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BUNDLES WITH PERIODIC MAPS AND MOD p CHERN POLYNOMIAL

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ABSTRACT. Suppose that $E \to B$ is a vector bundle with a linear periodic map of period p; the map is assumed free on the outside of the 0-section. A polynomial $c_E(y)$, called a mod p Chern polynomial of E, is defined. It is analogous to the Stiefel-Whitney polynomial defined by Dold for real vector bundles with the antipodal involution. The mod p Chern polynomial can be used to measure the size of the periodic coincidence set for fibre preserving maps of the unit sphere bundle of E into another vector bundle.

1. Introduction. In [2] Albrecht Dold defined the Stiefel-Whitney polynomials for vector bundles with the antipodal involution; it is a useful tool in studying "parametrized" Borsuk-Ulam type problems (compare also [4]). Given two vector bundles $E \to B$ and $E' \to B$, with $S(E) \to B$ being the unit sphere bundle of E, and a fibre preserving equivariant map $f : S(E) \to E'$, Dold used the Stiefel-Whitney polynomials to estimate the size of the zero set $Z(f) = f^{-1}0$ of f, where 0 is the zero section of E. Since f is equivariant, the zero set of f is the set of antipodal coincidences of f, i.e., the set $A(f) = \{x \in S(E) \mid fx = f(-x)\}$.

In this paper we make similar constructions for vector bundles with linear periodic fibre preserving maps of a prime period p that are free on the outside of the 0-section. Given such a bundle $E \to B$ over a base space B (which is assumed to be a CW-complex), we define a mod p Chern polynomial, $c_E(y)$, associated to E. Using the Chern polynomial, we obtain a lower bound on the size of the periodic coincidence set $A(f)$ for fibre preserving maps $f : S(E) \to E'$ of the unit sphere bundle of E into another vector bundle in terms of the (cohomology) dimension. The periodic coincidence set $A(f)$ is the set of points $x \in S(E)$ such that f maps the entire orbit of x to a single point.

In Section 7 we prove our result for equivariant maps $E \to W$, where W is another vector bundle over B with a linear periodic fibre preserving map of a prime period, free on the outside of the 0-section. In this case, $A(f) = Z(f)$. In Section 8 we give an estimation of the size of the periodic coincidence set for maps that are not necessarily equivariant; this requires some additional constructions.

Throughout the paper we will use cohomology of the Čech type. The Čech cohomology has a continuity property, which says that if a cohomology class vanishes on

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a closed set, then it vanishes on a neighborhood of this set. Throughout the paper (with the exception of the example in Section 2 below), we will use the cohomology with coefficients in $\mathbf{Z}_p$, and the coefficient group will be suppressed in the notation: $H^*(\ ) = H^*(\ ;\mathbf{Z}_p)$.

If G is a topological group, BG is a classyfying space for G, Y is a space, and $u \in H^*(BG)$, we denote by $u|Y$ the image of u in H^*Y under the homomorphism induced by a classifying map $Y \to BG$ ("the restriction" of u to Y).

If X is a G-space, we denote by $\overline{X} = X/G$ the orbit space of the action.

2. Some examples. The problem of extending Borsuk-Ulam type theorems from a single sphere to a bundle of spheres is nontrivial even for trivial bundles as the following example shows (in this case $p = 2$).

Example. Let $f : S^1 \times S^1 \to \mathbf{R}$ be a real-valued function. Then the coincidence set $A(f) = \{(x, y) \in S^1 \times S^1 \mid f(x, y) = f(-x, y)\}$ and $H^1(A(f); \mathbf{Z}_2)$ has $H^1(S^1; \mathbf{Z}_2) \cong \mathbf{Z}_2$ as a direct summand.

This is a consequence of Theorem 1 of [4]. This fact is related to a result by J. E. Connett [1], which was a motivation for [4].

The parametrized Borsuk-Ulam theorems (in the case of $p = 2$ or in the complex case, for $G = S^1$) can be applied to estimate the "size" (in terms of the cohomological dimension or the topological dimension) of the kernel of a linear vector preserving map from one vector bundle into another; compare [3] and [2], p. 282.

3. Linear periodic maps. Suppose that V is a finite-dimensional real vector space with a scalar product, and let $\tau : V \to V$ be an orthogonal periodic map of a period $p > 2$ without fixed points other than 0. Then the dimension of V must be even, and we can use p to define a complex structure on V as follows.

We first construct an orthogonal map $\sigma : V \to V$ of period $4p$ (a fourth root of τ). Given a vector $v \in V$, the vectors v and τv are never opposite for $p > 2$. They determine a two-dimensional subspace P (a plane) in V and a unit circle in it. They also determine an orientation in P along the smaller of the two arcs of the circle from v to τv. For any other vector w on that circle, let $b(v, w)$ be the unit vector bisecting the oriented pair v, w. Let $\sigma(v) = b(v, b(v, \tau(v)))$, so that $\sigma^4 = \tau$. Now we can set $i = \sigma^p$. Then i is linear and $i^2 = -1$; so i is a complex strtucture.

The map τ becomes a unitary map and can be diagonalized. The matrix of τ is a diagonal matrix whose entries are p-th roots of unity other than 1.

If $\dim V = 2n + 2$ and S^{2n+1} is the unit sphere in V, then the orbit space $S^{2n+1}/\mathbf{Z}_p$ of the $\mathbf{Z}_p$-action generated by τ is a lens space L_p^{2n+1}.

4. Comments on the classifying space for $\mathbf{Z}_p$. From now on we assume that p is a prime number greater than 2.

We would like to look closer at the cohomology of the classifying space for $\mathbf{Z}_p$ and compare it with the cohomology of BS^1.

A model for the classifying bundle for $\mathbf{Z}_p$ is the p-fold covering $\alpha : S^\infty \to B\mathbf{Z}_p = L_p^\infty$, where S^∞ is the unit sphere in the infinite-dimensional complex vector space $\mathbf{C}^\infty$, α is the orbit map of the standard action, multiplication by $\exp(\frac{2\pi\sqrt{-1}}{p})$, and the orbit space, L_p^∞, is an infinite-dimensional lens space.

A model for the classifying bundle for S^1 is the Hopf bundle $\beta : S^\infty \to P_\infty\mathbf{C}$ over the infinite-dimensional complex projective space $P^\infty\mathbf{C}$. We have a commutative diagram

$$\begin{array}{ccc} S^\infty & \xrightarrow{\alpha} & L_p^\infty \\ & \beta \searrow & \downarrow \pi \\ & & P_\infty \mathbf{C} \end{array}$$

It is well known that $H^*(L_p^\infty) \cong \Lambda[b] \otimes \mathbf{Z}_p[c]$, where $b \in H^1(L_p^\infty)$ and $c \in H^2(L_p^\infty)$ is the mod p reduction of the first Chern class. By comparing the Gysin sequences of the S^1-bundles α and β we see that the map

$$H^*(P_\infty \mathbf{C}) \cong \mathbf{Z}_p[c] \xrightarrow{\pi^*} \Lambda[b] \otimes \mathbf{Z}_p[c] \cong H^*(L_p^\infty)$$

is injective in even dimensions.

We will refer to b and c as *the universal characteristic classes of* $\mathbf{Z}_p$.

5. $\mathbf{Z}_p$-bundles. Suppose that $E \to B$ is a real vector bundle over B, with a Riemannian metric, of a real fibre dimension $2n+2$ and with a fibre preserving orthogonal action of $\mathbf{Z}_p$ that is free outside the zero section. By comments of Section 3, $E \to B$ can be given a complex structure so that the action of $\mathbf{Z}_p$ is unitary (compare [6], p. 149).

Let $S(E) \to B$ be the unit sphere bundle of E; its fibre is S^{2n+1}. Let $P(E) \to B$ be the (complex) projective bundle of $S(E) \to B$ and let $L(E) \to B$ be the lens space bundle, arising from the standard $\mathbf{Z}_p$-action, associated to E. The fibre of $P(E)$ is the complex projective space $P_n\mathbf{C}$, and the fibre of $L(E)$ is the lens space L_p^{2n+1}. $L(E)$ admits a free fibre preserving action of S^1.

The $H^*(B)$-modules $H^*(P(E))$ and $H^*(L(E))$ admit Leray-Hirsch bases. A basis for $H^*(P(E))$ is $\{1, u, \ldots, u^n\}$, where $u = c|P(E)$, the image of the first universal Chern class $c = c_1 \in H^2(P_\infty \mathbf{C})$ under the homomorphism induced by a classifying map $P(E) \to P_\infty \mathbf{C}$. A Leray-Hirsch basis for $H^*(L(E))$ is

$$\{1, s, t, st, t^2, \ldots, st^n\},$$

where $s = b|L(E)$ and $t = c|L(E)$ are the images of the universal classes b and c of $\mathbf{Z}_p$ under the homomorphism induced by a classifying map $L(E) \to L_p^\infty$.

Proposition 1. *$L(E)/S^1 \cong P(E)$, and the orbit map $\rho : L(E) \to L(E)/S^1 = P(E)$ induces a monomorphism $\rho^* : H^*(P(E)) \to H^*(L(E))$ of $H^*(B)$-algebras that is zero in odd dimensions and is an isomorphism of the even-dimensional subalgebras.*

Proof. This is a consequence of Propositions 1 and the Leray-Hirsch theorem ([7], p. 365). □

In particular, t^{n+1} can be written as a linear combination

$$t^{n+1} = \sum_{i=1}^{n+1} (-1)^i c_i t^{n-i+1},$$

where $c_i \in H^*(B)$ is the mod p Chern class of the bundle $E \to B$ reduced mod p (compare [7], p. 377).

6. Mod p Chern polynomial. Consider the algebra $\Lambda[x]\otimes H^*(B)[y]$ over $H^*(B)$.

Definition. Suppose that $E \to B$ is a bundle as in in Section 5 of a (real) fibre dimension $2n+2$, with an orthogonal fibre preserving periodic map of period p. The mod p Chern polynomial of E is

$$c_E(y) = \sum_{i=0}^{n+1} (-1)^i c_i y^{n+1-i} \in \Lambda[x] \otimes H^*(B)[y].$$

The map

$$e_E : \Lambda[x] \otimes (H^* B)[y] \to H^*(L(E)),$$
$$x \mapsto b|L(E) = s, \ y \mapsto c|L(E) = t$$

will be called the evaluation map; it is a homomorphism of $H^*(B)$-algebras.

Proposition 2. $\operatorname{Ker} e_E = (c_E(y))$*, the ideal in* $\Lambda[x] \otimes (H^*(B)[y]$ *generated by the Chern polynomial* $c_E(y)$.

Proof. First, $c_E(y)|L(E) = 0$, by the definition.

Now let $p(x,y)$ be a polynomial in $\Lambda[x] \otimes H^*(B)[y]$ such that $p(x,y)|L(E) = 0$. By dividing $c_E(y)$ into $p(x,y)$ we can write $p(x,y)$ as $p(x,y) = q(x,y)\cdot c_E(y) + r(x,y)$ where $\deg(r(x,y)) < 2n+2$. Since $c_E(t) = 0$ and the set $\{1, s, t, st, \dots, st^n\}$ is linearly independent, $r(x,y) = 0$. □

7. The zero set for equivariant maps. Suppose that $E \to B$ is a real vector bundle over B of a real fibre dimension $2n+2$ with a fibre preserving orthogonal action of $\mathbf{Z}_p$, free outside the zero section, as in Section 5. Let $S(E)$ be the unit sphere bundle of E, let $W \to B$ be another real vector bundle over B, also with a fibrewise action of $\mathbf{Z}_p$, free outside the zero section (so that the fibre dimension of W must be even). Let $g : S(E) \to W$ be an equivariant fibre preserving map.

The zero set $Z(g) = \{x \in S(E) \mid gx = 0\}$ is an equivariant subset of $S(E)$.

Proposition 3. *If a polynomial* $p(x,y) \in \Lambda[x] \otimes H^*(B)[y]$ *vanishes on* $\overline{Z(g)}$*, then there exists a polynomial* $q(x,y) \in \Lambda[x] \otimes H^*(B)[y]$ *such that*

$$p(x,y) \cdot c_W(y) = q(x,y) \cdot c_E(y).$$

In terms of ideals, if $e_{Z(g)} : \Lambda[x] \otimes H^*(B)[y] \to H^*(\overline{Z(g)})$ *is the evaluation map for* $Z(g)$*, then*

$$\operatorname{Ker}(e_{Z(g)}) \cdot (c_W(y)) \subset (c_E(y)).$$

Proof. Suppose that $p(x,y)$ vanishes on $\overline{Z(g)}$. Then, by the continuity of Čech cohomology, there is an invariant neighborhood N of $Z(g)$ in $S(E)$ such that $p(x,y)|\overline{N} = 0$. By the exactness of the cohomology sequence of the pair $(L(E), \overline{N})$, there is a $u \in H^*(L(E), \overline{N})$ such that $p(x,y)|L(E) = i^*u$, where $i : L(E) \to (L(E), \overline{N})$ is the inclusion.

On the other hand, the map $g_0 : S(E) - Z(g) \to W - 0$ induced by g is an equivariant map and an $H^*(B)$-algebra homomorphism. If $\overline{g_0}$ is the map of the orbit spaces induced by g_0, then $c_W(y)|(L(E) - \overline{Z(g)}) = \overline{g}_0^*(c_W(y)|(\overline{W-0}))$, since $\overline{g}_0^*$ is a homomorphism of $H^*(B)$-algebras. But $W - 0$ is fibrewise equivariantly homotopically equivalent to the sphere bundle $S(W)$ of W; so $\overline{W-0} \simeq L(W)$, and hence $c_W(y)|(\overline{W-0}) = 0$. By exactness of the cohomology sequence of $(L(E), L(E) - \overline{Z(g)})$, there is a $v \in H^*(L(E), L(E) - \overline{Z(g)})$ such that $i^*v = c_W(y)|L(E)$. But then $u \cdot v \in H^*(L(E), \overline{N} \cup (L(E) - \overline{Z(g)}))) = H^*(L(E), L(E)) = 0$.

So $(p(x,y)\cdot c_W(y))|L(E)=0$. By Proposition 5, $p(x,y)\cdot c_W(y)$ must be a multiple of $c_E(y)$. □

8. Estimating the size of the coincidence set. Suppose that $E\to B$ is a real vector bundle over B of a real fibre dimension $2n+2$ with a fibre preserving orthogonal action of $\mathbf{Z}_p$, free outside the zero section, as in Section 5. Suppose that $\tau: E\to E$ is a generator of the action. Let $S(E)$ be the unit sphere bundle of E, let $F\to B$ be another real vector bundle over B of fibre dimension k and let $f: S(E)\to F$ be a fibre preserving map. We do not assume that F has a $\mathbf{Z}_p$-action, or, even if it has one, that f is equivariant.

Definition. The periodic coincidence set $A(f)$ is the set of points $x\in S(E)$ such that f maps the entire $\mathbf{Z}_p$-orbit of x to a single point.

$A(f)$ is an equivariant subset of $S(E)$.

To estimate the size of $A(f)$ we will first "symmetrize" f to have an equivariant fibre preserving map from E to another $\mathbf{Z}_p$–bundle.

Let $V=p\cdot F=F\oplus\cdots\oplus F$ (p times). Then $\mathbf{Z}_p$ acts on V by cyclically permuting the coordinates, and the fixed point set of the action is the diagonal D in V; it is a subbundle of V. Let $D^\perp$ be the orthogonal complement of D in V. Then $D^\perp$ is an equivariant subbundle of D and the action on $D^\perp$ is orthogonal and free on the outside of the zero section. The linear projection "along the diagonal" defines an equivariant fibre preserving map $r:(V,V-D)\to(D^\perp,D^\perp-0)$, where 0 is the zero section of $D^\perp$; it is a fibrewise equivariant homotopy equivalence.

The map f induces an equivariant fibre preserving map $h: S(E)\to V$ by $z\mapsto(fz,f(\tau z),\ldots,f(\tau^{p-1}z))$, with $h^{-1}D=A(f)$. The map h followed by the projection r induces an equivariant fibre preserving map $(g,g_0):(S(E),S(E)-A(f))\to(D^\perp,D^\perp-0)$. Moreover, $Z(g)=A(f)$.

The fibre dimension $D^\perp$ is $k(p-1)$, which is even. By the comments in Section 3, $D^\perp$ admits a complex structure of a complex dimension $\frac{k(p-1)}{2}$ and hence a mod p Chern polynomial, $c_{D^\perp}(y)$. Applying Proposition 4 to $W=c_{D^\perp}$ we obtain the following result.

Proposition 4. *If a polynomial $p(x,y)\in\Lambda[x]\otimes H^*(B)[y]$ vanishes on $\overline{A(f)}$, then there exists a polynomial $q(x,y)\in\Lambda[x]\otimes H^*(B)[y]$ such that*

$$p(x,y)\cdot c_{D^\perp}(y)=q(x,y)\cdot c_E(y).$$

Again, in terms of ideals,

$$\mathrm{Ker}(e_{A(f)})\cdot(c_{D^\perp}(y))\subset(c_E(y)).$$

The degree of $c_E(y)$ is $2n+2$ and the degree of $c_{D^\perp}(y)$ is $k(p-1)$. Thus the evaluation map is nonzero for all polynomials whose degree is less than or equal to $2n+1-k(p-1)$. This yields the following theorem.

Theorem.

$$\sum_{i=0}^{n-\frac{k(p-1)}{2}}(H^*(B))y^i\oplus\sum_{i=0}^{n-\frac{k(p-1)}{2}}(H^*(B))xy^i\ \xrightarrow{e_{A(f)}}\ H^*(\overline{A(f)}),$$

$$x\mapsto s|\overline{A(f)},\quad y\mapsto t|\overline{A(f)}$$

is a monomorphism.

In other words, for all polynomials $p(x, y)$ *whose total degree in* x *and* y *is smaller than* $2n + 2 - k(p-1)$*,* $p(x, y)|\overline{A(f)} \neq 0$.

In terms of the cohomology dimension, we have

Corollary. $\text{cohom.dim}\overline{A(f)} \geq \text{cohom.dim}B + 2n + 1 - k(p-1)$.

Comment. The inequality in the Corollary is sharp, because it is sharp in the classical case of the Borsuk-Ulam theorem, for $B = point$.

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