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PARAMETRIC DECOMPOSITION OF POWERS OF IDEALS VERSUS REGULARITY OF SEQUENCES

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ABSTRACT. Let $Q = (a_1, a_2, \cdots, a_s)$ ($\subsetneq A$) be an ideal in a Noetherian local ring A. Then the sequence $a_1, a_2, \cdots, a_s$ is A-regular if every a_i is a non-zerodivisor in A and if $Q^n = \bigcap_\alpha (a_1^{\alpha_1}, a_2^{\alpha_2}, \cdots, a_s^{\alpha_s})$ for all integers $n \geq 1$, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s)$ runs over the elements of the set $\Lambda_{s,n} = \{(\alpha_1, \alpha_2, \cdots, \alpha_s) \in \mathbb{Z}^s \mid \alpha_i \geq 1 \text{ for all } 1 \leq i \leq s \text{ and } \sum_{i=1}^s \alpha_i = s + n - 1\}$.

1. INTRODUCTION

Throughout this note let A denote a commutative ring with the non-zero multiplicative identity. Let $a_1, a_2, \cdots, a_s \in A$ ($s \geq 1$) and $Q = (a_1, a_2, \cdots, a_s)$ in A. For each integer $n \geq 1$ we put

$$\Lambda_{s,n} = \{(\alpha_1, \alpha_2, \cdots, \alpha_s) \in \mathbb{Z}^s \mid \alpha_i \geq 1 \text{ for all } 1 \leq i \leq s \text{ and } \sum_{i=1}^{s} \alpha_i = s + n - 1\}.$$

Let $Q(\alpha) = (a_1^{\alpha_1}, a_2^{\alpha_2}, \cdots, a_s^{\alpha_s})$ for each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s) \in \Lambda_{s,n}$. Then $Q^n \subseteq \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$, and W. Heinzer, L. J. Ratliff, Jr. and K. Shah ([HRS, Theorem 2.4]) proved, among other things, that the equality holds true for all $n \geq 1$ if the sequence $a_1, a_2, \cdots, a_s$ is A-regular. The purpose of this note is to study the question of when the converse holds true, and our conclusion is stated as follows.

Theorem (1.1). *Let A be a Noetherian local ring and assume that $Q \subsetneq A$. Then the following two conditions are equivalent:*

(1) *The sequence $a_1, a_2, \cdots, a_s$ is A-regular.*
(2) *Every a_i is a non-zerodivisor in A and the equality*

$$Q^n = \bigcap_{\alpha \in \Lambda s,n} Q(\alpha)$$

holds true for all integers $n \geq 1$.

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The condition in Theorem (1.1) (2) that every a_i is a non-zerodivisor in A is not superfluous. (Consider the case $s = 1$ or the case where $a_i = 0$ for all $1 \leq i \leq s$. See Example (3.6) for a more non-trivial example.)

The condition in Theorem (1.1) (2) is a local condition. The following result gives a global version of Theorem (1.1).

Corollary (1.2). *Let A be a Noetherian ring and $Q \subsetneq A$. Assume that every a_i is a non-zerodivisor in A. Then the following two conditions are equivalent:*

(1) $\mathrm{grade}(Q, A) = s$.
(2) $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$ *for all* $n \geq 1$.

Our proof of Theorem (1.1) is based on the induction on s, which we will discuss in Section 3 (Proposition (3.4)). For this purpose we need some preliminary steps, including a brief proof of part of [HRS, Theorem 2.4], which we shall summarize in Section 2.

2. Some lemmata

We put $Q_i = (a_1, a_2, \cdots, a_i)$ for each $0 \leq i \leq s$. Let us begin with the following.

Lemma (2.1). *Suppose that $s \geq 2$ and $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$ for all $n \geq 1$. Then $(a_s^\ell) \cap Q_{s-1}^m \subseteq Q^{\ell+m}$ for all $\ell, m \geq 1$.*

Proof. Let $x \in (a_s^\ell) \cap Q_{s-1}^m$ and assume that $x \notin Q^{\ell+m} = \bigcap_{\alpha \in \Lambda_{s,\ell+m}} Q(\alpha)$. We choose $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s) \in \Lambda_{s,\ell+m}$ so that $x \notin Q(\alpha)$. Then $\alpha_s \geq \ell + 1$, since $x \in (a_s^\ell)$. Therefore, $\sum_{i=1}^{s-1} \alpha_i = [s + (\ell + m) - 1] - \alpha_s \leq s + m - 2$. Let $\delta = (s + m - 2) - \sum_{i=1}^{s-1} \alpha_i$. Then, since $(\alpha_1 + \delta, \alpha_2, \cdots, \alpha_{s-1}) \in \Lambda_{s-1,m}$, we get

$$x \in Q_{s-1}^m \subseteq (a_1^{\alpha_1+\delta}, a_2^{\alpha_2}, \cdots, a_{s-1}^{\alpha_{s-1}}) \subseteq Q(\alpha),$$

which is impossible. □

The following result (2.2) is due to [HRS]. We give a brief proof for the sake of completeness. Our proof might be of some interest, because it is totally different from the one given in [HRS].

Proposition (2.2). *Suppose the sequence $a_1, a_2, \cdots, a_s$ is A-regular. Then*

$$Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$$

for all $n \geq 1$.

Proof. Assume $Q^n \subsetneq \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$ and choose $a \in \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$ so that $a \notin Q^n$. Let ℓ be the largest integer satisfying the condition $a \in Q^\ell$. Then $0 \leq \ell < n$. Let $G = \bigoplus_{n \geq 0} Q^n/Q^{n+1}$ be the associated graded ring of Q and put $X_i = a_i \bmod Q^2$ $(1 \leq i \leq s)$. Then $X_1, X_2, \cdots, X_s$ are algebraically independent over $k = A/Q$ and $G = k[X_1, X_2, \cdots, X_s]$. We put $Y = a \bmod Q^{\ell+1}$. Then $Y \neq 0$ and $\deg Y = \ell < n$. Let $\alpha \in \Lambda_{s,n}$. Then, since $a \in Q(\alpha)$ and the sequence $X_1^{\alpha_1}, X_2^{\alpha_2}, \cdots, X_s^{\alpha_s}$ is G-regular, we get from [VV, Proposition 2.1] that $Y \in (X_1^{\alpha_1}, X_2^{\alpha_2}, \cdots, X_s^{\alpha_s})$. (The result [VV, Proposition 2.1] holds true with no extra assumption on base rings A.)

Therefore, in order to produce a contradiction, we have only to check the parametric decomposition

$$(X_1, X_2, \cdots, X_s)^n = \bigcap_{\alpha \in \Lambda_{s,n}} (X_1^{\alpha_1}, X_2^{\alpha_2}, \cdots, X_s^{\alpha_s})$$

of ideals $(X_1, X_2, \cdots, X_s)^n$ in the polynomial ring $G = k[X_1, X_2, \cdots, X_s]$.

Let $0 \le \beta_i \in \mathbb{Z}$ $(1 \le i \le s)$ and $c \in k$. Let $M = c\prod_{i=1}^s X_i^{\beta_i}$ and assume that $M \in \bigcap_{\alpha\in\Lambda_{s,n}}(X_1^{\alpha_1}, X_2^{\alpha_2}, \cdots, X_s^{\alpha_s})$ but $M \notin (X_1, X_2, \cdots, X_s)^n$. We put $m = \sum_{i=1}^s \beta_i$. Then, since $m \le n-1$, letting $\delta = (n-1)-m$, we get $(\beta_1+\delta+1, \beta_2+1, \cdots, \beta_s+1) \in \Lambda_{s,n}$. Consequently,

$$0 \ne c\prod_{i=1} X_i^{\beta_i} \in \bigcap_{\alpha\in\Lambda_{s,n}} (X_1^{\alpha_1}, X_2^{\alpha_2}, \cdots, X_s^{\alpha_s}) \subseteq (X_1^{\beta_1+\delta+1}, X_2^{\beta_2+1}, \cdots, X_s^{\beta_s+1}),$$

which is impossible. Hence $(X_1, X_2, \cdots, X_s)^n = \bigcap_{\alpha\in\Lambda_{s,n}}(X_1^{\alpha_1}, X_2^{\alpha_2}, \cdots, X_s^{\alpha_s})$. $\square$

3. Proof of Theorem (1.1)

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and assume that $Q = (a_1, a_2, \cdots, a_s) \subseteq \mathfrak{m}$. To prove Theorem (1.1) we need the following. Let us note a quick proof for the sake of completeness.

Lemma (3.1). *Let $a \in A$ and assume that $(0) : a^\ell \subseteq (a)$ for all $\ell \ge 1$. Then a is a non-zerodivisor in A.*

Proof. We may assume that $a \in \mathfrak{m}$. Let $x \in (0) : a^\ell$ and write $x = ay$ $(y \in A)$. Then $y \in (0) : a^{\ell+1}$, whence $(0) : a^\ell \subseteq a[(0) : a^{\ell+1}]$. Take $\ell \gg 0$ so that $(0) : a^\ell = (0) : a^{\ell+1}$. Then, since $(0) : a^{\ell+1} = a[(0) : a^{\ell+1}]$, by Nakayama's lemma we get $(0) : a^{\ell+1} = (0)$, whence a is a non-zerodivisor in A, as is $a^{\ell+1}$. $\square$

The following two results are the key for our proof of Theorem (1.1).

Lemma (3.2). *Suppose that $Q^n = \bigcap_{\alpha\in\Lambda s,n} Q(\alpha)$ for all $n \ge 1$. Then $Q_{s-1} : a_s = Q_{s-1}$ if a_s is a non-zerodivisor in A.*

Proof. We may assume that $s \ge 2$. By Lemma (3.1) it suffices to show that

$$Q_{s-1} : a_s^\ell \subseteq Q$$

for all $\ell \ge 1$. Since a_s is a non-zerodivisor in A, it is enough to check that $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q$. Assume the contrary and choose $m \ge 1$ so that $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q + Q_{s-1}^m$, but $(a_s^\ell) \cap Q_{s-1} \not\subseteq a_s^\ell Q + Q_{s-1}^{m+1}$. Then, since

$$(a_s^\ell) \cap Q_{s-1} \subseteq (a_s^\ell Q + Q_{s-1}^m) \cap (a_s^\ell) = a_s^\ell Q + [(a_s^\ell) \cap Q_{s-1}^m],$$

by Lemma (2.1) we get $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q + Q^{\ell+m}$. The following claim now readily shows $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q + Q_{s-1}^{m+1}$, which is impossible.

Claim (3.3). $Q^{\ell+m} \subseteq a_s^\ell Q + Q_{s-1}^{m+1}$.

Proof of Claim (3.3). Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s) \in \mathbb{Z}^s$ $(\alpha_i \ge 0)$ and assume that $\sum_{i=1}^s \alpha_i = \ell + m$. Then, if $\alpha_s \ge \ell$, we certainly have

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_s^{\alpha_s} = a_s^\ell (a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_{s-1}^{\alpha_{s-1}} a_s^{\alpha_s - \ell}) \in a_s^\ell Q,$$

because $\alpha_1 + \cdots + \alpha_{s-1} + (\alpha_s - \ell) = \sum_{i=1}^s \alpha_i - \ell = m \ge 1$. Suppose that $\alpha_s \le \ell - 1$. Then $a_1^{\alpha_1} \cdots a_{s-1}^{\alpha_{s-1}} \in Q_{s-1}^{m+1}$, because $\sum_{i=1}^{s-1}\alpha_i = \sum_{i=1}^s \alpha_i - \alpha_s \ge (\ell+m) - (\ell-1) =$

$m+1$. Hence $a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_s^{\alpha_s} \in a_s^\ell Q + Q_{s-1}^{m+1}$ in any case, so that $Q^{\ell+m} \subseteq a_s^\ell Q + Q_{s-1}^{m+1}$ as claimed. $\square$

Proposition (3.4). *Assume that* $Q^n = \bigcap_{\alpha \in \Lambda s,n} Q(\alpha)$ *for all* $n \geq 1$. *Then* $Q_i^n = \bigcap_{\alpha \in \Lambda i,n} Q_i(\alpha)$ *for all integers* $1 \leq i \leq s$ *and* $n \geq 1$.

Proof. We may assume that $s \geq 2$ and $i = s-1$. Let $x \in \bigcap_{\alpha \in \Lambda s-1,n} Q_{s-1}(\alpha)$ and assume $x \notin Q_{s-1}^n$. Choose the integer $\ell \geq 0$ so that $x \in Q_{s-1}^n + (a_s^\ell)$ but $x \notin Q_{s-1}^n + (a_s^{\ell+1})$. We write $x = y + a_s^\ell z$ ($y \in Q_{s-1}^n, z \in A$). We then have the following.

Claim (3.5). $x - y \in Q^{\ell+n} = \bigcap_{\alpha \in \Lambda_{s,\ell+n}} Q(\alpha)$.

Proof of Claim (3.5). Let $\alpha \in \Lambda_{s,\ell+n}$. Then, since $x - y = a_s^\ell z \in (a_s^\ell)$, to see that $x - y \in Q(\alpha)$, we may assume $\alpha_s \geq \ell+1$. Let $\delta = (s+n-2) - \sum_{i=1}^{s-1}\alpha_i$. Then, since $\alpha_s \geq \ell+1$ and $\sum_{i=1}^{s}\alpha_i = s + (\ell+n) - 1$, we have $\delta \geq 0$. Because $(\alpha_1 + \delta, \alpha_2, \cdots, \alpha_{s-1}) \in \Lambda_{s-1,n}$ and $y \in Q_{s-1}^n$, we get that $x - y \in (a_1^{\alpha_1+\delta}, a_2^{\alpha_2}, \cdots, a_{s-1}^{\alpha_{s-1}}) \subseteq Q(\alpha)$ (recall that $x \in \bigcap_{\alpha \in \Lambda_{s-1,n}} Q_{s-1}(\alpha)$). Thus $x - y \in Q(\alpha)$ for all $\alpha \in \Lambda_{s,\ell+n}$, so that we have $x - y \in Q^{\ell+n}$. $\square$

Consequently, we get $x \in Q_{s-1}^n + Q^{\ell+n} \subseteq Q_{s-1}^n + (a_s^{\ell+1})$, which is impossible. Hence $Q_{s-1}^n = \bigcap_{\alpha \in \Lambda_{s-1,n}} Q_{s-1}(\alpha)$ for all $n \geq 1$. $\square$

We are now in a position to finish the proof of Theorem (1.1).

Proof of Theorem (1.1). (2) $\Rightarrow$ (1) We may assume that $s \geq 2$ and our assertion holds true for $s-1$. Then by Lemma (3.2) and Proposition (3.4) the sequence $a_1, a_2, \cdots, a_{s-1}$ is A-regular and $Q_{s-1} : a_s = Q_{s-1}$, that is, the sequence $a_1, a_2, \cdots, a_s$ is A-regular.

(1) $\Rightarrow$ (2) This follows directly from Proposition (2.2) and is due to [HRS]. $\square$

The condition in Theorem (1.1) (2) that every a_i is a non-zerodivisor in A is not superfluous. Let us explore one example.

Example (3.6). Let $B = k[[X,Y,Z]]$ be the formal power series ring over a field k and put $A = B/(XY)$. Let x, y, and z denote respectively the reduction of X, Y, and Z mod (XY). Let $a_1 = z, a_2 = x$, and $Q = (a_1, a_2)$. Then the sequence a_1, a_2 is not A-regular, but the equality $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}} Q(\alpha)$ holds true for all $n \geq 1$.

Proof. Since $a_2 y = 0$, the sequence a_1, a_2 is not A-regular. Let $C = A/[(0) : a_2]$. Then $C = k[[X,Z]]$, since $(0) : a_2 = (y)$, so that the sequence $a_1 = z, a_2 = x$ is C-regular. Hence $(a_1,a_2)^n C = \bigcap_{\alpha \in \Lambda_{2,n}} (a_1^{\alpha_1}, a_2^{\alpha_2})C$ for all $n \geq 1$ (cf. Proposition (2.2)). Therefore, to see that $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}} Q(\alpha)$, it is enough to show $[(0) : a_2] \cap (a_1^n, a_2) \subseteq Q^n$. We will check that $[(0) : a_2] \cap (a_1^n, a_2) \subseteq a_1^n A$. Let $f \in [(0) : a_2] \cap (a_1^n, a_2)$ and write $f = a_1^n g + a_2 h$ ($g, h \in A$). Let $\overline{*}$ stand for the reduction mod $(0) : a_2$. Then $\overline{a_1^n}\,\overline{g} + \overline{a_2}\,\overline{h} = 0$, so that $\overline{h} \in a_1^n C$. Hence $f = a_1^n g + a_2 h \in a_1^n A$, because $h \in a_1^n A + [(0) : a_2]$. $\square$

Before closing this note, let us add one more result, which is an immediate consequence of Theorem (1.1).

Corollary (3.7). *Let A be a Noetherian local ring with $s = \dim A \geq 1$. Then the following two conditions are equivalent:*

(1) *A is a Cohen-Macaulay ring.*
(2) $\dim A/\mathfrak{p} = s$ *for all* $\mathfrak{p} \in \operatorname{Ass} A$ *and the ring* A *contains a system* $a_1, a_2, \cdots, a_s$ *of parameters, satisfying the equality*
$$Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$$
for all $n \geq 1$.

When this is the case, one has the equality $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$ *for all integers* $n \geq 1$ *and for every system* $a_1, a_2, \cdots, a_s$ *of parameters.*

In the forthcoming paper [GS] we shall give a characterization of Noetherian local rings A with $s = \dim A \geq 2$, in which the equality $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$ holds true for all integers $n \geq 1$ and for every parameter ideal $Q = (a_1, a_2, \cdots, a_s)$.

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