PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 4, Pages 1059–1066
S 0002-9939(03)07139-9
Article electronically published on July 14, 2003

ON INJECTIVE OR DENSE-RANGE OPERATORS LEAVING A GIVEN CHAIN OF SUBSPACES INVARIANT

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(Communicated by Joseph A. Ball)

With gratitude, dedicated to H. Hajiabolhassan, I. Mirfazeli, and F. Nouri

Abstract. In this paper we prove the existence of dense-range or one-to-one compact operators on a separable Banach space leaving a given finite chain of subspaces invariant. We use this result to prove that a semigroup $\mathcal{S}$ of bounded operators is reducible if and only if there exists an appropriate one-to-one compact operator K such that the collection $\mathcal{S}K$ of compact operators is reducible.

1. Introduction and Preliminary Results

We start by recalling some definitions and standard notation. Throughout this paper, unless otherwise stated, $\mathcal{X}$ stands for a separable real or complex Banach space. As is usual, by $\mathbb{F}$ we mean $\mathbb{R}$ or $\mathbb{C}$. The term *subspace* will be used to describe a closed subspace of a Banach space $\mathcal{X}$. The subspaces $\{0\}$ and $\mathcal{X}$ are called the trivial subspaces of $\mathcal{X}$; $\mathcal{B}(\mathcal{X})$ denotes the set (in fact the algebra) of bounded operators on $\mathcal{X}$; $\mathcal{B}_0(\mathcal{X})$ is used to denote the set (in fact the ideal) of compact operators on $\mathcal{X}$. A subspace $\mathcal{M}$ is *invariant* for a collection $\mathcal{F}$ of bounded operators if $T\mathcal{M} \subseteq \mathcal{M}$ $(T \in \mathcal{F})$. A collection $\mathcal{F}$ of bounded operators on a space of dimension greater than one is called *reducible* if it has a nontrivial invariant subspace. In case the dimension of the underlying space is one or zero, then the collection $\mathcal{F}$ is called *reducible* if $\mathcal{F} = \{0\}$ or $\mathcal{F} = \emptyset$, respectively, where $\emptyset$ denotes the empty set. For a collection $\mathcal{C}$ of vectors, $\langle \mathcal{C} \rangle$ is used to denote *the (not necessarily closed) linear manifold spanned by* $\mathcal{C}$.

We start off with a well-known lemma.

Lemma 1.1. *Let f, f_j $(1 \le j \le n)$ be linear functionals on a vector space $\mathcal{V}$. Then f is a linear combination of $f_1, ..., f_n$, i.e., $f \in \langle f_j \rangle_{j=1}^n$, iff $\bigcap_{j=1}^n \mathrm{Ker} f_j \subseteq \mathrm{Ker} f$.*

Proof. See [M], page 78. □

Corollary 1.2. *Let f, f_j $(1 \le j \le n)$ be linear functionals on a vector space $\mathcal{V}$, and let $\mathcal{W}$ be a subspace of $\mathcal{V}$. Then $f \in \langle f_1, ..., f_n, \mathcal{W}^\perp \rangle$ iff $\mathcal{W} \cap (\bigcap_{j=1}^n \mathrm{Ker} f_j) \subseteq \mathrm{Ker} f$. (Here $\mathcal{W}^\perp := \{f \in \mathcal{V}' : f(\mathcal{W}) = 0\}$.)*

Received by the editors October 15, 2002 and, in revised form, November 16, 2002.
2000 *Mathematics Subject Classification.* Primary 47A15, 47A46, 47D03.
Key words and phrases. Linear functional, invariant subspace, reducible, weak* topology.
The author gratefully acknowledges the support of an Izaak Walton Killam Memorial Scholarship at Dalhousie University as well as an NSERC PDF at the University of Toronto.

Proof. "$\Longrightarrow$" Obvious.

"$\Longleftarrow$" Let g, g_j $(1 \leq j \leq n)$ be, respectively, the restrictions of f, f_j $(1 \leq j \leq n)$ to the subspace $\mathcal{W}$. It easily follows from the hypothesis that

$$\bigcap_{j=1}^{n} \mathrm{Ker} g_j \subseteq \mathrm{Ker} g.$$

So by Lemma 1.1, $g \in \langle g_j \rangle_{j=1}^{n}$. Thus there are scalars c_j $(1 \leq j \leq n)$, such that $g = \sum_{j=1}^{n} c_j g_j$. Write

$$f = (f - \sum_{j=1}^{n} c_j f_j) + \sum_{j=1}^{n} c_j f_j = f' + \sum_{j=1}^{n} c_j f_j,$$

where $f' = f - \sum_{j=1}^{n} c_j f_j$. It is plain that

$$f'(\mathcal{W}) = (f - \sum_{j=1}^{n} c_j f_j)(\mathcal{W}) = (g - \sum_{j=1}^{n} c_j g_j)(\mathcal{W}) = 0.$$

Hence $f' \in \mathcal{W}^{\perp}$. It follows that $f \in \langle f_1, ..., f_n, \mathcal{W}^{\perp} \rangle$. □

Another preliminary result that we need is the following.

Proposition 1.3. *Let $\mathcal{X}$ be a separable Banach space. Then $\mathcal{X}^*$ is w*-separable.*

Proof. Clearly $\mathcal{X}^* = \bigcup_{n=1}^{\infty} B_n$ where B_n is the closed ball of radius $n (n \in \mathbb{N})$ in $\mathcal{X}^*$. By the Banach-Alaoglu Theorem, every B_n is w*-compact and by Theorem 5.5.1 of [C] every B_n is w*-metrizable, implying that every B_n is w*-separable. Now since every B_n is w*-separable, so is $\mathcal{X}^* = \bigcup_{n=1}^{\infty} B_n$. □

2. An Existence Result

In this section, for a given subspace of a Banach space we prove the existence of certain types of compact operators leaving the given subspace invariant. Then we use this fact to obtain a necessary and sufficient condition for reducibility of a semigroup of bounded operators.

Theorem 2.1. *Let $\mathcal{V}$ be a subspace of an infinite-dimensional separable Banach space $\mathcal{X}$.*

(a) *There exists a compact operator K in the norm closure of finite-rank operators on $\mathcal{X}$ whose range is dense and such that $\overline{K\mathcal{V}} = \mathcal{V}$.*

(b) *There exists a one-to-one compact operator K in the norm closure of finite-rank operators on $\mathcal{X}$ that leaves $\mathcal{V}$ invariant.*

Remark. It is worth mentioning that for (real or complex) Hilbert spaces we can obtain stronger results: there exists a one-to-one compact normal operator K whose range is dense and such that $\overline{K\mathcal{V}} = \mathcal{V}$. To see this, find an orthonormal basis for the given subspace $\mathcal{V}$ and extend the orthonormal basis to an orthonormal basis $\mathcal{B}$ for $\mathcal{X}$, and let K be a diagonal compact normal operator relative to $\mathcal{B}$. It is plain that K has dense range and that $\overline{K\mathcal{V}} = \mathcal{V}$. Now since K is normal and has dense range, it follows that K is one-to-one, which is what we wanted.

Proof. (a) Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of $\mathcal{V}$. Enlarge $\{x_i\}_{i=1}^{\infty}$ to a dense subset $\{x_i\}_{i=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty}$ of $\mathcal{X}$. Without loss of generality, we may assume that $x_i \neq 0, y_i \notin \mathcal{V}$ for all $i \in \mathbb{N}$. If necessary, by choosing subsequences of $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$, we may suppose that $\{x_i\}_{i=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty}$ is independent, $\overline{\langle \{x_i\}_{i=1}^{\infty} \rangle} = \mathcal{V}$, and that $\overline{\langle \{x_i\}_{i=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty} \rangle} = \mathcal{X}$.

Let f_1 be any functional such that $f_1(x_1) \neq 0$. Set $T_1 := x_1 \otimes f_1$ defined by $T_1(x) = f_1(x)x_1$. By the Hahn-Banach Theorem there is a linear functional f_2 such that $f_2(x_1) = 0$ but $f_2(x_2) \neq 0$. Define $T_2 := x_2 \otimes f_2$. Suppose that f_j's $(1 \leq j \leq k-1)$ are chosen. Since $\{x_i\}_{i=1}^{k}$ is independent, by the Hahn-Banach Theorem there is a functional f_k such that $f_k(x) = 0$ for all $x \in \langle x_1, ..., x_{k-1} \rangle$ but $f_k(x_k) \neq 0$. Define $T_k := x_k \otimes f_k$. Note that $T_k(x_j) = f_k(x_j)x_k$ and hence $T_k(x_j) = 0$ for $k > j$ and $T_k(x_k) = f_k(x_k)x_k \neq 0$.

Now, again by the Hahn-Banach Theorem, there is a linear functional f_1' such that $f_1'(x) = 0$ for all $x \in \mathcal{V}$ but $f_1'(y_1) \neq 0$. Define $T_1' := y_1 \otimes f_1'$. Suppose that f_j''s $(1 \leq j \leq k-1)$ are chosen. So by the Hahn-Banach Theorem there is a functional f_k' such that $f_k'(x) = 0$ for all $x \in \langle y_1, ..., y_{k-1}, \mathcal{V} \rangle$ but $f_k'(y_k) \neq 0$. Set $T_k' := y_k \otimes f_k'$. Note that $T_k'(x) = 0$ for all $x \in \mathcal{V}$. In particular, $T_k'(x_j) = 0$ for all positive integers j and k. Also note that $T_k' y_j = f_k'(y_j)y_k$, and hence $T_k' y_j = 0$ for $k > j$ and $T_k' y_k = f_k'(y_k)y_k \neq 0$.

Now define

$$K := \sum_{j=1}^{\infty} \left(\frac{T_j}{2^j \|T_j\|} + \frac{T_j'}{2^j \|T_j'\|} \right).$$

That K is well-defined is obvious since the series defining K is absolutely convergent, hence convergent. We need to show that K is compact, has dense range, and that $\overline{K\mathcal{V}} = \mathcal{V}$. That K is compact is obvious because K is the norm limit of finite rank, hence compact, operators. To verify the remaining claims, first of all we note that, by construction,

$$Kx_j = \sum_{i=1}^{j} c_{ij} x_i, \ j \geq 1$$

with $c_{ij} \in \mathbb{F}$ and $c_{jj} \neq 0$. Thus

$$K\mathcal{V} \ \supseteq \ \langle x_j \rangle_{j=1}^{\infty}.$$

On the other hand, since $T_j'(x) = 0$ for all $x \in \mathcal{V}$ and all positive integers j, and since $\mathcal{V}$ is closed, it easily follows from the series defining K that $Kx \in \mathcal{V}$ for all $x \in \mathcal{V}$. Thus $K\mathcal{V} \subseteq \mathcal{V}$. So we can write

$$\langle x_j \rangle_{j=1}^{\infty} \ \subseteq \ K\mathcal{V} \subseteq \mathcal{V}.$$

Thus

$$\mathcal{V} = \overline{\langle x_j \rangle_{j=1}^{\infty}} \ \subseteq \ \overline{K\mathcal{V}} \subseteq \overline{\mathcal{V}} = \mathcal{V}.$$

Hence $\overline{K\mathcal{V}} = \mathcal{V}$.

It follows from the series defining K that

$$Ky_j = x_j' + \sum_{i=1}^{j} c_{ij}' y_i, \ j \geq 1$$

with $c_{ij}' \in \mathbb{F}$ and $c_{jj}' \neq 0$ and where $x_j' \in \mathcal{V} = \overline{K\mathcal{V}}$. Thus $\overline{K\mathcal{X}} \ \supseteq \ \langle y_j \rangle_{j=1}^{\infty}$. Obviously $\overline{K\mathcal{X}} \ \supseteq \ \overline{K\mathcal{V}} = \mathcal{V}$. Hence

$$\overline{K\mathcal{X}} \ \supseteq \ \langle \langle y_j \rangle_{j=1}^{\infty}, \mathcal{V} \rangle.$$

Therefore

$$\overline{K\mathcal{X}} \ \supseteq \ \{x_i\}_{i=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty}.$$

So

$$\overline{K\mathcal{X}} \supseteq \overline{\langle \{x_i\}_{i=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty} \rangle} = \mathcal{X},$$

i.e., the range of K is dense in $\mathcal{X}$. We note that in the proof above we assumed that both $\mathcal{V}$ and $\mathcal{X}/\mathcal{V}$ are infinite-dimensional. If either $\mathcal{V}$ or $\mathcal{X}/\mathcal{V}$ is finite-dimensional, we end up having finitely many of either x_i's or y_i's in which case the proof can easily be adjusted. This completes the proof.

(b) Let $\mathcal{V} \subseteq \mathcal{X}$ be given. Consider $\mathcal{V}^{\perp} = \{f \in \mathcal{X}^* : f(\mathcal{V}) = 0\}$. Note that $\mathcal{V}^{\perp}$ is a weak* closed, thus a norm closed, subspace of $\mathcal{X}^*$. Also note that $\mathcal{X}^*$ is w*-separable by Proposition 1.3. Let $\{f_i\}_{i=1}^{\infty}$ be a w*-dense subset of $\mathcal{V}^{\perp}$. Enlarge $\{f_i\}_{i=1}^{\infty}$ to a w*-dense subset $\{f_i\}_{i=1}^{\infty} \cup \{g_i\}_{i=1}^{\infty}$ of $\mathcal{X}^*$. With no loss of generality, we may assume that $f_i \neq 0$ and $g_i \notin \mathcal{V}^{\perp}$ for all $i \in \mathbb{N}$. If necessary, by choosing subsequences of $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$, we may suppose that $\{f_i\}_{i=1}^{\infty} \cup \{g_i\}_{i=1}^{\infty}$ is independent, $\overline{\langle \{f_i\}_{i=1}^{\infty} \rangle}^{w*} = \mathcal{V}^{\perp}$, and that $\overline{\langle \{f_i\}_{i=1}^{\infty} \cup \{g_i\}_{i=1}^{\infty} \rangle}^{w*} = \mathcal{X}^*$.

Let $x_1 \in \mathcal{X}$ be such that $f_1(x_1) \neq 0$. Set $S_1 := f_1 \otimes x_1$ defined on $\mathcal{X}^*$ by $S_1 g = g(x_1) f_1$. We take notice of the fact that $S_1 = T_1^*$ where $T_1 = x_1 \otimes f_1$ defined on $\mathcal{X}$ by $T_1 x = f_1(x) x_1$, and that $\|S_1\| = \|T_1\|$. It follows from Lemma 1.1 that Ker $f_1 \not\subseteq$ Ker f_2. Thus there exists $x_2 \in \mathcal{X}$ such that $f_1(x_2) = 0$ but $f_2(x_2) \neq 0$. Set $S_2 = f_2 \otimes x_2$. As before take notice of the fact that $S_2 = T_2^*$ where $T_2 = x_2 \otimes f_2$, and that $\|T_2\| = \|S_2\|$. Now suppose that x_j's $(1 \leq j \leq k-1)$ are chosen. Again it follows from Lemma 1.1 that $\bigcap_{j=1}^{k-1} \mathrm{Ker} f_j \not\subseteq \mathrm{Ker} f_k$. So there exists $x_k \in \mathcal{X}$ such that $f_j(x_k) = 0$ for all $1 \leq j \leq k-1$, but $f_k(x_k) \neq 0$. Set $S_k = f_k \otimes x_k$ and note that $S_k = T_k^*$ where $T_k = x_k \otimes f_k$, and that $\|T_k\| = \|S_k\|$. Note that $S_k f_j = f_j(x_k) f_k$; thus $S_k f_j = 0$ for $k > j$ and $S_k f_k = f_k(x_k) f_k \neq 0$.

Since $g_1 \notin \mathcal{V}^{\perp}$, it follows from the definition that there exists $y_1 \in \mathcal{V}$ such that $g_1(y_1) \neq 0$. Set $S'_1 = g_1 \otimes y_1$ and notice that $S'_1 = T_1^{'*}$ where $T'_1 = y_1 \otimes g_1$, and that $\|T'_1\| = \|S'_1\|$. Suppose that y_j's $(1 \leq j \leq k-1)$ are chosen. Since $g_k \notin \langle g_1, ..., g_{k-1}, \mathcal{V}^{\perp} \rangle$, it follows from Corollary 1.2 that there is $y_k \in \mathcal{V} \cap (\bigcap_{j=1}^{k-1} \mathrm{Ker}\ g_j)$ such that $g_k(y_k) \neq 0$. Set $S'_k = g_k \otimes y_k$ and notice that $S'_k = T_k^{'*}$ where $T'_k = y_k \otimes g_k$, and that $\|T'_k\| = \|S'_k\|$. We note that $S'_k f = f(y_k) g_k = 0$ for all $f \in \mathcal{V}^{\perp}$. In particular, $S'_k f_j = 0$ for all $k, j \geq 1$, and that $S'_k g_j = g_j(y_k) g_k$; thus $S'_k g_j = 0$ for $k > j$ and $S'_k g_k = g_k(y_k) g_k \neq 0$.

Define

$$S := \sum_{j=1}^{\infty} \left(\frac{S_j}{2^j \|S_j\|} + \frac{S'_j}{2^j \|S'_j\|}\right).$$

Note that $S = T^*$ where

$$T = \sum_{j=1}^{\infty} \left(\frac{T_j}{2^j \|T_j\|} + \frac{T'_j}{2^j \|T'_j\|}\right).$$

The operators S and T are compact since they are norm limits of finite rank operators. We claim that the range of S is w*-dense in $\mathcal{X}^*$ and that $\overline{S\mathcal{V}^{\perp}}^{w*} = \mathcal{V}^{\perp}$.

Now, we note that by construction we can write

$$S f_j = \sum_{i=1}^{j} c_{ij} f_i, \ j \geq 1,$$

with $c_{ij} \in \mathbb{F}$ and $c_{jj} \neq 0$. Thus

$$S\mathcal{V}^{\perp} \supseteq \langle f_j \rangle_{j=1}^{\infty}.$$

Since $S = T^*$, S is w* to w* continuous by Theorem 3.1.11 on page 287 of [M]. Let $h \in \mathcal{V}^\perp$ be an arbitrary element. Since $\langle f_i \rangle_{i=1}^\infty$ is w*-dense in $\mathcal{V}^\perp$, it follows that $h = \text{w}^*\text{-}\lim_j h_j$ for some sequence $\{h_j\}_{j=1}^\infty$ in $\langle f_i \rangle_{i=1}^\infty$. Now since S is w* to w* continuous, it follows that $Sh = \text{w}^*\text{-}\lim_j Sh_j$ but $Sh_j \in \mathcal{V}^\perp$ by the series defining S (note that $S'_k f = f(y_k) g_k = 0$ for all $f \in \mathcal{V}^\perp$, and that $\mathcal{V}^\perp$ is w*-closed). Therefore $Sh \in \mathcal{V}^\perp$. So since h was arbitrary, we conclude that

$$\langle f_i \rangle_{i=1}^\infty \subseteq S\mathcal{V}^\perp \subseteq \mathcal{V}^\perp.$$

Hence

$$\mathcal{V}^\perp = \overline{\{f_i\}_{i=1}^\infty}^{w*} \subseteq \overline{S\mathcal{V}^\perp}^{w*} \subseteq \overline{\mathcal{V}^\perp}^{w*} = \mathcal{V}^\perp,$$

implying that $\overline{S\mathcal{V}^\perp}^{w*} = \mathcal{V}^\perp$. Now it follows from the series defining S that

$$Sg_j = f'_j + \sum_{i=1}^{j} c'_{ij} g_i, \ j \geq 1,$$

with $c'_{ij} \in \mathbb{F}$ and $c'_{jj} \neq 0$, where $f'_j \in \mathcal{V}^\perp = \overline{S\mathcal{V}^\perp}^{w*}$ (note that $f'_j \in \mathcal{V}^\perp$ follows from the fact that $\mathcal{V}^\perp$ is norm closed). Thus $\overline{S\mathcal{X}^*}^{w*} \supseteq \langle g_j \rangle_{j=1}^\infty$. Obviously

$$\overline{S\mathcal{X}^*}^{w*} \supseteq \overline{S\mathcal{V}^\perp}^{w*} = \mathcal{V}^\perp.$$

It follows that

$$\overline{S\mathcal{X}^*}^{w*} \supseteq \langle \langle g_j \rangle_{j=1}^\infty, \mathcal{V}^\perp \rangle.$$

So we can write

$$\overline{S\mathcal{X}^*}^{w*} \supseteq \{f_i\}_{i=1}^\infty \cup \{g_i\}_{i=1}^\infty.$$

Thus

$$\overline{S\mathcal{X}^*}^{w*} \supseteq \overline{\langle \{f_i\}_{i=1}^\infty \cup \{g_i\}_{i=1}^\infty \rangle}^{w*} = \mathcal{X}^*,$$

i.e., $\overline{T^*\mathcal{X}^*}^{w*} = \mathcal{X}^*$. So by Theorem 3.1.17 of [M], T is 1-1.

That T leaves $\mathcal{V}$ invariant is not that difficult to see. We have $T_k x = f_k(x) x_k$, and so for all $x \in \mathcal{V}$ and for all $k \geq 1$, we have $T_k x = 0$. Now letting $x \in \mathcal{V}$ be arbitrary, we can write

$$Tx = \sum_{j=1}^\infty \frac{T_j x}{2^j \|T_j\|} + \frac{T'_j x}{2^j \|T'_j\|} = \sum_{j=1}^\infty \frac{g_j(x) y_j}{2^j \|T'_j\|} \in \mathcal{V},$$

for $y_k \in \mathcal{V}, \ k \in \mathbb{N}$ and note that $\mathcal{V}$ is a closed subspace of $\mathcal{X}$. Thus $T\mathcal{V} \subseteq \mathcal{V}$. So $T : \mathcal{X} \longrightarrow \mathcal{X}$ is compact, 1-1, and leaves $\mathcal{V}$ invariant. Again in the proof above we assumed that both $\mathcal{V}^\perp$ and $\mathcal{X}^*/\mathcal{V}^\perp$ are infinite-dimensional. If either of them is finite-dimensional, just as we mentioned in the proof of part (a), we end up having finitely many f_i's or g_i's, in which case the proof can easily be adjusted. □

Remark. For a given subspace $\mathcal{V}$ of a (real or complex) Banach space $\mathcal{X}$, we do not know whether there exists a one-to-one compact operator K in the norm closure of finite-rank operators on $\mathcal{X}$ whose range is dense and such that $\overline{K\mathcal{V}} = \mathcal{V}$.

Theorem 2.2. *Let $\mathcal{X}$ be an infinite-dimensional separable Banach space, and let $\mathcal{F}$ be a finite chain of subspaces.*

(a) *There exists a compact operator K in the norm closure of finite-rank operators on $\mathcal{X}$ whose range is dense and such that $\overline{K\mathcal{V}} = \mathcal{V}$ for all $\mathcal{V} \in \mathcal{F}$.*

(b) *There exists a one-to-one compact operator K in the norm closure of finite-rank operators on $\mathcal{X}$ that leaves $\mathcal{V}$ invariant for all $\mathcal{V} \in \mathcal{F}$.*

Proof. (a) Suppose that

$$\mathcal{V}_0 := \{0\} \subset \mathcal{V}_1 \subset ... \subset \mathcal{V}_n \subset \mathcal{V}_{n+1} := \mathcal{X}$$

is a given finite chain of subspaces. It is easily seen that there is a countable independent set

$$\bigcup_{i=1}^{n+1} \{x_{ij}\}_{1 \leq j \leq \dim(\mathcal{V}_i/\mathcal{V}_{i-1})}$$

of elements of $\mathcal{X}$ such that $\bigcup_{i=1}^{k} \{x_{ij}\}_{1 \leq j \leq \dim(\mathcal{V}_i/\mathcal{V}_{i-1})}$ is dense in $\mathcal{V}_k$ for all $1 \leq k \leq n+1$ and such that $x_{ij} \notin \mathcal{V}_{i-1}$ for all $1 \leq i \leq n+1$, $1 \leq j \leq \dim(\mathcal{V}_i/\mathcal{V}_{i-1})$. The rest of the proof is similar to that of Theorem 2.1. We omit the details.

(b) Just as in the proof of the (a) part of the theorem, the idea of the proof of part (b) is identical to that of Theorem 2.1(b). □

Remark. 1. Again if the underlying space happens to be a real or complex Hilbert space, then, just as we saw in the remark preceding the proof of Theorem 2.1, a similar argument shows that there exists a one-to-one compact normal operator K whose range is dense and such that $\overline{K\mathcal{V}} = \mathcal{V}$ for all $\mathcal{V} \in \mathcal{F}$ where $\mathcal{F}$ is the given finite chain of subspaces.

2. Similarly, by adjusting the proof of part (a) of Theorem 2.2, it is not difficult to see that for a given countable chain $\mathcal{F}$ of subspaces of an infinite-dimensional separable Banach space $\mathcal{X}$, of the form

$$\mathcal{V}_0 := \{0\} \subset \mathcal{V}_1 \subset ... \subset \mathcal{V}_n \subset ... \subset \mathcal{V}_\infty := \mathcal{X},$$

there always exists a compact operator K in the norm closure of finite-rank operators whose range is dense and such that $\overline{K\mathcal{V}} = \mathcal{V}$ for all $\mathcal{V} \in \mathcal{F}$.

Theorem 2.3. *Let $\mathcal{X}$ be an infinite-dimensional separable Banach space, and $\mathcal{S}$ a semigroup of operators in $\mathcal{B}(\mathcal{X})$. Then the following are equivalent:*

(i) The semigroup $\mathcal{S}$ is reducible.

(ii) There exists a one-to-one compact operator K in the norm closure of finite-rank operators such that the collection

$$\mathcal{S}K = \{AK : A \in \mathcal{S}\}$$

is reducible.

(iii) There exists a compact operator K in the norm closure of finite-rank operators whose range is dense and such that the collection

$$K\mathcal{S} = \{KA : A \in \mathcal{S}\}$$

is reducible.

Proof. Let $\mathcal{A}$ denote the algebra generated by the semigroup $\mathcal{S}$. We note that $\mathcal{A} = \langle \mathcal{S} \rangle$. That being noted, it suffices to prove the assertion for an algebra $\mathcal{A}$ of operators in $\mathcal{B}(\mathcal{X})$. We prove that $(i) \Longleftrightarrow (ii)$ and $(i) \Longleftrightarrow (iii)$.

$(i) \Longrightarrow (ii)$ Suppose that $\mathcal{A}$ is reducible. So there exists a nontrivial subspace $\{0\} \subsetneqq \mathcal{V} \subsetneqq \mathcal{X}$ such that $\mathcal{A}\mathcal{V} \subseteq \mathcal{V}$. By Theorem 2.1(b), there exists a one-to-one compact operator K such that $K\mathcal{V} \subseteq \mathcal{V}$. It is plain that $\mathcal{A}K\mathcal{V} \subseteq \mathcal{V}$.

$(i) \Longrightarrow (iii)$ Can be proven in a similar manner as in $(i) \Longrightarrow (ii)$ except that one has to use Theorem 2.1(a).

$(ii) \Longrightarrow (i)$ Let $\{0\} \subsetneqq \mathcal{V} \subsetneqq \mathcal{X}$ be a nontrivial subspace of $\mathcal{X}$ such that $\mathcal{A}K\mathcal{V} \subseteq \mathcal{V}$ where K is a 1-1 compact operator. Choose $0 \neq x \in \mathcal{V}$ and set $\mathcal{W} := \overline{\mathcal{A}Kx}$. If

$\mathcal{W} = 0$, then $\{0\} \subsetneqq \langle Kx \rangle \subsetneqq \mathcal{X}$, and $\langle Kx \rangle$ is a nontrivial closed subspace of $\mathcal{A}$. If $\mathcal{W} \neq 0$, then $\{0\} \subsetneqq \mathcal{W} \leq \mathcal{V} \subsetneqq \mathcal{X}$ is obviously a nontrivial closed subspace of $\mathcal{A}$. So in any event, $\mathcal{A}$ is reducible.

$(iii) \Longrightarrow (i)$ Let $\{0\} \subsetneqq \mathcal{V} \subsetneqq \mathcal{X}$ be a nontrivial subspace of $\mathcal{X}$ such that $K\mathcal{A}\mathcal{V} \subseteq \mathcal{V}$ where K is a compact operator whose range is dense. Choose $0 \neq x \in \mathcal{V}$ and set $\mathcal{W} := \overline{\mathcal{A}x}$. If $\mathcal{W} = 0$, then $\langle x \rangle$ is a closed nontrivial subspace of $\mathcal{A}$. If $\mathcal{W} \neq 0$, then $\{0\} \subsetneqq \mathcal{W} \subsetneqq \mathcal{X}$ is obviously a nontrivial closed subspace of $\mathcal{A}$. That $\mathcal{W} \subsetneqq \mathcal{X}$ follows from the fact that K has dense range and that $K\overline{\mathcal{A}x} \subseteq K\overline{\mathcal{A}\mathcal{V}} \subseteq \overline{K\mathcal{A}\mathcal{V}} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{V}$. So in any event $\mathcal{A}$ is reducible. □

Remark. 1. Let $\mathcal{S}$ be as in Theorem 2.3, and $A, B \in \mathcal{B}(\mathcal{X})$ where A is 1-1 and B is a dense-range bounded operator. It follows from the proof of the preceding theorem that $\mathcal{S}$ is reducible if $\mathcal{S}A$ or $B\mathcal{S}$ is reducible.

2. It is shown in [Y1] (see Theorem 4.2 of [Y1]) that triangularizability of $\mathcal{S}A$ implies reducibility of $\mathcal{S}$ provided that $\operatorname{rank}(A) \geq 2$. As pointed out in the remark following Theorem 4.2 of [Y1], the assertion we just mentioned holds on finite-dimensional vector spaces over general fields.

3. By Lemma 2.5.1 of [Y2], reducibility of $T\mathcal{S}|_{\overline{T\mathcal{X}}}$ implies that of the semigroup $\mathcal{S}$ provided that $T \neq 0$. The finite-dimensional counterpart of this lemma is Lemma 2.2.1 of [Y2] which says: *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, and $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$, and T a nonzero linear transformation in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = T\mathcal{V}$ is the range of T.*

We conclude with the following questions, which we have not been able to resolve.

Let $\mathcal{X}$ be an infinite-dimensional separable real or complex Banach space, and $\mathcal{C}$ an arbitrary chain of subspaces.

1. Does there exists a compact operator K on $\mathcal{X}$ whose range is dense and such that $\overline{K\mathcal{V}} = \mathcal{V}$ for all $\mathcal{V} \in \mathcal{C}$?

2. Does there exists a one-to-one compact operator K on $\mathcal{X}$ that leaves the chain $\mathcal{C}$ invariant, i.e., $K\mathcal{V} \subseteq \mathcal{V}$ for all $\mathcal{V} \in \mathcal{C}$?

Note that since every chain of subspaces can be extended to a maximal chain of subspaces, it suffices to answer the above questions for maximal chains of subspaces.

Acknowledgment

I would like to thank my supervisor Professor Heydar Radjavi for his encouragement and very helpful comments. I would also like to thank the referee for the helpful comments.

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