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SELF-NORMALIZING SYLOW SUBGROUPS

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Abstract. Using the classification of finite simple groups we prove the following statement: Let $p > 3$ be a prime, Q a group of automorphisms of p-power order of a finite group G, and P a Q-invariant Sylow p-subgroup of G. If $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q)$ is trivial, then G is solvable. An equivalent formulation is that if G has a self-normalizing Sylow p-subgroup with $p > 3$ a prime, then G is solvable. We also investigate the possibilities when $p = 3$.

1. Introduction

A well-known result by Glauberman and Thompson states that a finite nonabelian simple group cannot have a self-normalizing Sylow p-subgroup for $p > 3$. (See for instance, [4, Thm. X.8.13].)

In this paper we extend this result to show the following general statement:

Theorem 1.1. *Let p be an odd prime and P a Sylow p-subgroup of the finite group G. If $p = 3$, assume that G has no composition factors of type $\mathrm{L}_2(3^f)$, $f = 3^a$ with $a \geq 1$.*

(1) *If $P = \mathbf{N}_G(P)$, then G is solvable.*
(2) *If $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$, then $G/O_{p'}(G)$ is solvable.*

Note that the second result implies the first since it is well known ([7], [1, Lemma 12.1]) that if H is a group of automorphisms of R with $\gcd(|H|, |R|) = 1$ and $\mathbf{C}_R(H) = 1$, then R is solvable. We then apply this result to P acting on $O_{p'}(G)$. We will say more about this in the next section.

If G is a simple group with $p \geq 5$, it was an observation of Thompson that this followed quite easily from a result of Glauberman. See [4, Thm. X.8.15].

An easy consequence of the previous theorem (or our proof) is the extension of this result to $p = 3$.

Corollary 1.2. *If p is an odd prime and G is a nonabelian finite simple group, then $\mathbf{N}_G(P) \neq P\mathbf{C}_G(P)$.*

Proof. By the theorem, we need only consider $p = 3$ and $G = \mathrm{L}_2(3^{3^a})$. Then the split torus acts nontrivially on a Sylow 3-subgroup. $\square$

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We point out the following consequence (see [4] for a weaker result).

Corollary 1.3. *Let G be a finite group and suppose that $p_i, i = 1, 2$ are distinct primes dividing the order of G. If P_i is a Sylow p_i-subgroup of G, then one of the P_i cannot be self-normalizing.*

Proof. Let G be a minimal counterexample to the assertion. Let N be a minimal normal subgroup of G. If $P_i \not\leq N$ for $i = 1, 2$, then G/N is another counterexample. So by minimality we may assume that $P_1 \leq N$, say. Then $G = \mathbf{N}_G(P_1)N = P_1N = N$. So G is simple and obviously nonabelian. Since at least one of the p_i is odd, we can apply the previous result. □

2. Some preliminary results

In this section, we discuss some results that we will use in the proof of Theorem 1.1. The first is the old result of Burnside that $\mathbf{N}_G(P)$ controls fusion in $Z(P)$ for P a Sylow subgroup.

Lemma 2.1. *Let P be a Sylow p-subgroup of the finite group G. If $x, y \in Z(P)$ are conjugate in G, then x, y are conjugate in $\mathbf{N}_G(P)$.*

Proof. Suppose that $y^g = x$ for some $g \in G$. Then P and P^g are both Sylow p-subgroups of $\mathbf{C}_G(x)$. Thus, $P^{gc} = P$ for some $c \in \mathbf{C}_G(x)$. So $y^{gc} = x^c = x$ as required. □

The only place that the classification of finite simple groups comes in for $p \geq 5$ is the following:

Theorem 2.2. *If L is a finite simple group and* $\mathrm{Out}(L)$ *is divisible by an odd prime, then L is a Chevalley group.*

As noted, we will use the following result (really only the special case where Q is a p-group). We sketch the proof.

Theorem 2.3. *If Q is a π-group acting fixed point freely on a π'-group R, then R is solvable.*

Proof. Let $G = RQ$. The hypotheses are equivalent to the fact that Q is self-normalizing in G. Since any two Hall π-subgroups are conjugate in G and in any quotient of G (by the Schur-Zassenhaus theorem), it follows that Q is self-normalizing in G/R_1 where R_1 is any normal subgroup of G contained in R. So by induction on $|R|$, we may assume that R is a minimal normal subgroup of G. If R is solvable, the result follows. This implies that R is a direct product of t copies of a nonabelian simple group L and that Q permutes these copies transitively. Let Q_1 be the normalizer of L. We see easily that $\mathbf{C}_R(Q) \cong \mathbf{C}_L(Q_1)$ and so by induction $t = 1$, i.e., R is a nonabelian simple group. In particular, Q has odd order. Thus, R is a Chevalley group and Q is a cyclic group of field automorphisms, whence the fixed points are nontrivial (since they contain the corresponding Chevalley group over a subfield). □

Of course, the proof shows a bit more – for example, the same conclusion holds if we assume that $\mathbf{C}_Q(R)$ has odd order or is a 2-group. See [7]. The critical case when R is simple is handled in [1] (where it was shown that the set of fixed points has even order).

The following result will also be useful. We will primarily use this result for the classical groups.

Lemma 2.4. *Let L be a simple Chevalley group defined over the field of q elements. Let $x \in L$ be a semisimple element.*

(1) *If L is not twisted, or a Suzuki or Ree group, then x^q and x are conjugate in L.*
(2) *If $L = \mathrm{U}_n(q)$, ${}^2D_n(q)$ with n odd, or ${}^2E_6(q)$, then x^q is conjugate to x^{-1}.*
(3) *If $L = {}^2D_n(q)$ with n even, then x^{q^2} is conjugate to x.*
(4) *If $L = {}^3D_4(q)$, then x^{q^3} is conjugate to x.*

Proof. There is no harm in assuming that L is simply connected.

In all cases, the absolutely irreducible representations of L over an algebraically closed field of natural characteristic are known; see, for example, [2]. If L is not twisted, or a Suzuki or Ree group, then every such representation is defined over $\mathbb{F}_q$, whence x and x^q have the same Brauer character for every such irreducible representation. Thus, x and x^q are conjugate, proving (1).

Similarly, if L is twisted by a graph automorphism of order d, every irreducible representation in the natural characteristic is defined over $\mathbb{F}_{q^d}$, giving (3) and (4).

Moreover, for types A_n, D_n and E_6, the weight of the dual of an irreducible representation is minus the image under the longest element of the Weyl group (see [2, 31.6]). If -1 is not contained in the Weyl group (that is, in types A_n, D_n with n odd, and E_6), then this is the same as the image under the graph automorphism of order 2.

Thus it follows that x^q and x^{-1} have the same character in every irreducible representation of a twisted group $\mathrm{U}_n(q)$, ${}^2D_n(q)$ (n odd) or ${}^2E_6(q)$. □

Let us remark that if L is classical, the lemma can be proved in a more elementary fashion by considering the action on the natural module. Also, in the proof of Theorem 1.1, this result could be replaced by less elegant arguments using the structure of the normalizer of maximal tori.

Note that the conclusion of (2) is not valid for $L = {}^2D_n(q)$ with n even, as can be seen already on elements of order 8 in ${}^2D_2(3) \cong \mathrm{SL}_2(9)$.

3. Reduction to the almost simple case

In this section we reduce the proof of Theorem 1.1 to the case of almost simple groups.

Let G be a minimal counterexample to the theorem (i.e., to either part of the theorem). Let P be a Sylow p-subgroup of G, $C = \mathbf{C}_G(P)$ and $N = \mathbf{N}_G(P)$. Note that if H is a proper subgroup of G containing P, then H satisfies the assumptions of the theorem as well, and hence, by minimality of G, the conclusion.

Step 1. $O_{p'}(G) = 1$.

Suppose not and consider $G/O_{p'}(G)$. It is still true that $N = PC$ in the quotient and so, by minimality, $G/O_{p'}(G)$ is solvable. We only need to show that either $O_{p'}(G)$ is solvable or $\mathbf{C}_{O_{p'}(G)}(P) \neq 1$. This is Theorem 2.3.

Step 2. $O_p(G) = 1$.

Suppose not and let $A = O_p(G)$. Then G/A satisfies the theorem and, in particular, G/B is solvable where $B/A = O_{p'}(G/A)$. This implies that $E(G) \leq B$ and so is a p'-group, whence $E(G) = 1$. Another application of Step 1 implies that

$F^*(G) = A$. Thus, $C_G(P) \leq C_G(A) \leq A \leq P$ and so P is self-normalizing. Thus, P/A is self-normalizing in G/A and, by minimality, G/A and so G is solvable.

Step 3. G is almost simple.

Let M be a minimal normal subgroup of G. By Steps 1 and 2 we have $M = L^t$ with L a nonabelian simple subgroup and by Step 1, p divides $|L|$. Thus, $G = MP$ by minimality.

Moreover, M is the unique minimal normal subgroup of G (since $O_p(G) = 1$).

There is no harm in enlarging P to assume that $G = MQ$ where Q is a full Sylow p-subgroup of $\mathrm{Aut}(M)$ (for we are asserting that there exists a p'-element $x \in M$ with $[x, Q] \leq Q \cap M = P \cap M$ and $[x, Q] \neq 1$ – but then $[x, Q \cap M] \neq 1$ and so $[x, P] \neq 1$).

Then $Q = P_1 \wr S$ where P_1 is a Sylow p-subgroup of $\mathrm{Aut}(L)$ and S is a Sylow p-subgroup of the symmetric group on t letters. In this case, if $t > 1$ we see that by induction, we can choose $x_1 \in L$ so that $1 \neq [x_1, P_1] \leq P_1 \cap L$ and taking $y := (x_1, \ldots, x_1) \in L^t$ we find that $N \neq PC$.

So we now have that G is almost simple with socle L, $P \cap L \neq 1$, $G = LP$ and that every overgroup of P other than G satisfies the theorem.

4. The almost simple case

We now handle the almost simple case. So recall that $G = LP$ with L simple and P a Sylow p-subgroup. We still assume that every proper subgroup of G satisfies the theorem.

Claim: L is a Chevalley group in characteristic r and $\mathrm{Out}(L)$ has order divisible by p.

If not, then by Theorem 2.2, $G = L$. If $p > 3$, the result is the Glauberman-Thompson theorem. If $p = 3$, we need to invoke the classification. If $L = A_n, n \geq 5$, then any element of order 3 is conjugate to its inverse and so by the Burnside result, every element of order 3 in $Z(P)$ is conjugate to its inverse, whence the result. If L is sporadic, we just inspect to see that $\mathbf{N}_G(P) \neq P\mathbf{C}_G(P)$.

Claim: $r \neq p$.

Assume $r = p$ and let R be a proper parabolic subgroup of L. Note that G contains no diagonal automorphisms nor graph automorphisms except possibly in the case $L = D_4(3^a)$ and $p = 3$.

We exclude that case momentarily. It follows that $\mathbf{N}_G(R)$ contains P and G/L is cyclic generated by a field automorphism x. By induction, $\mathbf{N}_G(R)$ must satisfy the theorem, and so the Levi complement in R must be solvable or involve only $\mathrm{L}_2(3^{3^a})$. This implies that either L is a rank one group or $r = 3$ and L is a rank two group. Note that by the exclusion of the case $\mathrm{L}_2(3^{3^a})$, there exists $1 \neq g \in \mathbf{C}_T(x)$ where T is a nontrivial torus in the normalizer of $P \cap L$ over the fixed field of the field automorphism. Then $g \in \mathbf{N}_G(P)$ and does not centralize P, whence the claim.

Now consider the case $L = D_4(3^a)$ and $p = 3$. The same argument applies except that we need to take $g \in T$ defined over the prime field and commuting with a graph automorphism normalizing the given Borel subgroup.

Claim: -1 is not in the Weyl group of L.

Otherwise every r'-element of L is conjugate to its inverse in L. In particular, $z \in L \cap Z(P)$ is conjugate to its inverse against Burnside's lemma.

Claim: P does not contain graph automorphisms.

The only case to consider here is the triality automorphism for $D_4(q)$, but this is ruled out by the previous claim.

Claim: P does not contain diagonal automorphisms.

If so, then $L = \mathrm{L}_n^{\epsilon}(q)$ with p dividing $\gcd(q-\epsilon 1, n)$. It follows that the normalizer of P is contained in the normalizer $N(T_1)$ where T_1 is the torus of order $(q-\epsilon 1)^{n-1}$. Since this surjects onto $\mathfrak{S}_n$ with n a multiple of p, we have a contradiction (by induction for $n > 3$ and by inspection for $n = 3$).

It follows now from the two previous claims that $P = \langle P \cap L, \sigma\rangle$ where σ is a field automorphism. So we may write $L = L(q^{p^a})$ with σ a field automorphism of order p^a and q a prime power prime to p. Set $q_0 = q^{p^a}$.

Claim: $P \cap L$ is not abelian.

Suppose this is the case. Then $L_0 = \mathbf{C}_L(\sigma)$ has an abelian Sylow p-subgroup $P_0 := L \cap \mathbf{C}_P(\sigma) \le Z(P)$. Since $\mathbf{N}_G(P)$ controls fusion in $Z(P)$, it follows that $\mathbf{N}_G(P_0) = \mathbf{C}_G(P_0)$; whence by the normal p-complement theorem [3, Hauptsatz IV.2.6], P_0 has a normal p-complement in L_0. On the other hand $O^p(L_0)$ is a Chevalley group over a smaller field. In particular, this must be solvable, whence $p = 3$ and $L_0 = \mathrm{L}_2(3)$, i.e., $L = \mathrm{L}_2(3^{3^a})$, a case excluded by hypothesis.

At this point, we are left with the groups of type A_n^{ϵ}, D_n^{ϵ} and E_6^{ϵ}, with nonabelian Sylow p-subgroup. In particular, p divides the order of the Weyl group of L.

Claim: L is not of type $E_6^{\epsilon}(q^{p^a})$ with $p = 5$.

Since we may assume the Sylow 5-subgroup of L is nonabelian, it follows (for example from Springer–Steinberg [6, Th. 5.16 and Cor. 5.17] and the order formula) that $\mathbf{N}_G(P) \le K := T(W \times \langle\sigma\rangle)$, where T is the split torus (of order $(q_0-1)^6$ or $(q_0+1)^6$, respectively) where W is the Weyl group of L. Since $W = \mathrm{U}_4(2).2$ is not a counterexample, the result holds.

Claim: L is not $E_6^{\epsilon}(q^{p^a})$ with $p = 3$.

The same argument applies as for $p = 5$ unless 3 does not divide $q-\epsilon 1$. In those cases, by Lemma 2.4 any element of order 3 in L is conjugate to its inverse and so the Burnside result applies.

Claim: $L \ne \mathrm{L}_n^{\epsilon}(q^{p^a})$.

First consider $\epsilon = +$. Then by Lemma 2.4 every element $x \in L \cap Z(P)$ is conjugate to x^q. So Burnside's fusion result applies unless $x^q = x$, i.e., $p|(q-1)$. Then $\mathbf{N}_G(P) \le K := T(\mathfrak{S}_n \times \langle\sigma\rangle)$ where T is the split torus. Moreover, $n \ge p$ (otherwise $L \cap P$ is abelian). Since $\mathfrak{S}_n$ is not a counterexample to the theorem, the result holds except possibly if $n \le 4$ and $p = 3$. It is still the case for $H = \mathfrak{S}_n, n = 3, 4$ that $\mathbf{N}_H(S) = \mathfrak{S}_3 \ne S\mathbf{C}_H(S)$ with S a Sylow 3-subgroup.

If $\epsilon = -$, the same argument applies with a slight twist. Taking x as above, x^q is conjugate to x^{-1}, whence $p|(q+1)$ or Burnside's fusion result applies.

Claim: L is not $D_n^{\pm}(q_0), n \ge 4$.

It is more convenient to work in the corresponding subgroup of the classical group (and it suffices to show the result there). Let V be the natural module. We may also assume that n is odd since for n even, $-1 \in W$.

Let $x \in Z(P) \cap L$. So $x \in \mathrm{SO}_{2n}^{\epsilon}(q_0)$. Let U be a nontrivial irreducible module for $\langle x\rangle$. If U is nondegenerate, then it is of $-$ type and x^q is conjugate to x^{-1}. Thus, $x^q = x^{-1}$ or Burnside applies. Thus, $q+1$ is a multiple of p. In this case, $P \le (\Omega_2^-(q_0) \wr \mathfrak{S}_d)\langle\sigma\rangle$ with $d = n$ or $n-1$ depending upon the type of space. If $p < d$, then $P \cap L$ is abelian. If $p \geq d$, then the result follows by induction (noting that $\mathfrak{S}_3$ and $\mathfrak{S}_4$ have Sylow 3-subgroups with normalizers that are not counterexamples).

If U is totally singular, then x is conjugate to x^q, whence either Burnside applies or $p|(q-1)$. Then $P \leq (\Omega_2^+(q_0) \wr \mathfrak{S}_d)\langle\sigma\rangle$ and we argue as above.

We now have dealt with all the possibilities. This proves Theorem 1.1.

5. Examples

The following examples show that the results fail often when $p = 2$ and that we do need to exclude certain possibilities when $p = 3$.

Example 5.1. Let G be an untwisted group of Lie type of rank l over the field with $q = 2$ elements. Then the Borel subgroup coincides with its unipotent radical, since the maximally split torus has order $(q-1)^l = 1$. The smallest non-solvable example is $\mathrm{L}_3(2)$ with a self-normalizing Sylow 2-subgroup. The isomorphism to $\mathrm{L}_2(7)$ shows that examples also arise in groups of Lie type over fields of odd characteristic.

Also, the Sylow 2-subgroups of the alternating groups $\mathfrak{A}_n$, $n \geq 6$, and of the symmetric groups $\mathfrak{S}_n$, $n \geq 5$, are self-normalizing.

Finally, all sporadic groups except J_1, J_2, J_3, Suz and HN have self-normalizing Sylow 2-subgroups.

Example 5.2. Let $G = \mathrm{L}_2(q)$, q odd. Then the Borel subgroup is an extension of its unipotent radical by the maximally split torus of order $(q-1)/2$. In particular, the Sylow 3-subgroup is self-normalizing if and only if $q = 3$, when $G = \mathrm{L}_2(3) \cong \mathfrak{A}_4$ is solvable. But we get a non-solvable example as follows. Assume that $q = 3^f$ with $f = 3^a$ a power of 3. Then the field automorphism σ of G of order f centralizes just $\mathrm{L}_2(3)$. Thus the extension of G by $Q = \langle\sigma\rangle$ shows that we need to exclude composition factors $\mathrm{L}_2(3^{3^a})$ in the hypotheses of Theorem 1.1 for $p = 3$.

Example 5.3. Let $G = \mathrm{L}_{p+1}(q)$ with $p|(q-1)$. Let P be a Sylow p-subgroup of G. Then $Z(P)$ is cyclic of order p and $\mathbf{N}_G(P) \leq \mathbf{C}_G(Z(P))$. This shows that there may be no fusion in the center of P.

Note added in proof. After the completion of this paper, we learned of a preprint by Menegazzo and Tamburini [5] from 2001 containing our Corollary 1.2, that is, the extension of the Glauberman-Thompson result to the prime $p = 3$.

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