A SIMPLE PROOF THAT SUPER-REFLEXIVE SPACES ARE $K$-SPACES

FÉLIX CABELLO SÁNCHEZ

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Abstract. We demonstrate the title.

A quasi-Banach space $Z$ is called a $K$-space if every extension of $Z$ by the ground field splits; that is, whenever $X$ is a quasi-Banach space having a line $L$ such that $X/L$ is isomorphic to $Z$, $L$ is complemented in $X$ (and so, $X = L \oplus Z$). These spaces play an important rôle in the theory of extensions of (quasi) Banach spaces [1], [2].

The property of being a $K$-space is closely related to the behaviour of quasi-linear functionals. Recall that a homogeneous functional $f : Z \to \mathbb{K}$ is said to be quasi-linear if there is a constant $Q$ such that

$$|f(x + y) - f(x) - f(y)| \leq Q(\|x\| + \|y\|) \quad (x, y \in Z).$$

The least possible constant in the preceding inequality shall be denoted by $Q(f)$.

It is well known [1] that $Z$ is a $K$-space if and only if each quasi-linear functional on $Z$ can be approximated by a true linear (but not necessarily continuous!) functional $\ell : Z \to \mathbb{K}$ in the sense that the distance

$$\text{dist}(f, \ell) \overset{\text{def}}{=} \inf\{K \geq 0 : |f(x) - \ell(x)| \leq K\|x\| \text{ for all } x \in Z\}$$

is finite.

The main examples of $K$-spaces are supplied by Kalton and co-workers: for instance, $L_p$ spaces ($0 < p \leq \infty$) are $K$-spaces if and only if $p \neq 1$ ([1], [4], [5], [6]). Also, $B$-convex spaces (Banach spaces having nontrivial type $p > 1$) are $K$-spaces and so are quotients of Banach $K$-spaces.

In this short note, we present a very simple proof that super-reflexive Banach spaces are $K$-spaces. Of course this is contained in Kalton’s result for $B$-convexity. Nevertheless, I believe that a simpler proof for this particular case is interesting because, in the presence of some unconditional structure (e.g., for Banach lattices), $B$-convexity is equivalent to super-reflexivity.

Mini-Theorem. Every super-reflexive space is a $K$-space.

Proof. Suppose on the contrary that $Z$ is super-reflexive and there exists a quasi-linear function $f : Z \to \mathbb{K}$ such that $\text{dist}(f, \ell) = \infty$ for all linear maps $\ell : Z \to \mathbb{K}$.
Let $\mathcal{F}$ denote the family of all finite-dimensional subspaces of $Z$. For each $E \in \mathcal{F}$, let $f_E$ denote the restriction of $f$ to $E$. It is clear that $Q(f_E) \leq Q(f)$. Put
\[ \delta_E = \text{dist}(f_E, E^*) = \inf \{ \text{dist}(f_E, \ell) : \ell \in E^* \}. \]
Obviously, $\delta_E$ is finite for all $E \in \mathcal{F}$. The hypothesis means that $\delta_E \to \infty$ with respect to the natural (inclusion) order in $\mathcal{F}$. In particular, $\delta_E > 0$ for $E$ large enough. Now, for each $E \in \mathcal{F}$, take $\ell_E \in E^*$ such that $\text{dist}(f_E, \ell_E) = \delta_E$ and let
\[ g_E = \delta_E^{-1}(f_E - \ell_E) \] (if $\delta_E = 0$, take $g_E = 0$). Clearly, $|g_E(x)| \leq \|x\|$ provided $x \in E$. Also, it is clear that $Q(g_E) \to 0$ as $E$ increases in $\mathcal{F}$.

Let $\mathcal{U}$ be any ultrafilter refining the Fréchet filter on $\mathcal{F}$, and let $\mathcal{F}_\mathcal{U}$ denote the ultraproduct of $\mathcal{F}$ with respect to $\mathcal{U}$. Define $g : \mathcal{F}_\mathcal{U} \to \mathbb{K}$ by
\[ g[(x_E)]_{\mathcal{U}} = \lim_{\mathcal{U}(E)} g_E(x_E), \]
where $[(x_E)]_{\mathcal{U}}$ denotes the class of $(x_E)$ in $\mathcal{F}_\mathcal{U}$.

Obviously, $g$ is a (well-defined) bounded linear functional on $\mathcal{F}_\mathcal{U}$ and, in fact, $\|g\| \leq 1$. Since $Z$ is super-reflexive, $\mathcal{F}_\mathcal{U}$ is reflexive and we have $(\mathcal{F}_\mathcal{U})^* = (\mathcal{F}^*)_{\mathcal{U}}$, where $\mathcal{F}^* = \{ E^* : E \in \mathcal{F} \}$ (see [7]). It follows that $g = [(\ell_E^*)]_{\mathcal{U}}$, where $\ell_E^* \in E^*$ and $\|\ell_E^*\| \leq 1$ for all $E$. Hence,
\[ \lim_{\mathcal{U}(E)} g_E(x_E) = \lim_{\mathcal{U}(E)} \ell_E^*(x_E) \]
and so
\[ \lim_{\mathcal{U}(E)} \text{dist}(g_E, \ell_E^*) = 0. \]
In particular, for every $\varepsilon > 0$, the set $S = \{ E \in \mathcal{F} : 0 < \text{dist}(g_E, \ell_E^*) < \varepsilon \}$ belongs to $\mathcal{U}$. But, for $E \in S$, one has
\[ \text{dist}(f_E, \ell_E + \delta_E \ell_E^* \leq \varepsilon \delta_E < \delta_E, \]
a contradiction. \hfill \Box

References


Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071 Badajoz, Spain
E-mail address: fcbello@unex.es