by a
Speiser graph. Nevanlinna asked if the type of the surface is determined by the
mean excess of the graph: whether mean excess zero implies that the surface
is parabolic, and negative mean excess implies that the surface is hyperbolic.
Teichmüller gave an example of a hyperbolic simply-connected Riemann sur-
face whose mean excess is zero, disproving the first of these implications. We
give an example of a simply-connected parabolic Riemann surface with nega-
tive mean excess, thus disproving the other part. We also construct an example
of a complete, simply-connected, parabolic surface with nowhere positive cur-
vature such that the integral of curvature in any disk about a fixed basepoint
is less than $-\epsilon$ times the area of disk, where $\epsilon > 0$ is some constant.

1. Introduction

The uniformization theorem states that for every simply-connected Riemann
surface $X$ there exists a conformal map $\varphi : X_0 \to X$, where $X_0$ is either the complex
plane $\mathbb{C}$, the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, or the extended complex plane
(Riemann sphere) $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and these possibilities are mutually exclusive [1].
The map $\varphi$ is called the uniformizing map. A simply-connected Riemann surface
$X$ is said to have hyperbolic, parabolic, or elliptic type, according to whether it is
conformally equivalent to $\mathbb{D}$, $\mathbb{C}$, or $\overline{\mathbb{C}}$, respectively. In what follows, we assume that
$X$ is not compact, thus excluding the elliptic case from consideration.

We are interested in the application of the Uniformization Theorem to the fol-
lowing construction. A surface spread over the sphere is a pair $(X, \psi)$, where $X$ is
topological surface and $\psi : X \to \overline{\mathbb{C}}$ a continuous, open and discrete map. The
map $\psi$ is called a projection. Two such surfaces $(X_1, \psi_1)$, $(X_2, \psi_2)$ are equivalent,
if there exists a homeomorphism $\phi : X_1 \to X_2$, such that $\psi_1 = \psi_2 \circ \phi$. According
to a theorem of Stoïlow [10], a continuous open and discrete map $\psi$ near each point
$z_0$ is homeomorphically equivalent to a map $z \mapsto z^k$. The number $k = k(z_0)$ is
called the local degree of $\psi$ at $z_0$. If $k \neq 1$, $z_0$ is called a critical point and $\psi(z_0)$
a critical value. The set of critical points is a discrete subset of $X$. The theorem
of Stoilow implies that there exists a unique conformal structure on $X$ that makes $\psi$ into a meromorphic function. If $X$ is simply-connected, what is the type of the Riemann surface so obtained? This is one version of the type problem. Equivalent surfaces have the same type.

Rolf Nevanlinna’s problem concerns a particular class of surfaces spread over the sphere, denoted by $F_q$. Let $\{a_1, \ldots, a_q\}$ be distinct points in $\overline{\mathbb{C}}$.

**Definition.** A surface $(X, \psi)$ belongs to the class $F_q = F(a_1, \ldots, a_q)$ if $\psi$ restricted to the complement of $\psi^{-1}(\{a_1, \ldots, a_q\})$ is a covering map onto its image $\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_q\}$.

Assume that $(X, \psi) \in F_q$ and $X$ is noncompact. We fix a Jordan curve $L$, visiting the points $a_1, \ldots, a_q$ in cyclic order. The curve $L$ is usually called a base curve. It decomposes the sphere into two simply-connected regions $H_1$, the region to the left of $L$, and $H_2$, the region to the right of $L$. Let $L_i, i = 1, 2, \ldots, q$, be the arc of $L$ from $a_i$ to $a_{i+1}$ (with indices taken modulo $q$). Let us fix points $p_1$ in $H_1$ and $p_2$ in $H_2$, and choose $q$ Jordan arcs $\gamma_1, \ldots, \gamma_q$ in $\mathbb{C}$, such that each arc $\gamma_i$ has $p_1$ and $p_2$ as its endpoints, and has a unique point of intersection with $L$, which is in $L_i$. We take these arcs to be interiorwise disjoint, that is, $\gamma_i \cap \gamma_j = \{p_1, p_2\}$ when $i \neq j$. Let $\Gamma'$ denote the graph embedded in $\overline{\mathbb{C}}$, whose vertices are $p_1$, $p_2$, and whose edges are $\gamma_i$, $i = 1, \ldots, q$, and let $\Gamma = \psi^{-1}(\Gamma')$. We identify $\Gamma$ with its image in $\mathbb{R}^2$ under an orientation-preserving homeomorphism of $X$ onto $\mathbb{R}^2$. The graph $\Gamma$ has the following properties:

1. $\Gamma$ is infinite, connected;
2. $\Gamma$ is homogeneous of degree $q$;
3. $\Gamma$ is bipartite.

A graph, properly embedded in the plane and satisfying these properties is called a Speiser graph, also known as a line complex. The vertices of a Speiser graph $\Gamma$ are traditionally marked by $\times$ and $\circ$, such that each edge of $\Gamma$ connects a vertex marked $\times$ with a vertex marked $\circ$. Such a marking exists, since $\Gamma$ is bipartite. Each face of $\Gamma$, i.e., connected component of $\mathbb{R}^2 \setminus \Gamma$, has either a finite even number of edges along its boundary, in which case it is called an algebraic elementary region, or infinitely many edges, in which case it is called a logarithmic elementary region. Two Speiser graphs $\Gamma_1$, $\Gamma_2$ are said to be equivalent, if there is a sense-preserving homeomorphism of the plane that takes $\Gamma_1$ to $\Gamma_2$.

The above construction is reversible. Suppose that the faces of a Speiser graph $\Gamma$ are labelled by $a_1, \ldots, a_q$, so that when going counterclockwise around a vertex $\times$, the indices are encountered in their cyclic order, and around $\circ$ in the reversed cyclic order. We fix a simple closed curve $L \subset \overline{\mathbb{C}}$ passing through $a_1, \ldots, a_q$. Let $H_1, H_2, L_1, \ldots, L_q$ be as before. Let $\Gamma^*$ be the planar dual of $\Gamma$. If $e$ is an edge of $\Gamma^*$ from a face of $\Gamma$ marked $a_j$ to a face of $\Gamma$ marked $a_{j+1}$, let $\psi$ map $e$ homeomorphically onto the corresponding arc $L_j$ of $L$. This defines $\psi$ on the edges and vertices of $\Gamma^*$. We then extend $\psi$ to the faces of $\Gamma^*$ in the obvious way. This defines a surface spread over the sphere $(\mathbb{R}^2, \psi) \in F(a_1, \ldots, a_q)$. See [6] for further details.

For a Speiser graph $\Gamma$, Nevanlinna introduces the following characteristics. Let $V(\Gamma)$ denote the set of vertices of the graph $\Gamma$. To each vertex $v \in V(\Gamma)$ we assign the number

$$E(v) = 2 - \sum_{f \in F(v)} (1 - 1/k_f),$$
where $F(v)$ denotes the set of faces containing $v$ on their boundary and $2k_f$ is the number of edges on the boundary of $f$, $k_f \in \{1, 2, \ldots, \infty\}$. The function $E : V(\Gamma) \to \mathbb{R}$ is called the \textit{excess}. This definition is motivated by considering the curvature, as follows. The $\psi$-pullback of the spherical metric $2|dw|/(1 + |w|^2)$ is generally singular, i.e., it may degenerate on $\psi^{-1}(\{a_1, \ldots, a_q\})$. The surface $X$, endowed with the pullback metric, is a spherical polyhedral surface, which is a particular kind of orbifold. The \textit{integral curvature} $\omega$ on $X$ is a signed Borel measure, so that for each Borel subset $B \subset X$, $\omega(B)$ is the area of $B$ with respect to the pullback metric minus $2\pi \sum_z (k_z - 1)$, where the sum is over all critical points $z \in B$ and $k_z$ is the local degree of $\psi$ at $z$.

Each vertex of $\Gamma$ represents a hemisphere, and each face of $\Gamma$ with $2k$ edges represents a critical point, where $k$ is the local degree of $f$ at this point. Therefore, each vertex of $\Gamma$ has positive integral curvature $2\pi$, and each face with $2k$ edges has negative integral curvature $-2\pi(k-1)$. We assign the negative curvature evenly to all the vertices of the face. A face with infinitely many edges contributes $-2\pi$ to each vertex on its boundary. The curvature assigned to every $v \in V(\Gamma)$ is exactly $2\pi E(v)$.

Nevanlinna also defines the mean excess of a Speiser graph $\Gamma$. We fix a base vertex $v_0 \in V(\Gamma)$, and consider an exhaustion of $\Gamma$ by a sequence of finite graphs $\Gamma_i$, where $\Gamma_i$ is the ball of combinatorial radius $i$, centered at $v$. By averaging $E$ over all the vertices of $\Gamma_i$, and taking the limit, we obtain the mean excess, denoted $E_m = E_m(\Gamma)$, provided that the limit exists. If the limit does not exist, we consider the \textit{upper mean excess} $\overline{E}_m$ and \textit{lower mean excess} $\underline{E}_m$, which are the upper (lim sup) and lower (lim inf) limits, respectively.

\textbf{Nevanlinna’s Problem} [5]. Does $\underline{E}_m \geq 0$ imply that the surface $X$ with the pullback complex structure is parabolic? Conversely, does $\overline{E}_m < 0$ imply that it is hyperbolic?

Teichmüller [11] constructed an example of a surface with hyperbolic type, for which the mean excess is zero, thus giving a negative answer to the first question. We will shortly prove that the answer to the other question is negative as well, by constructing a parabolic surface $(\mathbb{R}^2, \psi) \in F_3$ with $E_m < 0$.

In Section 3 we shall construct an example of a nonpositively curved, simply-connected, complete, parabolic surface, whose curvature in any ball about a fixed basepoint is less than a negative constant times the area of the ball.

2. Counterexample

P. Doyle [3] proved that the surface $(X, \psi)$ is parabolic if and only if a certain modification of the Speiser graph is recurrent. (See [4] and [9] for background on recurrence and transience of infinite graphs.) In the particular case where $k_f$ is bounded, the recurrence of the Speiser graph itself is equivalent to $(X, \psi)$ being parabolic. Though we will not really need this fact, it is not too hard to see that in a Speiser graph satisfying $E_m < 0$ the number of vertices in a ball grows exponentially with the radius. Thus, we may begin searching for a counterexample by considering recurrent graphs with exponential growth. A very simple standard example of this sort is a tree constructed as follows. In an infinite 3-regular tree $T_3$, let $v_0, v_1, \ldots$ be an infinite simple path. Let $T$ be the set of vertices $u$ in $T_3$ such that $d(u, v_n) \leq n$ for all sufficiently large $n$. Note that there is a unique infinite
Figure 1. The surfaces $S(v)$. (a) shows $S(v)$ for a leaf $v$, where $s = 2$. (b) shows $S(v)$ for a degree 3 vertex.

Figure 2. The Speiser graph with $s = 2$. The disk $S(v_0)$ is the left “eye” in $S(v_1)$.

simple path in $T$ starting from any vertex $u$. This implies that $T$ is recurrent. It is straightforward to check that the number of vertices of $T$ in the ball $B(v_0, r)$ grows exponentially with $r$.

Our Speiser graph counterexample is a simple construction based on the tree $T$. Fix a parameter $s \in \{1, 2, \ldots \}$, whose choice will be discussed later. To every leaf (degree-one vertex) $v$ of $T$ associate a closed disk $S(v)$ and on it draw the graph indicated in Figure 1(a), where the number of concentric circles, excluding $\partial S(v)$, is $s$. If $v$ is not a leaf, then it has degree 3. We then associate to it the graph indicated in Figure 1(b), drawn on a triply connected domain $S(v)$. We combine these to form the Speiser graph $\Gamma$ as indicated Figure 2 by pasting the outer boundary of the surface corresponding to each vertex into the appropriate inner boundary component of its parent. Here, the parent of $v$ is the vertex $v'$ such that $d(v', v_n) = d(v, v_n) - 1$ for all sufficiently large $n$.

Every vertex of $\Gamma$ has degree 4 and every face has 2, 4 or 6 edges on its boundary. Therefore, $\Gamma$ is a Speiser graph. Consequently, as discussed above, there is a surface spread over the sphere $X = (\mathbb{R}^2, \psi)$ whose Speiser graph is $\Gamma$. It is immediate to verify that $\Gamma$ is recurrent, for example, by the Nash-Williams criterion. Doyle’s Theorem [3] then implies that $X$ is parabolic. Alternatively, one can arrive at the
same conclusion by noting that there is an infinite sequence of disjoint isomorphic
annuli on $(\mathbb{R}^2, \Gamma)$ separating any fixed point from $\infty$, and applying extremal length.
(See [11, 12] for the basic properties of extremal length.)

We now show that $\overline{E}_m < 0$ for $\Gamma$. Note that the excess is positive only on vertices
on the boundary of 2-gons, which arise from leaves in $T$. On the other hand, every vertex of degree 3 in $T$ gives rise to vertices in $\Gamma$ with negative excess. Take as a basepoint for $\Gamma$ a vertex $w_0 \in S(v_1)$ with negative excess, say. It is easy to see that there are constants, $a > 1, c > 0$ such that the number $n^-_r$ of negative excess vertices in the combinatorial ball $B(w_0, r)$ about $w_0$ satisfies $c a^r \leq n^-_r \leq a^r/c$.

If $w$ is a vertex with positive excess, then there is a unique vertex $\sigma(w)$ with negative excess closest to $v$: in fact, if $w \in S(v)$, then $\sigma(w)$ is the closest vertex to $w$ on $\partial S(v)$, and the (combinatorial) distance from $w$ to $\sigma(w)$ is our parameter $s$. The map $w \mapsto \sigma(w)$ is clearly injective. This implies that the number $n^+_r$ of positive excess vertices in $B(w_0, r)$ satisfies $n^+_r + s \leq n^-_r$, $r \in \{0, 1, 2, \ldots\}$. By choosing $s$ sufficiently large, we may therefore arrange to have the total excess in $B(w_0, r)$ to be less than $-c a^r$, for some $\epsilon > 0$ and every $r \in \{0, 1, 2, \ldots\}$. It is clear that the number of vertices with zero excess in $B(w_0, r)$ is bounded by a constant (which may depend on $s$) times $n^-_r$. Hence, $\overline{E}_m < 0$ for $\Gamma$.

By allowing $s$ to depend on the vertex in $T$, if necessary, we may arrange to have $E_m = \overline{E}_m$; that is, $E_m$ exists, while maintaining $E_m < 0$. We have thus demonstrated that the resulting surface is a counterexample in Nevanlinna’s problem.

3. A NONPOSITIVE CURVATURE EXAMPLE

We now construct an example of a simply-connected, complete, parabolic surface $Y$ of nowhere positive curvature, with the property

$$\int_{D(a, r)} \text{curvature} < -\epsilon \text{area}(D(a, r)),$$

for some fixed $a \in Y$ and every $r > 0$, where $D(a, r)$ denotes the open disc centered at $a$ of radius $r$, and $\epsilon > 0$ is some fixed constant.

Consider the surface $\mathbb{C} = \mathbb{R}^2$ with the metric $|dz|/y$ in $P = \{z = x + iy: y \geq 1\}$, and $\exp(1 - y)|dz|$ in $Q = \{y < 1\}$. We denote this surface by $Y$. Let $\beta$ denote the curve $\{y = 1\}$ in $Y$, i.e., the common boundary of $P$ and $Q$.

Let $Q'$ denote the universal cover of $\{z \in \mathbb{C}: |z| > 1\}$. Note that $Q$ is isometric to $Q'$ via the map $z \mapsto \exp(itz + 1)$. Hence, the curvature is zero on $Q$, and the geodesic curvature of $\partial Q$ is $-1$. The geodesic curvature of $\partial P$ is $1$. Consequently, $Y$ has no concentrated curvature on $\beta$. The surface $Y$ is thus a “surface of bounded curvature”, also known as an Aleksandrov surface (see [2, 11]). The curvature measure of $Y$ is absolutely continuous with respect to area; the curvature of $Y$ is $-1$ (times area measure) on $P$ and $0$ on $Q$.

The surface $Y$ is parabolic, and the uniformizing map is the identity map onto $\mathbb{R}^2$ with the standard metric.

We will now prove [1] with $a = i$. Set $\beta_r = D(a, r) \cap \beta$. Note that the shortest path in $Y$ between any two points on $\beta$ is contained in $P$, and is the arc of a circle orthogonal to $\{y = 0\}$. Using the Poincaré disc model, it is easy to see that there exists a constant $c > 0$, such that

$$ce^{r/2} \leq \text{length} \beta_r \leq e^{r/2}/c,$$
where the right inequality holds for all \( r \), and the left for all sufficiently large \( r \). By considering the intersection of \( D(a, r) \) with the strip \( 1 < y < 2 \) it is clear that
\[
(3) \quad O(1) \text{ area}(P \cap D(a, r)) \geq \text{length } \beta_r,
\]
for all sufficiently large \( r \).

Consider some point \( p \in Q \), and let \( p' \) be the point on \( \beta \) closest to \( p \). It follows easily (for example, by using the isometry of \( Q \) and \( Q' \)) that if \( q \) is any point in \( \beta \), then \( dq(p, q) = dq(p, p') + dq(p', q) + O(1) \). Consequently, if \( d(p, a) \leq r \), then there is an \( s \in [0, r] \) such that \( p' \in \beta_s \) and \( dq(p, p') \leq r - s + O(1) \). Furthermore, it is clear that the set of points \( p \) in \( Q \) such that \( p' \in \beta_s \) and \( dq(p, p') \leq t \) has area \( O(t^2 + t) \times \text{length } \beta_s \). Consequently,
\[
\text{area}(Q \cap D(a, r)) \leq O(1) \sum_{j=0}^{r} (j + 1)^2 \text{length } \beta_{r-j}.
\]
Using (2), we have
\[
(4) \quad \text{area}(Q \cap D(a, r)) \leq O(1) \text{ length } \beta_r,
\]
for all sufficiently large \( r \).

Now, combining (3) and (4), we obtain (1) for all sufficiently large \( r \). It therefore holds for all \( r \).

**Acknowledgements**

The authors are grateful to B. Davis, D. Drasin, and especially to A. Eremenko for helpful discussions and their interest in this work.

**References**


Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
*E-mail address*: itai@math.weizmann.ac.il

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
*E-mail address*: smerenko@math.purdue.edu

Microsoft Research, One Microsoft Way, Redmond, Washington 98052
*E-mail address*: schramm@microsoft.com