PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 4, Pages 1103–1111
S 0002-9939(03)07150-8
Article electronically published on September 18, 2003

MINIMAL DISPLACEMENT AND RETRACTION PROBLEMS IN INFINITE-DIMENSIONAL HILBERT SPACES

KRZYSZTOF BOLIBOK

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We give the first constructive example of a Lipschitz mapping with positive minimal displacement in an infinite-dimensional Hilbert space H. We use this construction to obtain an evaluation from below of the minimal displacement characteristic in the space H. In the second part we present a simple and constructive proof of existence of a Lipschitz retraction from a unit ball B onto a unit sphere S in the space H, and we improve an evaluation from above of a retraction constant $k_0(H)$.

1. Introduction

Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with the closed unit ball B and the unit sphere S. For any $k \geq 0$, let $L(k)$ denote the class of Lipschitz mappings with constant k. If K is a bounded, closed and convex subset of X, then by d_T we will denote minimal displacement of T,

$$d_T = \inf_{x \in K} \|x - Tx\|,$$

where $T : K \to K$. This notion for Lipschitz mappings was introduced by Goebel [7] in 1973. In this paper he gave examples of sets K and Lipschitz mappings T with positive d_T. The problem for which sets there exists a Lipschitz mapping with positive minimal displacement was studied by many authors, and a final solution was given by Lin and Sternfeld [12]. They proved that if K is bounded, closed and convex but noncompact, then there exists a Lipschitz mapping T such that $d_T > 0$. On the other hand, very little is known about qualitative properties of minimal displacement. Goebel in the above-mentioned paper introduced some functions describing the minimal displacement problem. We recall two of them. The first, denoted by φ_X, is defined as follows:

$$\varphi_X(k) = \sup\{d_T : K \subset X, r(K) = 1, T : K \to K, T \in L(k), k \geq 1\},$$

where $r(K)$ is the Chebyschev radius of K, i.e.,

$$r(K) = \inf[\sup\{\|x - y\| : x \in K\} : y \in X].$$

Received by the editors November 6, 2001 and, in revised form, December 10, 2002.
2000 *Mathematics Subject Classification.* Primary 47H09, 47H10.
Key words and phrases. Lipschitz mappings, minimal displacement, Lipschitz retraction.
This research was supported in part by KBN grant 2 PO3A 029 15.

The second ψ_X denotes the minimal displacement characteristic of X,

$$\psi_X(k) = \sup\{d_T : T : B \to B, T \in L(k), k \geq 1\}.$$

It is known that for any space X,

$$\psi_X(k) \leq \varphi_X(k) \leq 1 - \frac{1}{k}.$$

Moreover, the following holds:

$$\begin{aligned} \psi_X(1-\alpha+\alpha k) &\geq \alpha\psi_X(k) \text{ for any } \alpha \in [0,1], \\ \lim_{k\to\infty} \psi_X(k) &= 1 \text{ for any space } X. \end{aligned}$$

There are some "square" spaces like $c_0, C[0,1]$ for which $\psi_X(k) = \varphi_X(k) = 1 - \frac{1}{k}$. The two functions are equal in the Hilbert space H and the following estimate holds:

$$\psi_H(k) = \varphi_H(k) \leq \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}}.$$

In 1979 Nowak [13] proved that in some infinite-dimensional Banach spaces there exist Lipschitz retractions of unit balls onto unit spheres. Four years later Benyamini and Sternfeld [1] extended the above result to all infinite-dimensional Banach spaces. This result leads directly to the definition of the so-called retraction constant $k_0(X)$, being the infimum of the set of all numbers $k > 1$ for which there exists a retraction $R : B \to S$ belonging to $L(k)$. It known that $k_0(X) \geq 3$ for any space X. For the space H we know that $k_0(H) > 4.5$. On the other hand, Komorowski and Wośko [11] gave an example of a Lipschitz retraction with Lipschitz constant less than 64.5. Unfortunately their example is not completely constructive. For a wider discussion of the topics mentioned above we refer the reader to [3]-[6], [9]-[11], [14], and [15].

2. Minimal displacement

For more than twenty years since Goebel's paper there have been no results on the minimal displacement problem in the space H. In 1996 the author [2] proved the following evaluation from below of the function ψ_H.

Theorem 2.1. *In an infinite-dimensional Hilbert space H,*

$$\psi_H(k) \geq \sup_{\epsilon\in(0,2)} \left(1 - \frac{2+\epsilon}{\sqrt{1+\epsilon(\epsilon+2)k^2} - 1} - \epsilon(k+1)\right).$$

Unfortunately this evaluation is very imprecise. Below we present a better estimation. Let us consider the Hilbert space $L^2(0,1)$ and let S^+ denote the nonnegative part of the unit sphere S, i.e.,

$$S^+ = \{x \in S : x(t) \geq 0 \text{ for } t \in (0,1)\}.$$

By d_T denote minimal displacement also in this setting (S^+ is not convex). We start with the following lemma.

Lemma 2.2. *For every $\alpha \geq 2$ there exists a mapping $T : S^+ \to S^+$ from the class $L\left(\alpha^{\frac{3}{2}}\right)$ such that $d_T = \frac{\sqrt{2}\left(\sqrt{\alpha}-1\right)}{\sqrt{\alpha+1}}$.*

Proof. Let $\alpha \geq 2$ and let the mapping $T : S^+ \to S^+$ be defined by an equality

$$\int_0^t [(Tx)(s)]^2 \, ds = \left(\int_0^t [x(s)]^2 \, ds \right)^{\alpha} \qquad \text{for every } t \in (0,1).$$

Observe that the map is well defined and that it can be written equivalently as

$$(Tx)(t) = \sqrt{\alpha} x(t) \|x\|_t^{\alpha-1},$$

where

$$\|x\|_t = \sqrt{\int_0^t [x(s)]^2 \, ds.}$$

First we show that T is a Lipschitz mapping.

$$\begin{aligned} \|Tx - Ty\|^2 &= \alpha \int_0^1 \left[x(t) \|x\|_t^{\alpha-1} - y(t) \|y\|_t^{\alpha-1} \right]^2 dt \\ &\leq \alpha \int_0^1 \left[x(t) \left| \|x\|_t^{\alpha-1} - \|y\|_t^{\alpha-1} \right| + |x(t) - y(t)| \, \|y\|_t^{\alpha-1} \right]^2 dt \\ &\leq \alpha \int_0^1 \left[(\alpha - 1) x(t) \left| \|x\|_t - \|y\|_t \right| + |x(t) - y(t)| \right]^2 dt \\ &\leq \alpha \int_0^1 \left[(\alpha - 1) x(t) \|x - y\| + |x(t) - y(t)| \right]^2 dt. \end{aligned}$$

Furthermore using standard arguments such as the triangle and Cauchy-Schwartz inequalities, we get

$$\|Tx - Ty\|^2 \leq \alpha^3 \|x - y\|^2.$$

Hence $T \in L\left(\alpha^{\frac{3}{2}}\right)$. The minimal displacement of T can be written as follows:

$$\begin{aligned} \|x - Tx\|^2 &= \|x\|^2 - 2 \langle x, Tx \rangle + \|Tx\|^2 \\ &= 2 - 2 \int_0^1 x(t) (Tx)(t) \, dt \\ &= 2 - 2\sqrt{\alpha} \int_0^1 [x(t)]^2 \left(\int_0^t [x(s)]^2 \, ds \right)^{\frac{\alpha-1}{2}} dt. \end{aligned}$$

Setting

$$z = \int_0^t [x(s)]^2 \, ds,$$

we get

$$\|x - Tx\|^2 = 2 - 2\sqrt{\alpha} \int_0^1 z^{\frac{\alpha-1}{2}} dz = \frac{2\left(\sqrt{\alpha} - 1\right)^2}{\alpha + 1} > 0.$$

Observe that $\|x - Tx\| = \text{const}$ for every $x \in S^+$. Moreover, when $\alpha \to \infty$, then $\|x - Tx\| \to \sqrt{2} = \operatorname{diam} S^+$. □

Theorem 2.3. *Let H be an infinite-dimensional Hilbert space. Then*

$$\psi_H \left(\sqrt{2} (\alpha + 1) \alpha^{\frac{3}{2}} \right) \geq \frac{\alpha - 1}{\alpha + 1} \quad \text{for } \alpha \geq 2.$$

Proof. For any $r > 0$, let $B^+(r)$ be the nonnegative part of the ball $B(r)$ of radius r, i.e., the set

$$B^+(r) = \{x \in B(r) : x(t) \geq 0 \text{ for } t \in (0,1)\}.$$

Define a mapping $T_1 : B \to B^+(1)$ by

$$(T_1 x)(t) = |x(t)|.$$

Obviously, $T_1 \in L(1)$. Let δ denote the following expression:

$$\delta = d_T \sqrt{1 - \left(\frac{d_T}{2}\right)^2} = \frac{\alpha - 1}{\alpha + 1},$$

where d_T is the minimal displacement of the map T from the previous lemma. Furthermore, we define a map $\widetilde{T}$ such that $d_{\widetilde{T}} = \delta$. But now let a mapping $T_2 : B^+(1) \to B^+(1)$ be given as

$$(T_2 x)(t) = \begin{cases} x(t) & \text{for } \|x\| \geq 1 - \delta, \\ 1 - \delta - \|x\| + x(t) & \text{for } \|x\| < 1 - \delta. \end{cases}$$

Obviously when $\|x\| \geq 1 - \delta$ and $\|y\| \geq 1 - \delta$, then $T_2 \in L(1)$. When $\|x\| < 1 - \delta$ and $\|y\| < 1 - \delta$, we get

$$\begin{aligned} \|T_2 x - T_2 y\|^2 &= \int_0^1 (x(t) - y(t) + \|y\| - \|x\|)^2 \, dt \\ &\leq 2 \int_0^1 (x(t) - y(t))^2 \, dt + 2 \int_0^1 (\|y\| - \|x\|)^2 \, dt \\ &\leq 4 \|x - y\|^2. \end{aligned}$$

Hence

$$\|T_2 x - T_2 y\| \leq 2 \|x - y\|.$$

Let us estimate $\inf_{x \in B^+} \|T_2 x\|$. Observe that if $\|x\| \geq 1 - \delta$, then

$$\|T_2 x\| = \|x\| \geq 1 - \delta.$$

If $\|x\| < 1 - \delta$, then

$$\begin{aligned} \|T_2 x\|^2 &= \int_0^1 (1 - \delta - \|x\| + x(t))^2 \, dt \\ &= (1 - \delta - \|x\|)^2 + 2(1 - \delta - \|x\|) \int_0^1 x(t) \, dt + \int_0^1 (x(t))^2 \, dt \\ &\geq 2 \|x\|^2 - 2(1 - \delta) \|x\| + (1 - \delta)^2. \end{aligned}$$

The above minimum is attained for $\|x\| = \frac{1-\delta}{2}$ and is equal to $\frac{(1-\delta)^2}{2}$, which finally shows that

$$\inf_{x \in B^+} \|T_2 x\| \geq \frac{1 - \delta}{\sqrt{2}}.$$

Let $C = B^+(1) \setminus B^+\left(\frac{1-\delta}{\sqrt{2}}\right)$. Suppose that a mapping $T_3 : \overline{C} \to S^+$ is the radial projection, i.e.,

$$(T_3 x)(t) = \frac{x(t)}{\|x\|}.$$

From the properties of the radial projection we get that $T_3 \in L\left(\frac{\sqrt{2}}{1-\delta}\right)$. We define the above-mentioned mapping $\widetilde{T} : B \to S^+$ as the composition

$$\widetilde{T} = T \circ T_3 \circ T_2 \circ T_1,$$

where T is the mapping from the previous lemma. It is easy to see that

$$\widetilde{T} \in L\left(\frac{2\sqrt{2}\alpha^{\frac{3}{2}}}{1-\delta}\right) \quad \text{for } \alpha \geq 2,$$

or equivalently, after simple calculations,

$$\widetilde{T} \in L\left(\sqrt{2}\,(\alpha+1)\,\alpha^{\frac{3}{2}}\right) \quad \text{for } \alpha \geq 2.$$

The minimal displacement of a map $\widetilde{T}$ can be evaluated as follows:

$$d_{\widetilde{T}} = \inf_{x\in B}\left\|x - \widetilde{T}x\right\| \geq \inf_{x\in B^+(1)}\left\|x - \widetilde{T}x\right\| = \inf_{x\in B^+(1)}\left\|x - T\circ T_3\circ T_2 x\right\|.$$

Observe that if $x \in B^+(1)$ and $\|x\| < 1-\delta$, then

$$\left\|x - \widetilde{T}x\right\| > \delta.$$

If $x \in B^+(1)$ and $\|x\| \geq 1-\delta$, then

$$\begin{aligned}
\left\|x - \widetilde{T}x\right\| &= \left\|x - T\circ T_3\circ T_2 x\right\| = \left\|x - T\circ T_3 x\right\| = \left\|x - T\left(\tfrac{x}{\|x\|}\right)\right\| \\
&\geq \left\|\tfrac{x}{\|x\|} - T\left(\tfrac{x}{\|x\|}\right)\right\|\sqrt{1-\left(\frac{\left\|\frac{x}{\|x\|} - T\left(\frac{x}{\|x\|}\right)\right\|}{2}\right)^2} \\
&= d_T\sqrt{1-\left(\tfrac{d_T}{2}\right)^2} = \delta = \tfrac{\alpha-1}{\alpha+1},
\end{aligned}$$

which finally shows that

$$\psi_{L^2(0,1)}\left(\sqrt{2}\,(\alpha+1)\,\alpha^{\frac{3}{2}}\right) \geq \frac{\alpha-1}{\alpha+1} \quad \text{for } \alpha \geq 2.$$

Observe that if $\alpha \to \infty$, then $\frac{\alpha-1}{\alpha+1} \to 1$, which shows that the above estimate is good for large α. □

3. Retraction

In this section we give the first completely constructive example of a Lipschitz retraction in the space H. Let $P : X \to B$ be the radial projection, i.e.,

$$Px = \begin{cases} x & \text{if } x \in B, \\ \frac{x}{\|x\|} & \text{if } x \notin B. \end{cases}$$

Denoting by $P(X)$ the Lipschitz constant of such a projection, it is known that $P(H) = 1$ and $1 \leq P(X) \leq 2$ for any space X [16]. We start with the following technical lemma, which holds in all infinite-dimensional Banach spaces.

Lemma 3.4. *Suppose that there exists a homotopy $G : [0,1]\times S \to S$ such that for any $x \in S$, the following conditions hold:*

$$G(0,x) = x \quad \textit{and} \quad G(1,x) \equiv x_0 \in S.$$

Assume that the homotopy G is Lipschitz with constants M and N, i.e., for every $x, y \in S$ and $c, d \in [0,1]$, the following condition holds:

$$\|G(c,x) - G(d,y)\| \leq M\,|c-d| + N\,\|x-y\|.$$

Then there exists a retraction $R : B \to S$ of a class $R \in L\left(\frac{P(X)N}{r}\right)$, where $r \in (0,1)$ is a solution of the equation

$$\frac{P(X)N}{r} = \frac{M - P(X)N\ln r}{1-r}.$$

Proof. Define a retraction $R : B \to S$ for $r \in (0,1)$ as follows:

$$Rx = \begin{cases} x_0 & \text{if } \|x\| \le r, \\ G\left(1 - f(\|x\|), \frac{x}{\|x\|}\right) & \text{if } \|x\| > r, \end{cases}$$

where f is any increasing, convex and differentiable function on the interwal $[r,1]$ such that $f(r) = 0, f(1) = 1$. Observe that the retraction R is Lipschitz. For any $x, y \in B$ such that $\|x\| \ge r, \|y\| \ge r$ we have

$$\begin{aligned} \|Rx - Ry\| &= \left\| G\left(1 - f(\|x\|), \tfrac{x}{\|x\|}\right) - G\left(1 - f(\|y\|), \tfrac{y}{\|y\|}\right) \right\| \\ &\le M\left|f(\|x\|) - f(\|y\|)\right| + N\left\|\tfrac{x}{\|x\|} - \tfrac{y}{\|y\|}\right\| \\ &\le Mf'(\max\{\|x\|, \|y\|\})\left|\|x\| - \|y\|\right| + \tfrac{P(X)N}{\max\{\|x\|,\|y\|\}}\|x - y\| \\ &\le \max_{s\in[r,1]}\left\{Mf'(s) + \tfrac{P(X)N}{s}\right\}\|x - y\|. \end{aligned}$$

To optimize the Lipschitz constant of a retraction R, observe that

$$\int_r^1 \left(Mf'(t) + \frac{P(X)N}{t}\right)dt = M - P(X)N\ln r$$

does not depend on the choice of a function f. The Lipschitz constant will be minimal if the function satisfies a condition

$$Mf'(t) + \frac{P(X)N}{t} = \frac{M - P(X)N\ln r}{1-r}.$$

We get

$$\|Rx - Ry\| \le \frac{P(X)N}{r}\|x - y\|,$$

where r is a solution of the equation

$$\frac{P(X)N}{r} = \frac{M - P(X)N\ln r}{1-r}.$$

Observe that such a solution exists and is unique because the above functions are continuous on the interval $(0,1)$ and one is strictly increasing while the other is strictly decreasing.

Now we can proceed to the construction of a Lipschitz retraction in the Hilbert space $L^2(0,1)$. □

Construction 3.5. Define a homotopy $G : [0,1] \times S \to S$ as follows:

$$G(c,x) = \frac{(1-c)x + cT \circ T_1 x}{\|(1-c)x + cT \circ T_1 x\|},$$

where the map T is the map from Lemma 2.2 and the map T_1 is from the proof of Theorem 2.3. We show that the homotopy G is well defined and

$$\inf_{c\in[0,1]}\inf_{x\in S}\|(1-c)x + cT \circ T_1 x\| > 0.$$

Observe that

$$\inf_{c\in[0,1]}\inf_{x\in S}\|(1-c)\,x+cT\circ T_1x\| = \inf_{c\in[0,1]}\inf_{x\in S^+}\|(1-c)\,x-cTx\|$$

and

$$\begin{aligned} d_T^2 &= \inf_{x\in S^+}\|x-Tx\|^2 \\ &\le \|x-Tx\|^2 \\ &= 2\|x\|^2-2\langle x,Tx\rangle+\|Tx\|^2 \\ &= 2-2\langle x,Tx\rangle . \end{aligned}$$

Using this estimate we get

$$\begin{aligned} \|(1-c)\,x-cTx\|^2 &= (1-c)^2\|x\|^2-2c\,(1-c)\langle x,Tx\rangle+c^2\|Tx\|^2 \\ &= 1-c\,(1-c)\left(4-d_T^2\right) \\ &\ge \tfrac{d_T^2}{4}>0. \end{aligned}$$

The homotopy G is Lipschitz. Indeed,

$$\begin{aligned} \|G(c,x)-G(d,y)\| &\le \|G(c,x)-G(d,x)\|+\|G(d,x)-G(d,y)\| \\ &\le \frac{2\,|c-d|}{\inf\limits_{c\in[0,1]}\inf\limits_{x\in S}\|(1-c)\,x+cT\circ T_1x\|} \\ &\quad+\sup_{c\in[0,1]}\frac{1+c\,(k-1)}{\inf\limits_{z\in S^+}\|(1-c)\,z-cTz\|}\,\|x-y\|, \end{aligned}$$

where $k=\alpha^{\frac{3}{2}}$. Since for $c\in[0,1]$ we have

$$\|(1-c)\,x-cTx\| = \|cx-(1-c)\,Tx\|,$$

we obtain

$$\begin{aligned} \|H(c,x)-H(d,y)\| &\le \tfrac{4}{d_T}\,|c-d|+\sup_{c\in\left[\frac{1}{2},1\right]}\tfrac{1+c(k-1)}{\inf\limits_{z\in S^+}\|(1-c)z-cTz\|}\,\|x-y\| \\ &= M(\alpha)\,|c-d|+N(\alpha)\,\|x-y\|, \end{aligned}$$

where (after calculations)

$$M(\alpha)=\frac{2\sqrt{2\,(\alpha+1)}}{\sqrt{\alpha}-1}\quad\text{and}\quad N(\alpha)=\sup_{c\in\left[\frac{1}{2},1\right]}\frac{1+c\left(\alpha^{\frac{3}{2}}-1\right)}{\sqrt{1-2c\,(1-c)\,\frac{\left(\sqrt{\alpha}+1\right)^2}{\alpha+1}}}.$$

Now define a mapping $T_2 : B^+(r)\to B^+(r)$ by

$$(T_2x)(t)=r-\|x\|+x(t).$$

From the proof of Theorem 2 we conclude that

$$T_2\in L(2)\quad\text{and}\quad\inf_{x\in B^+(r)}\|T_2x\|\ge\frac{r}{\sqrt{2}}.$$

Now we can define a retraction $R : B\to S$ by

$$(Rx)(t)=\begin{cases}\left(T\circ\left(\frac{T_2}{\|T_2\|}\right)\circ T_1x\right)(t) & \text{if } \|x\|\le r,\\ G\left(1-f(\|x\|),\frac{x(t)}{\|x\|}\right) & \text{if } \|x\|>r,\end{cases}$$

where f is any increasing, convex and differentiable function defined on the interval $[r,1]$ such that $f(r)=0$ and $f(1)=1$.

Observe that if $\|x\| \leq r$ and $\|y\| \leq r$ we get

$$\begin{aligned} \|Rx - Ry\| &\leq \left\| T \circ \left(\tfrac{T_2}{\|T_2\|}\right) \circ T_1 x - T \circ \left(\tfrac{T_2}{\|T_2\|}\right) \circ T_1 y \right\| \\ &\leq \tfrac{2\sqrt{2}k}{r} \|x - y\| = 2\sqrt{2} \tfrac{\alpha^{\frac{3}{2}}}{r} \|x - y\| . \end{aligned}$$

If $\|x\| > r$ and $\|y\| > r$, using the previous lemma we have

$$\|Rx - Ry\| \leq \frac{N(\alpha)}{r} \|x - y\| ,$$

where r is a solution of the equation

$$\frac{N(\alpha)}{r} = \frac{M(\alpha) - N(\alpha) \ln r}{1 - r}.$$

Numerical experiments show that for $\alpha = 2.535$ we get a retraction with the minimal Lipschitz constant slightly less than 32.26. This implies the following.

Corollary 3.6. *In the infinite-dimensional Hilbert space H,*

$$k_0(H) < 32.26.$$

The problem of exact evaluation of $k_0(X)$ for at least one space is still open.

References

[1] Y. Benyamini and Y. Sternfeld, *Spheres in infinite-dimensional normed spaces are Lipschitz contractible*, Proc. Amer. Math. Soc. **88** (1983), 439-445. MR **85h:**46028

[2] K. Bolibok, *Constructions of Lipschitzian mappings with non-zero minimal displacement in spaces $L^1(0,1)$ and $L^2(0,1)$*, Ann. Univ. Marie Curie-Sklodowska Sect. A **50** (1996), 25-31. MR **98j:**47123

[3] K. Bolibok, *Construction of a Lipschitz retraction in the space c_0,* Ann. Univ. Marie Curie-Sklodowska Sect. A. **51** (1997), 43-46. MR **99m:**46029

[4] K. Bolibok and K. Goebel, *A note on minimal displacement and retraction problems*, J. Math. Anal. Appl. **206** (1997), 308-314. MR **97m:**47078

[5] C. Franchetti, *Lipschitz maps and the geometry of the unit ball in normed spaces*, Arch. Math. **46** (1986), 76-84. MR **87g:**46025

[6] M. Furi and M. Martelli,*On the minimal displacement of points under alpha-Lipschitz maps in normed spaces*, Boll. Un. Mat. Ital. **9** (1974), 791-799. MR **51:**6509

[7] K. Goebel, *On the minimal displacement of points under Lipschitz mappings*, Pacific J. Math. **45** (1973), 151-163. MR **48:**7050

[8] K. Goebel, *Metric environment of the topological fixed point theorems*, "Handbook of Metric Fixed Point Theory", Kluwer, Dordrecht, 2001, 577-611. MR **2003e:**47096

[9] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990. MR **92c:**47070

[10] K. Goebel and T. Komorowski, *Retracting balls onto spheres and minimal displacement problems*, Fixed Point Theory and Applications, Pitman Research Notes in Math., Longman Sci. Tech., Harlow, 1991, 155-172. MR **93b:**47110

[11] T. Komorowski and J. Wośko, *A remark on the retracting of a ball onto a sphere in an infinite-dimensional Hilbert space*, Math. Scand. **67** (1990), 223-226. MR **92e:**58012

[12] P. K. Lin and Y. Sternfeld, *Convex sets with the Lipschitz fixed point property are compact*, Proc. Amer. Math. Soc. **93** (1985), 633-639. MR **86c:**47074

[13] B. Nowak, *On the Lipschitz retraction of the unit ball in infinite-dimensional Banach spaces onto its boundary*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **27** (1979), 861-864. MR **82g:**58008

[14] S. Reich, *Minimal displacement of points under weakly inward pseudo-Lipschitzian mappings I*, Atti. Accad. Naz. Linzei Rend. Cl. Sci. Fis. Mat. Natur. **59** (1975), 40-44. MR **56:**9345

[15] S. Reich, *Minimal displacement of points under weakly inward pseudo-Lipschitzian mappings II*, Atti. Accad. Naz. Linzei Rend. Cl. Sci. Fis. Mat. Natur. **60** (1976), 95-96. MR **58:**7264

[16] R. L. Thele, *Some results on the radial projection in Banach spaces*, Proc. Amer. Math. Soc. **42** (1974), 483-486. MR **48:**6892

Institute of Mathematics, Maria Curie - Skłodowska University, 20-031 Lublin, Poland

E-mail address: bolibok@golem.umcs.lublin.pl