THE SHARP LOWER BOUND
FOR THE FIRST POSITIVE EIGENVALUE
OF A SUB-LAPLACIAN
ON A PSEUDO-HERMITIAN MANIFOLD

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Abstract. This paper studies, using the Bochner technique, a sharp lower bound of the first eigenvalue of a subelliptic Laplace operator on a strongly pseudoconvex CR manifold in terms of its pseudo-Hermitian geometry. For dimensions greater than or equal to 7, the lower bound under a condition on the Ricci curvature and the torsion was obtained by Greenleaf. We give a proof for all dimensions greater than or equal to 5. For dimension 3, the sharp lower bound is proved under a condition which also involves a distinguished covariant derivative of the torsion.

1. Introduction and main results

Let $M$ be a $(2n+1)$-dimensional strongly pseudoconvex CR manifold and $H(M)$ the structure bundle, where $H(M)$ is a subbundle of the complexified tangent bundle $T_{\mathbb{C}}(M)$ of which each fiber is an $n$-dimensional complex vector space. Let $\theta$ be a real nonvanishing one-form on $M$ that annihilates $H(M) \oplus \overline{H(M)}$. Then, $(M, \theta)$ is a strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [9]. Locally, one can choose $n$ complex one-forms $\theta^\alpha$, so that $(\theta, \theta^\alpha, \overline{\theta^\alpha})$ form a basis of complex covectors and

$$d\theta = i \theta^\alpha \wedge \overline{\theta^\beta}, \quad \theta^\alpha = \overline{\theta^\alpha}.$$  

The local coframe $(\theta, \theta^\alpha, \overline{\theta^\alpha})$ is uniquely determined up to

$$\theta = \theta', \quad \theta^\alpha = \delta^\beta_\alpha U^\alpha_\beta, \quad \overline{\theta^\alpha} = \delta^\beta_\alpha U^\alpha_\beta$$

where

$$U^\alpha_\beta U^\alpha_\gamma = \delta^\gamma_\beta, \quad U^\alpha_\beta = \overline{U^{\alpha}_\beta}.$$  

If we compare the dual frame

$$X_0 = \overline{X}_0, \quad X_\alpha = \overline{X}_\alpha$$

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to \((\theta, \theta^\alpha, \theta^\beta)\), then the transformation (1.2) gives

\[
X_0' = X_0, \quad X'_\alpha = U^\beta_\alpha X_\beta, \quad X'_\beta = U^\alpha_\beta X_\alpha
\]

which singles out a unique transversal \(X_0\) to \(H(M) \oplus \overline{H(M)}\). Furthermore,

\[
d\theta^\alpha = \theta^\beta \wedge w^\alpha_\beta + \theta \wedge \tau^\alpha
\]

where \(w^\alpha_\beta\) are connection 1-forms that are skew-Hermitian:

\[
w^\alpha_\beta = -w^\beta_\alpha
\]

and the \(\tau^\alpha\) are torsion 1-forms of type \((0,1)\):

\[
\tau^\alpha = A^{\alpha\beta} \theta^\beta, \quad A^{\alpha\beta} = A^{\beta\alpha}.
\]

Moreover, if we define curvature 2-forms \(\Omega^\alpha_\beta\) by

\[
\Omega^\alpha_\beta = dw^\alpha_\beta - w^\gamma_\beta \wedge w^\alpha_\gamma - i\theta^\beta \wedge \tau^\alpha + i\tau^\beta \wedge \theta^\alpha,
\]

then

\[
\Omega^\alpha_\beta = R_{\alpha\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta + \lambda_{\alpha\beta\gamma\delta} \wedge \theta,
\]

where \(\lambda_{\alpha\beta\gamma\delta}\) are 1-forms and the curvature tensor components \(R_{\alpha\beta\gamma\delta}\) satisfy

\[
R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}.
\]

Let

\[
\Gamma^\alpha_\beta = \omega^\alpha_\beta(X_j), \quad \Gamma^\alpha_\beta = \omega^{\alpha\beta}(X_j)
\]

where \(\alpha, \beta \in I\) with \(I = \{1, \ldots, n\}\) and \(j \in \{0\} \cup I \cup \overline{I}\). Then \(R_{\alpha\beta\gamma\delta}\) can also be written as

\[
R_{\alpha\beta\gamma\delta} = X_\rho(\Gamma^\beta_\delta) - X_\rho(\Gamma^\gamma_\delta) - \Gamma^\gamma_\delta - \Gamma^\beta_\delta + \Gamma^\delta_\rho \Gamma^\alpha_\beta - \Gamma^\gamma_\delta \Gamma^\rho_\beta + i\delta^{\alpha\beta} \Gamma^\gamma_\delta + i\delta^{\rho\beta} \Gamma^\alpha_\gamma.
\]

For \(X = \sum_{\alpha=1}^n x^\alpha X_\alpha\) where \(x^\alpha\) are local functions, we let

\[
\text{Ric}(X, X) = R_{\alpha\beta} x^\alpha x^\beta, \quad R_{\alpha\beta\gamma\delta} = g^{\alpha\delta} R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta},
\]

and

\[
\text{Tor}(X, X) = i(A_{\alpha\beta\gamma\delta} x^\alpha x^\beta - A_{\alpha\beta} x^\alpha x^\beta).
\]

Then the covariant derivatives and the sub-Laplacian of a function \(f\) on \(M\) are given by

\[
f_j = X_j f, \quad f_{\alpha\beta} = X_j f_{\alpha\gamma} - \Gamma^\gamma_\delta \gamma^\alpha_\beta f, \quad f_{\alpha\beta} = X_j f_{\alpha\gamma} - \Gamma^\gamma_\delta \gamma^\alpha_\beta f
\]

and

\[
\Delta f = 2\text{Re}(\text{tr}(\pi + D^2 f)) = \sum_{\alpha} \partial_{\alpha\gamma} f_{\alpha\gamma} + f_{\alpha\gamma}.
\]

Let \(\lambda_1\) be the first positive eigenvalue of \(\Delta\). We shall prove the following theorems.

**Theorem 1.1.** Let \(n \geq 2\) and let \(M\) be a \((2n + 1)\)-dimensional strongly pseudo-convex pseudo-Hermitian manifold in the sense of Webster. If

\[
\text{Ric}(X, X) + (n/2) \text{Tor}(X, X) \geq k_0 g_m(X, X)
\]

for all \(m \in M\) and \(X \in H_m(M)\), for some positive constant \(k_0\), then \(\lambda_1 \geq \frac{n k_0}{n+1}\).
Theorem 1.2. Let $M$ be a 3-dimensional strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster. Let

$$\text{Ric}_m(X, X) + \frac{1}{2} \text{Tor}_m(X, X) - \frac{3}{k_0} B_m^2(X, X) \geq k_0 g_m(X, X),$$

for all $m \in M$, $X \in H_m(M)$, and for some positive constant $k_0$, where

$$B_m^2(x_1 X_1, x_1 X_1) = 2 |A_{11}|^2 |x_1|^2 - \text{Re} X_0(A_m^1 x_1^2 - 2 \text{Re} A_m^1 \Gamma_{10} x_1^2).$$

Then $\lambda_1 \geq \frac{4m}{n}$.

The above two theorems are sharp when $M$ is the unit sphere in $\mathbb{C}^{n+1}$, in which case the torsion vanishes and $\lambda_1 = \frac{4m}{n+2} k_0 = n$.

Observing that $(A_m^1)^0 = X_0 A_m^1 - 2 A_m^1 \Gamma_0^T = X_0 A_m^1 + 2 A_m^1 \Gamma_1^0$, we remark that while the pseudo-Hermitian case differs from the Riemannian case in that torsion enters into the picture in addition to the Ricci curvature, the 3-dimensional pseudo-Hermitian case differs from the higher-dimensional cases in that the first covariant derivative of the torsion along the distinguished transversal $X_0$ also plays a role in Theorem 1.2.

2. Proof of Theorem 1.1

We shall start to prove Theorem 1.1.

Proof. Let $(X_0, X_\alpha, X_\bar{\alpha})$ be a local frame given by (1.4). Let $X_\alpha^*$ be the adjoint of $X_\alpha$ with respect to $dv$. Then

$$X_\alpha^* = -X_\alpha + (\sum_\beta \Gamma_{\beta\bar{\alpha}}) \text{ and } \tilde{\Delta} = -\sum_\alpha (X_\alpha^* X_\alpha + X_\bar{\alpha} X_\bar{\alpha}).$$

Let $\tilde{\nabla} f = \sum_\alpha f_\alpha X_\alpha \in \Gamma(H(M)) \text{ and } \tilde{d} f = f_\alpha \theta^\alpha + f_{\bar{\alpha}} \bar{\theta}^{\bar{\alpha}}$. We recall the following formulae proved in [3].

Bochner formula:

$$\frac{1}{2} \Delta |\tilde{\nabla} f|^2 = \| \pi_+ D^2 f \|^2 + \| \pi_- D^2 f \|^2 + \text{Re} (\tilde{\nabla} f, \tilde{\nabla} (\tilde{\Delta} f)) + (\text{Ric} + (n-2)/2 \text{Tor}) (\tilde{\nabla} f, \tilde{\nabla} f) + i(D^2 f)(X_0, (\tilde{d} f)^*)^*,$$

(2.3) \[ \int_M i(D^2 f)(X_0, (\tilde{d} f)^*)^* dv = \int_M \|\pi_+ D^2 f\|^2 - \|\pi_- D^2 f\|^2 - \text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f) dv \]

and

(2.4) \[ \int_M i(D^2 f)(X_0, (\tilde{d} f)^*)^* dv = \int_M -\frac{4}{n} |\text{tr}(\pi_+ D^2 f)|^2 + \frac{1}{n} (\tilde{\Delta} f)^2 + \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) dv, \]

where $\|\pi_+ D^2 f\|^2, \|\pi_- D^2 f\|^2, \text{tr}(\pi_+ D^2 f)$ and $D^2 f(X_0, (\tilde{d} f)^*)^*$ are locally given by $\sum f_{\beta\bar{\alpha}} f_{\bar{\beta}\alpha}, \sum f_{\beta\alpha} f_{\bar{\beta}\bar{\alpha}}, \sum f_{\beta\bar{\alpha}}$ and $\sum f_{\beta\alpha} f_{\alpha 0} - f_{\alpha 0}$, respectively.
If \( f \) is a non-constant real-valued function so that \( \Delta f = -\lambda_1 f \), then \( \int_M f dv = 0 \) and

\[
(2.5) \quad \lambda_1 \int_M |f|^2 dv = - \int_M (f, \Delta f) dv = 2 \int_M |\nabla f|^2 dv.
\]

Consequently, for any \( c \in [0, 1] \), \((1 - c) \times (2.3) + c \times (2.4) \) gives

\[
\int_M i(D^2f)(X_0, (\bar{d}f)^*) dv
\]

\[
(2.6) = \int_M \left[ \frac{2(1-c)}{n} \|\pi_+ D^2 f\|^2 - \frac{4c}{n} (\text{tr}(\pi_+ D^2 f))^2 - \frac{2(1-c)}{n} \|\pi_- D^2 f\|^2 + \frac{2\lambda_1 c}{n} |\nabla f|^2 - \frac{2(1-c)}{n} \text{Ric}(\nabla f, \nabla f) + c \text{Tor}(\nabla f, \nabla f) \right] dv.
\]

By the Cauchy-Schwarz inequality,

\[
(2.7) \quad \|\pi_+ D^2 f\|^2 \geq \frac{1}{n} |\text{tr}(\pi_+ D^2 f)|^2.
\]

Then

\[
0 = \int_M \frac{1}{2} \Delta |\nabla f|^2 dv
\]

\[
= \int_M \left[ \|\pi_+ D^2 f\|^2 + \|\pi_- D^2 f\|^2 - \lambda_1 \text{Re}(\nabla f, \nabla f) + (\text{Ric} + \frac{(n-2)}{2} \text{Tor})(\nabla f, \nabla f) + \frac{2(1-c)}{n} \|\pi_+ D^2 f\|^2 - \frac{4c}{n} (\text{tr}(\pi_+ D^2 f))^2 - \frac{2(1-c)}{n} \|\pi_- D^2 f\|^2 + \frac{2\lambda_1 c}{n} |\nabla f|^2 - \frac{2(1-c)}{n} \text{Ric}(\nabla f, \nabla f) + c \text{Tor}(\nabla f, \nabla f) \right] dv
\]

\[
\geq \int_M \left[ \left( \frac{1}{n} + \frac{2(1-c)}{n^2} - \frac{4c}{n} \right) |\text{tr}(\pi_+ D^2 f)|^2 + \left( 1 - \frac{2(1-c)}{n} \right) \|\pi_- D^2 f\|^2 - \lambda_1 (1 - \frac{2c}{n}) |\nabla f|^2 + ((1 - \frac{2(1-c)}{n}) \text{Ric} + \frac{(n-2+2c)}{2} \text{Tor})(\nabla f, \nabla f) \right] dv.
\]

To get rid of the \( |\text{tr}(\pi_+ D^2 f)|^2 \) term, we solve \( \frac{1}{n} + \frac{2(1-c)}{n^2} - \frac{4c}{n} = 0 \) for \( c \) and let \( c = \frac{n+2}{2(n+2n)} \). For this choice of \( c \), the above inequality becomes

\[
0 \geq \int_M \left[ \frac{2(n-1)}{1+2n} \|\pi_- D^2 f\|^2 - \frac{(n+1)\lambda_1}{n} |\nabla f|^2 + (\text{Ric} + \frac{n}{2} \text{Tor})(\nabla f, \nabla f) \right] dv.
\]

By the hypothesis, \((\text{Ric} + \frac{n}{2} \text{Tor})(\nabla f, \nabla f) \geq k_0 |\nabla f|^2 \). Hence

\[
0 \geq \int_M \left[ \frac{2(n-1)}{2n+1} \|\pi_- D^2 f\|^2 + \left( k_0 - \frac{n+1}{n} \lambda_1 \right) |\nabla f|^2 \right] dv,
\]

which implies that for \( n \geq 2 \), \( \frac{n+1}{n} \lambda_1 \geq k_0 \). Therefore, we have proved Theorem 1.1. \( \square \)
3. Proof of Theorem 1.2

For \( n = 1 \), we need more on the pseudo-Hermitian geometry of \( M \). First we compute the curvature 2-form \( \Omega_1^1 \) in (1.9) to get \( \lambda_1^1 \) in (1.10) explicitly. The result is

\[
(3.1) \quad \Omega_1^1 = d\omega_1^1 = R_{1\Gamma_1\Gamma} \theta_1^1 \wedge \theta^\Gamma + W_{1\Gamma_1} \theta_1^1 \wedge \theta - W_{1\Gamma_1} \theta^\Gamma \wedge \theta
\]

where

\[
R_{1\Gamma_1\Gamma} = X_1 \Gamma_{11}^1 - X_0 \Gamma_{11}^1 + \Gamma_{11}^1 \Gamma_{11}^1 - \Gamma_{11}^1 \Gamma_{11}^1 + i \Gamma_{11}^1,
\]

\[
W_{1\Gamma_1} = X_1 \Gamma_{10}^1 - X_0 \Gamma_{11}^1 + \Gamma_{11}^1 \Gamma_{10}^1 - A_{11} \Gamma_{11}^1
\]

and \( W_{1\Gamma_1} = W_{1\Gamma_1} \). There is another curvature 2-form \( \Omega^1 \) defined by

\[
(3.2) \quad \Omega^1 = d\tau^1 - \tau^1 \wedge \omega^1.
\]

Explicit computation gives

\[
(3.3) \quad \Omega^1 = (X_1 A_{1\Gamma} + 2 A_{1\Gamma} \Gamma_{11}^1) \theta_1^1 \wedge \theta^\Gamma - |A_{1\Gamma}|^2 \theta_1^1 \wedge \theta + (-X_0 A_{1\Gamma} - 2 A_{1\Gamma} \Gamma_{10}^1) \theta^\Gamma \wedge \theta.
\]

It was shown in [9] that the coefficient of \( \theta_1^1 \wedge \theta^\Gamma \) in \( \Omega^1 \) is equal to \( W_{1\Gamma_1} \) in \( \Omega^1 \). Equating the two expressions of \( W_{1\Gamma_1} \), we get

\[
(3.4) \quad X_1 A_{1\Gamma} = X_0 \Gamma_{11}^1 - X_0 \Gamma_{10}^1 + \Gamma_{11}^1 \Gamma_{10}^1 - A_{1\Gamma} \Gamma_{11}^1.
\]

With this extra information, we will be able to prove the following lemma. For convenience, we shall henceforth write \( A = A_{1\Gamma} \).

**Lemma 3.1.** Let \( \tilde{\Delta} f = -\lambda_1 f \) and \( f_0 = X_0 f \). Then

\[
(3.5) \quad \frac{1}{2} \int_M \tilde{\Delta} f_0^2 \, dv = -\lambda_1 \int_M f_0^2 \, dv + 2 \int_M |X_1 f_0|^2 \, dv - 4 \text{Re} \int_M A f_1 X_1 f_0 \, dv
\]

(both sides being zero).

**Proof.**

\[
\frac{1}{2} \tilde{\Delta} (f_0^2) = \frac{1}{2} \left( (f_0^2)_\Gamma + (f_0^2)_\gamma \right)
\]

\[
= \frac{1}{2} \left[ X_\Gamma (f_0^2)_1 + X_1 (f_0^2)_\Gamma + X_1 (f_0^2)_\Gamma - X_0 (f_0^2)_\gamma \right]
\]

\[
= X_\Gamma (f_0 X_1 f_0) - \Gamma_{11}^1 (f_0 X_1 f_0) - \Gamma_{11}^1 (f_0 X_1 f_0) - \Gamma_{11}^1 (f_0 X_1 f_0)
\]

\[
= 2 |X_1 f_0|^2 + 2 f_0 \text{Re} (X_\Gamma X_1 f_0) - 2 f_0 \text{Re} (\Gamma_{11}^1 X_1 f_0).
\]
Using the Lie bracket \([X_0, X_1] = \Gamma^1_{10} X_1 - \overline{\mathcal{A}} X_{T_f}\) (1.15) and (1.7), we get

\[
X_{T_f} X_1 f_0 = X_T X_0 f_0 + X_T [X_1, X_0] f \\
= X_0 X_T X_1 f + [X_T, X_0] X_1 f + X_T [X_1, X_0] f \\
= X_0 (f_1 + \Gamma^1_{1T} f_1) + (AX_1 - \Gamma^1_{10} X_T) X_1 f + X_T (\overline{\mathcal{A}} X_T - \Gamma^1_{10} X_1) f \\
= X_0 f_1 + X_0 (\Gamma^1_{1T} f_1 + \Gamma^1_{1T} X_0 f_1 + (AX_1 X_1 + \overline{\mathcal{A}} X_T X_T) f \\
\quad - \Gamma^1_{10} X_T X_1 f + X_T (\overline{\mathcal{A}}) X_T f - X_T (\Gamma^1_{10}) X_1 f - \Gamma^1_{10} X_T X_1 f \\
= X_0 f_1 + X_0 (\Gamma^1_{1T} f_1 + \Gamma^1_{1T} X_1 f_0 + \Gamma^1_{1T} (\overline{\mathcal{A}} X_T + \Gamma^1_{10} X_1) f \\
\quad + (AX_1 X_1 + \overline{\mathcal{A}} X_T X_T) f + X_T (\overline{\mathcal{A}}) X_T f - X_T (\Gamma^1_{10}) X_1 f \\
= X_0 f_1 + [X_0 (\Gamma^1_{1T}) - X_T (\Gamma^1_{10}) + \Gamma^1_{1T} \Gamma^1_{10}] f_1 + (X_T (\overline{\mathcal{A}}) - \Gamma^1_{1T} \overline{\mathcal{A}}) f \\
\quad + (AX_1 X_1 + \overline{\mathcal{A}} X_T X_T) f + \Gamma^1_{1T} X_1 f_0.
\]

Thus,

\[
2 \text{Re} X_T X_1 f_0 = X_0 (f_1 + f_{T_{1T}}) + 2 \text{Re} [X_0 (\Gamma^1_{1T}) - X_T (\Gamma^1_{10}) + \Gamma^1_{1T} \Gamma^1_{10} + X_1 A - A \Gamma^1_{1T}] f_1 \\
\quad + 2(AX_1 X_1 + \overline{\mathcal{A}} X_T X_T) f + 2 \text{Re} \Gamma^1_{1T} X_1 f_0.
\]

Therefore,

\[
\frac{1}{2} \Delta f_0^2 = 2 |X_1 f_0|^2 - 2 f_0 \text{Re} \left( \Gamma^1_{1T} X_1 f_0 \right) \\
\quad + f_0 \left[ X_0 (f_1 + f_{T_{1T}}) + 2 \text{Re} [X_0 (\Gamma^1_{1T}) - X_T (\Gamma^1_{10}) \\
\quad + \Gamma^1_{1T} \Gamma^1_{10} + X_1 A - A \Gamma^1_{1T}] f_1 \\
\quad + 2(AX_1 X_1 + \overline{\mathcal{A}} X_T X_T) f + 2 \text{Re} \Gamma^1_{1T} X_1 f_0 \right] \\
= 2 |X_1 f_0|^2 + 2 f_0 (AX_1 X_1 + \overline{\mathcal{A}} X_T X_T) f \\
\quad + f_0 X_0 \Delta f + 2 f_0 \text{Re} [X_0 (\Gamma^1_{1T}) - X_T (\Gamma^1_{10}) + \Gamma^1_{1T} \Gamma^1_{10} + X_1 A - A \Gamma^1_{1T}] f_1.
\]

Using \(X^* = -X + \Gamma^1_{1T}\), we get

\[
2 \int_M f_0 AX_1 X_1 f_1 dv = 2 \int_M X^* (f_0 A) f_1 dv \\
= -2 \int_M X_1 (f_0 A) f_1 dv + 2 \int_M \Gamma^1_{1T} f_0 A dv \\
= -2 \int_M X_1 (A) f_1 f_0 dv - 2 \int_M A f_1 X_1 f_0 dv + 2 \int_M \Gamma^1_{1T} A f_0 f_1 dv.
\]
Then, by (3.4) with $A = A_{\mathcal{T}}$,

\[
2 \int_M f_0 AX_1 X_1 f \, dv + \int_M \left( \Gamma_{\mathcal{T}}^1 \Gamma_{\mathcal{T}}^1 - X_{\mathcal{T}} \Gamma_{\mathcal{T}}^1 + X_0 \Gamma_{\mathcal{T}}^1 + X_1 A - A \Gamma_{\mathcal{T}}^1 \right) f_1 f_0 \, dv \\
= -2 \int_M A f_1 X_1 f_0 \, dv + \int_M \left( \Gamma_{\mathcal{T}}^1 \Gamma_{\mathcal{T}}^1 - X_{\mathcal{T}} \Gamma_{\mathcal{T}}^1 + X_0 \Gamma_{\mathcal{T}}^1 \right) \\
\quad + X_0 (\Gamma_{\mathcal{T}}^1) - X_1 A + A \Gamma_{\mathcal{T}}^1 \right) f_1 f_0 \, dv \\
= -2 \int_M A f_1 X_1 f_0 \, dv.
\]

Therefore,

\[
\int_M \frac{1}{2} \Delta f_0^2 \, dv = 2 \int_M |X_1 f_0|^2 \, dv - \lambda_1 \int_M (f_0)^2 \, dv - 4 \text{Re} \int_M A f_1 X_1 f_0 \, dv
\]

and the proof of the lemma is complete. \hfill \Box

Notice that

\[
i(f_{\mathcal{T}} f_{10} - f_{1} f_{70}) \\
= if_{\mathcal{T}}(X_0 f_{1} - \Gamma_{10}^1 f_{1}) - if_{1}(X_0 f_{\mathcal{T}} - \Gamma_{70}^1 f_{\mathcal{T}}) \\
= if_{\mathcal{T}}(X_1 f_0 + (\Gamma_{10}^1 f_{1} - \overline{A} f_{\mathcal{T}}) - \Gamma_{10}^1 f_{1}) \\
\quad - if_{1}(X_{\mathcal{T}} f_0 + (\Gamma_{70}^1 f_{\mathcal{T}} - A f_{1}) - \Gamma_{70}^1 f_{\mathcal{T}}) \\
= i(f_{\mathcal{T}} X_1 f_0 - f_1 X_{\mathcal{T}} f_0) + i(A f_1^2 - \overline{A} f_{\mathcal{T}}^2) \\
= i(f_{\mathcal{T}} X_1 f_0 - f_1 X_{\mathcal{T}} f_0) + \text{Tor}(\nabla f, \nabla f),
\]

and

\[
X_1^* = -X_{\mathcal{T}} + \Gamma_{1}^1_{\mathcal{T}} \quad \text{and} \quad [X_1, X_{\mathcal{T}}] = -i X_0 - \Gamma_{1}^1_{\mathcal{T}} X_1 + \Gamma_{1}^1_{\mathcal{T}} X_{\mathcal{T}}.
\]

By (3.7)

\[
\text{Im} \int_M f_{\mathcal{T}} X_1 f_0 \, dv = \text{Im} \int_M (-X_1 X_{\mathcal{T}} f + \Gamma_{1}^1_{\mathcal{T}} f_{\mathcal{T}}) f_0 \, dv \\
= \text{Im} \int_M i \frac{1}{2} X_0 f_0 \, dv \\
= \frac{1}{2} \int_M f_0^2 \, dv.
\]

Hence, for $n = 1$, by (2.2), (3.6), (3.8) and the equation

\[
\int_M |f_{\mathcal{T}}|^2 \, dv = \int_M (\text{Re} f_{\mathcal{T}})^2 + (\text{Im} f_{\mathcal{T}})^2 \, dv \\
= \int_M \frac{1}{4} |\Delta f|^2 + \frac{1}{4} |f_0|^2 \, dv \\
= \int_M \frac{\lambda_1}{2} |\nabla f|^2 + \frac{1}{4} |f_0|^2 \, dv,
\]
we have
\[ 0 = \frac{1}{2} \int_M \bar{\Delta} |\bar{\nabla} f|^2 dv \]
\[ = \int_M \left( \frac{\lambda_1}{2} |\bar{\nabla} f|^2 + \frac{1}{4}|f_0|^2 \right) + |f_{11}|^2 - \lambda_1 |\bar{\nabla} f|^2 \]
\[+ \text{Ric}(\bar{\nabla} f, \bar{\nabla} f) - \frac{1}{2} \text{Tor}(\bar{\nabla} f, \bar{\nabla} f) + i(f_{10}f_{10} - f_{1}f_{10}) dv \]
\[= \int_M \left( \frac{\lambda_1}{2} |\bar{\nabla} f|^2 + \frac{1}{4}|f_0|^2 \right) + |f_{11}|^2 - \lambda_1 |\bar{\nabla} f|^2 \]
\[+ \left( \text{Ric} + \frac{1}{2} \text{Tor} \right)(\bar{\nabla} f, \bar{\nabla} f) + i(f_{10}f_{10} - f_{1}f_{10}) dv \]
\[= \int_M \left( -\frac{1}{2}\lambda_1 |\bar{\nabla} f|^2 + \frac{1}{4}|f_0|^2 + |f_{11}|^2 \right) \]
\[+ \left( \text{Ric} + \frac{1}{2} \text{Tor} \right)(\bar{\nabla} f, \bar{\nabla} f) - 2\text{Im}(f_{10}f_{10}) dv \]
\[= \int_M \left( -\frac{1}{2}\lambda_1 |\bar{\nabla} f|^2 + |f_{11}|^2 + \left( \text{Ric} + \frac{1}{2} \text{Tor} \right)(\bar{\nabla} f, \bar{\nabla} f) - \frac{3}{2} \text{Im}(f_{10}f_{10}) \right) dv. \]

By Lemma 3.1, where \( \int_M \bar{\Delta} f_0^2 dv = 0 \), and (3.8), we have
\[ (3.9) \quad 2 \int_M |X_1 f_0|^2 - 4\text{Re} \int_M A f_1 X_1 f_0 dv = \lambda_1 \int_M f_0^2 dv = 2\lambda_1 \text{Im} \int_M f_{10} f_1 f_0 dv. \]
Thus,
\[ (3.10) \quad \int_M |X_1 f_0|^2 = 2\text{Re} \int_M A f_1 X_1 f_0 dv + \lambda_1 \text{Im} \int_M f_{10} f_1 f_0 dv \]
\[ \leq 2\text{Re} \int_M A f_1 f_0 f_0 dv + \lambda_1 \left( \int_M |f_{10}|^2 dv \right)^{1/2} \left( \int_M |X_1 f_0|^2 dv \right)^{1/2} \]
\[ \leq 2\text{Re} \int_M A f_1 X_1 f_0 dv + \frac{\lambda_1}{2} \int_M |f_{10}|^2 dv + \frac{1}{2} \int_M |X_1 f_0|^2 dv. \]
Since
\[ (3.11) \quad \int_M X_0(A f_1^2) dv = 0, \]
we have
\[ (3.12) \quad \int_M A f_1 X_0 f_1 dv = -\frac{1}{2} \int_M X_0(A f_1^2) dv. \]
Thus,
\[ \text{Re} \int_M A f_1 X_1 f_0 dv \]
\[ = \text{Re} \int_M A f_1 (X_0 f_1 + \bar{A} X_1 f - \Gamma^1_{10} f_1) dv \]
\[ = \text{Re} \int_M (|A|^2 |f_1|^2 - \frac{1}{2} X_0(A f_1^2) - 2\Gamma^1_{10} f_1^2) dv \]
\[ = \int_M \frac{1}{2} B^2 (\bar{\nabla} f, \bar{\nabla} f) dv. \]
If there is no confusion, we shall simply write

\[(3.14)\]

\[B^2(\tilde{\nabla} f, \tilde{\nabla} f) = B^2 |f_1|^2.\]

(3.10)–(3.14) imply that

\[(3.15)\]

\[
\int_M |X_1 f_0|^2 dv \leq 4 \text{Re} \int_M A f_1 X_1 f_0 dv + \lambda_1^2 \int_M |f_1|^2 dv
\]

\[
\leq \int_M (2B^2 + \lambda_1^2) |f_1|^2 dv.
\]

Therefore,

\[(3.16)\]

\[
- \frac{3}{2} \text{Im} \int_M f_1 X_1 f_0 dv \\
\geq \frac{3}{2} \left( \int_M |f_1|^2 dv \right)^{1/2} \left( \int_M |X_1 f_0|^2 dv \right)^{1/2}
\]

\[
\geq - \frac{3b}{4} \int_M |f_1|^2 dv - \frac{3}{4b} \int_M |X_1 f_0|^2 dv
\]

\[
\geq - \frac{3b}{4} \int_M |f_1|^2 dv - \frac{3}{4b} \int_M (2B^2 + \lambda_1^2) |f_1|^2 dv.
\]

For simplicity, we will use the notation

\[(3.17)\]

\[\text{Ric}(\tilde{\nabla} f, \tilde{\nabla} f) + \frac{1}{2} \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) = k |f_1|^2.\]

Therefore,

\[
0 \geq - \frac{1}{2} \lambda_1 \int_M |f_1|^2 dv + \int_M k |f_1|^2 dv
\]

\[
- \frac{3b}{4} \int_M |f_1|^2 dv - \frac{3}{4b} \int_M (2B^2 + \lambda_1^2) |f_1|^2 dv
\]

\[
= - \lambda_1 \int_M \left( \frac{1}{2} + \frac{3\lambda_1}{4b} \right) |f_1|^2 dv + \int_M \left( k - \frac{3}{4b} - \frac{3B^2}{2b} \right) |f_1|^2 dv.
\]

Let \(b = k_0/2\). Then by (1.17) and (1.18),

\[
k - \frac{3B^2}{2b} = k - \frac{3B^2}{k_0} \geq k_0.
\]

Thus,

\[
\lambda_1 \geq \frac{(k_0 - \frac{3b}{2})}{\frac{1}{2} + \frac{3\lambda_1}{4b}} = \frac{(4k_0 - 3b)b}{2b + 3\lambda_1} = \frac{5k_0^2}{4(k_0 + 3\lambda_1)}.
\]

This holds if and only if \(12\lambda_1^2 + 4k_0 \lambda_1 \geq 5k_0^2\), i.e., \((2\lambda_1 - k_0)(6\lambda_1 + 5k_0) \geq 0\). Since \(\lambda_1 > 0\), we have \(6\lambda_1 + 5k_0 > 0\). Hence \(\lambda_1 \geq \frac{k_0}{2}\). Therefore, the proof of Theorem 1.2 is complete. \(\square\)

Finally, we remark that for \(n = 1\), (1.2) and (1.5) reduce to \(\theta = \theta', \theta^1 = e^{\alpha_0} \theta_0^1\) and \(X_0' = X_0, X_1' = e^{\alpha_0} X_1\) where \(\alpha \in \mathbb{R}\). Under these transformations, it can be checked that the quantities considered in Theorem 1.2 also have intrinsic meaning even though they are expressed locally.
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REFERENCES


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