SOME NUMERICAL INVARIANTS OF LOCAL RINGS

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Abstract. Let $R$ be a formal power series ring over a field of characteristic zero and $I \subseteq R$ any ideal. The aim of this work is to introduce some numerical invariants of the local rings $R/I$ by using the theory of algebraic $D$-modules. More precisely, we will prove that the multiplicities of the characteristic cycle of the local cohomology modules $H^i_I(R)$ and $H^p_p(H^i_I(R))$, where $p \subseteq R$ is any prime ideal that contains $I$, are invariants of $R/I$.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $n$ containing the field $k$, and let $A$ be a local ring that admits a surjective ring homomorphism $\pi : R \to A$. Set $I = \text{Ker}(\pi)$. G. Lyubeznik [9] defines a new set of numerical invariants of $A$ by means of the Bass numbers $\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H^p_I(R)) := \dim_k \text{Ext}^p_R(k, H^i_I(R))$. This invariant depends only on $A$, $i$, and $p$, but neither on $R$ nor on $\pi$. Completion does not change $\lambda_{p,i}(A)$; so one can assume $R = k[[x_1, \ldots, x_n]]$, with $x_1, \ldots, x_n$ independent variables.

Lyubeznik numbers can be described as the multiplicities of the characteristic cycle of the local cohomology modules $H^i_I(R)$. The aim of this work is to prove that the multiplicities of the characteristic cycle of the local cohomology modules $H^p_p(H^i_I(R))$, where $p \subseteq R$ is any prime ideal that contains $I$, are also invariants of $R/I$. Among these invariants we may find the Bass numbers $\mu_p(p, H^i_I(R)) := \dim_k(p) \text{Ext}^p_{R_p}(k(p), H^i_I(R_p))$.

2. The characteristic cycle

In the sequel, $D$ will denote the ring of differential operators corresponding to the formal power series ring $R = k[[x_1, \ldots, x_n]]$, where $k$ is a field of characteristic zero and $x_1, \ldots, x_n$ are independent variables. For details we refer to [4] and [5].

The ring $D$ has a natural increasing filtration given by the order such that the corresponding associated graded ring $gr(D)$ is isomorphic to the polynomial ring $R[\xi_1, \ldots, \xi_n]$.

Let $M$ be a finitely generated $D$-module equipped with a good filtration, i.e., an increasing sequence of finitely generated $R$-submodules such that the associated graded module $gr(M)$ is a finitely generated $gr(D)$-module. The characteristic ideal of $M$ is the ideal in $gr(D) = R[\xi_1, \ldots, \xi_n]$ given by $J(M) :=$
rad(Ann_{gr(D)}(gr(M))). One may prove that J(M) is independent of the good filtration on M. The characteristic variety of M is the closed algebraic set given by

\[ C(M) := V(J(M)) \subseteq \text{Spec}(gr(D)) = \text{Spec}(R[\xi_1, \ldots, \xi_n]). \]

The characteristic variety allows us to describe the support of a finitely generated D-module as an R-module. Let \( \pi : \text{Spec}(R[\xi_1, \ldots, \xi_n]) \rightarrow \text{Spec}(R) \) be the map defined by \( \pi(x, \xi) = x \). Then \( \text{Supp}_R(M) = \pi(C(M)) \).

The characteristic cycle of M is defined as

\[ CC(M) = \sum m_i V_i \]

where the sum is taken over all the irreducible components \( V_i = V(q_i) \) of the characteristic variety \( C(M) \), where \( q_i \in \text{Spec}(gr(D)) \) and \( m_i \) is the multiplicity of the module \( gr(M)_{q_i} \). Notice that the contraction of \( q_i \) to \( R \) is a prime ideal; so the variety \( \pi(V_i) \) is irreducible.

2.1. Bass numbers and characteristic cycle. Let \( p \in \text{Spec}(R) \) be a prime ideal. The Bass numbers \( \mu_p(I, H^n_I(R)) \) of the local cohomology modules \( H^n_I(R) \), where \( I \subseteq R \) is any ideal, can be described as the multiplicities of the characteristic cycle of \( H^n_p(H^n_I(R)) \). Namely, we have:

**Proposition 2.1.** Let \( I \subseteq R \) be an ideal, \( p \subseteq R \) a prime ideal, and let

\[ CC(H^n_p(H^n_I(R))) = \sum \lambda_{p, I, \alpha} V_{\alpha} \]

be the characteristic cycle of the local cohomology module \( H^n_p(H^n_I(R)) \). Then, the Bass numbers with respect to \( p \) of \( H^n_I(R) \) are

\[ \mu_p(I, H^n_I(R)) = \lambda_{p, I, \alpha}, \]

where \( \pi(V_{\alpha}) \) is the subvariety of \( X = \text{Spec}(R) \) defined by \( p \).

**Proof.** Let \( \widehat{R}_p \) be the completion with respect to the maximal ideal \( pR_p \) of the localization \( R_p \). Notice that \( \widehat{R}_p \) is a formal power series ring of dimension ht \( p \). Since Bass numbers are invariant by completion we have

\[ \mu_p(I, H^n_I(R)) = \mu_0(p, H^n_{\widehat{R}_p}(H^n_{\widehat{R}_p}(\widehat{R}_p))), \]

where the last assertion follows from [3] Lemma 1.4. By using [3] Theorem 3.4], we have

\[ H^n_{\widehat{R}_p}(H^n_{\widehat{R}_p}(\widehat{R}_p)) = E^{\mu_0}_p(H^n_{\widehat{R}_p}(\widehat{R}_p)). \]

So, its characteristic cycle is

\[ CC(H^n_{\widehat{R}_p}(H^n_{\widehat{R}_p}(\widehat{R}_p))) = \mu_0(p, H^n_{\widehat{R}_p}(\widehat{R}_p)) \quad V_{\alpha}, \]

where \( \pi(V_{\alpha}) \) is the subvariety of \( X' = \text{Spec} \widehat{R}_p \) defined by the ideal \( p \widehat{R}_p \). Notice that we have used the following fact (see [3] and [1] for details):

\[ CC(H^n_{\widehat{R}_p}(\widehat{R}_p)) = CC(E^{\mu_0}_p(\widehat{R}_p)) = V_{\alpha}. \]

Finally, by using the flatness of the morphism \( R \rightarrow \widehat{R}_p \), this characteristic cycle can be obtained from the characteristic cycle of \( H^n_p(H^n_I(R)) \). Namely, if

\[ CC(H^n_p(H^n_I(R))) = \sum \lambda_{p, I, \alpha} V_{\alpha} \]
2.2. Direct image. Some geometrical operations as the direct image have a key role in the theory of \( D \)-modules. Our aim in this section is to give a brief survey of this operation in the particular case of the injection of \( \mathbb{A}^n_k \) in \( \mathbb{A}^{n+1}_k \). The main references we will use in this section are [5] and [10].

Let \( D_{n+1} \) and \( D_n \) be the rings of differential operators corresponding to \( R' = k[[x_1, \ldots, x_{n+1}]] \) and \( R = k[[x_1, \ldots, x_n]] \), respectively. Let \( M \) be a \( D_n \)-module. The direct image corresponding to the injection is the \( D_{n+1} \)-module \( i_+(M) \) defined as

\[
i_+(M) = k[\partial_1] \hat{\otimes}_k M = M[\partial_1].
\]

The characteristic variety of \( i_+(M) \) can be computed from the characteristic variety of \( M \). Namely, we have

\[
C(i_+(M)) = \{(x, 0, \xi, \tau) \mid (x, \xi) \in C(M) \} \subseteq \text{Spec } (R'[[x_1, \ldots, x_n, \tau]]),
\]

where we have considered \( C(M) \subseteq \text{Spec } (R[[x_1, \ldots, x_n]]) \).

The following result is stated as is used in our work.

**Lemma 2.2.** Let \( p \subseteq R \) be a prime ideal that contains an ideal \( I \subseteq R \). The direct image of the local cohomology module \( H^p_I(R) \) is

\[
i_+(H^p_I(R)) = H^1_{(I)}(H^p_{I,R'}(R')).
\]

**Proof.** Let \( D_1 \) be the ring of differential operators corresponding to the formal power series ring \( k[[t]] \). For simplicity we will denote the local cohomology modules \( H^p_I(R) \) and \( H^p_{I,R'}(R') \) by \( N \) and \( N' \), respectively. Then we have

\[
H^1_{(I)}(N') = H^1_{(I)}(N \hat{\otimes}_k k[[t]]) = H^1_{(I)}(k[[t]]) \hat{\otimes}_k N = (D_1/D_1 \cdot (t)) \hat{\otimes}_k N = i_+(N).
\]

**Remark 2.3.** In general, let \( I_1, \ldots, I_s \) be a set of ideals of \( R \). Then, the direct image of the local cohomology module \( H^i_{I_1} \cdots (H^i_{I_s}(R)) \cdots ) \) is

\[
i_+(H^i_{I_1} \cdots (H^i_{I_s}(R)) \cdots ) = H^1_{(I)}(H^i_{I_1,R'}(\cdots (H^i_{I_s,R'}(R')) \cdots )).
\]

3. MULTIPLECTIES OF THE CHARACTERISTIC CYCLE

Let \( A \) be a ring that admits a presentation \( A \cong R/I \) for a given ideal \( I \subseteq R = k[[x_1, \ldots, x_n]] \). Recall that we have \( \text{Spec } (A) = \{ p \in \text{Spec } (R) \mid I \subseteq p \} \). Throughout this section, a prime ideal of \( A \) will also mean the corresponding prime ideal of \( R \) that contains \( I \).

Let \( R/I \) and \( R'/I' \) be two different presentations of the local ring \( A \). Then, for any prime ideal of \( A \), we will denote \( p' \in \text{Spec } (R') \) to be the prime ideal that corresponds to \( p \in \text{Spec } (R) \) by the isomorphism \( \text{Spec } (R/I) \cong \text{Spec } (R'/I') \).

**Theorem 3.1.** Let \( A \) be a local ring that admits a surjective ring homomorphism \( \pi : R \longrightarrow A \), where \( R = k[[x_1, \ldots, x_n]] \) is the formal power series ring. Set \( I = \ker \pi \), let \( p \subseteq A \) be a prime ideal and let

\[
CC(H^p_I(R)) = \sum \lambda_{p, p', i, \alpha} V_\alpha
\]
be the characteristic cycle of the local cohomology modules $H^p_\mathfrak{p}(H^n_{I^i}(R))$. Then the multiplicities $\lambda_{p,i,\alpha}$ depend only on $A$, $\mathfrak{p}$, $p$, $i$ and $\alpha$ but not on $R$ or $\pi$.

The proof of the theorem is inspired in the proof of [9] Theorem 4.1, but here we must be careful with the behavior of the characteristic cycle. So instead of [9] Lemma 4.3 we will use the following:

**Lemma 3.2.** Let $g : R' \rightarrow R$ be a surjective ring homomorphism, where $R'$ is a formal power series ring of dimension $n'$. Set $I' = \ker \pi g$ and let

$$CC(H^p_\mathfrak{p}(H^n_{I^i}(R))) = \sum \lambda_{p,i,\alpha} V_\alpha$$

be the characteristic cycle of the local cohomology modules $H^p_\mathfrak{p}(H^n_{I^i}(R))$. Then, the characteristic cycle of $H^p_\mathfrak{p}(H^n_{I'^i}(R'))$ is

$$CC(H^p_\mathfrak{p}(H^n_{I'^i}(R'))) = \sum \lambda_{p,i,\alpha} V_\alpha'$$

where $\pi(V_\alpha')$ is the subvariety of $X' = \text{Spec } R'$ defined by the defining ideal of $\pi(V_\alpha)$ contracted to $R'$.

**Proof.** $R$ is regular, and so $\ker g$ is generated by $n' - n$ elements that form part of a minimal system of generators of the maximal ideal $\mathfrak{m}' \subseteq R'$. By induction on $n' - n$ we are reduced to the case $n' - n = 1$. So $\ker g$ is generated by one element $f \in \mathfrak{m}' \setminus \mathfrak{m}'^2$. By Cohen’s structure theorem, $R' = k[[x_1, \ldots, x_n, t]]$ where we assume $f = t$ by a change of variables. We identify $R$ with the subring $k[[x_1, \ldots, x_n]]$ of $R'$. In particular, we have to consider $I' = IR' + (t)$ and $\mathfrak{p}' = \mathfrak{p}R' + (t)$.

By using Lemma 2.2 and the degeneration of Grothendieck’s spectral sequence $E_2^{p,q} = H^p(I)(H^q(M)) \implies H^{p+q}(I+)(M)$ we have

$$i_+(H^p_\mathfrak{p}(H^n_{I^i}(R))) = H^p_{I'}(H^n_{IR'/(R')}) = H^p_{IR'}(H^n_{I^i}(R'))$$

$$= H^p_{IR'}(H^n_{I^i+1}(R')) = H^p_{IR'}(H^n_{I'^i}(R'))$$

$$= H^p_{IR'+(t)}(H^n_{I'^i}(R')) = H^p_{IR'+(t)}(R'),$$

where the second to last assertion comes from the fact that $H^n_{I'^i}(R')$ is a $(t)$-torsion module. Then we are done by the results in Section 2.2. \qed

Now we continue the proof of Theorem 3.1.

**Proof.** Let $\pi' : R' \rightarrow A$ and $\pi'' : R'' \rightarrow A$ be surjections with $R' = k[[y_1, \ldots, y_{n'}]]$ and $R'' = k[[z_1, \ldots, z_{n''}]]$. Let $I' = \ker \pi'$ and let $I'' = \ker \pi''$. Let $R''' = R'' \otimes_k R''$ be the external tensor product, $\pi''' = \pi' \otimes_k \pi'' : R'' \otimes_k R'' \rightarrow A$ and $I''' = \ker \pi'''$.

By Lemma 3.2 if the characteristic cycle of $H^p_{\mathfrak{p}'}(H^n_{I'^i}(R'))$ is

$$CC(H^p_{\mathfrak{p}'}(H^n_{I'^i}(R'))) = \sum \lambda'_{p,i,\alpha} V_\alpha'$$

then the characteristic cycle of $H^p_{\mathfrak{p}''}(H^n_{I''+n'''}(R''))$ is

$$CC(H^p_{\mathfrak{p}''}(H^n_{I''+n'''}(R''))) = \sum \lambda''_{p,i,\alpha} V_\alpha''$$

where $\pi(V_\alpha'')$ is the subvariety of $X'' = \text{Spec } R'''$ defined by the defining ideal of $\pi(V_\alpha'')$ contracted to $R'''$.

By Lemma 3.2 if the characteristic cycle of $H^p_{\mathfrak{p}'''}(H^n_{I''+n'''}(R''))$ is

$$CC(H^p_{\mathfrak{p}'''}(H^n_{I''+n'''}(R''))) = \sum \lambda'''_{p,i,\alpha} V_\alpha'''$$

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then the characteristic cycle of $H^n_{\nu}(H^{n' + n'' - i}(R'))$ is

$$CC(H^n_{\nu}(H^{n' + n'' - i}(R'))) = \sum \lambda_{p, p, \alpha} V_{\alpha},$$

where $\pi(V_{\alpha})$ is the subvariety of $X'' = \text{Spec } R''$ defined by the defining ideal of $\pi(V_{\alpha})$ contracted to $R''$. In particular, we have $\lambda_{p, p, \alpha} = \lambda_{p, p, \alpha}$ for all $p$, $i$ and $\alpha$.

Remark 3.3. By the same arguments one may prove that the multiplicities of the characteristic cycle of the corresponding local cohomology modules can be computed by means of $V$.

Corollary 3.4. Let $A$ be a ring that admits a surjective ring homomorphism $\pi : R \rightarrow A$, where $R = k[[x_1, \ldots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let $p \subseteq A$ be a prime ideal. The Bass numbers $\mu_p(H^n_{\pi}^{-i}(R))$ depend only on $p$, $i$, and $\pi$.

When $p$ is the zero ideal, we obtain the invariance with respect to $R/I$ of the multiplicities of the characteristic cycle of $H^n_{\pi}^{-i}(R)$.

Corollary 3.5. Let $A$ be a local ring that admits a surjective ring homomorphism $\pi : R \rightarrow A$, where $R = k[[x_1, \ldots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let

$$CC(H^n_{\pi}^{-i}(R)) = \sum \sum m_{i, \alpha} V_{\alpha},$$

be the characteristic cycle of the local cohomology modules $H^n_{\pi}^{-i}(R)$. Then the multiplicities $m_{i, \alpha}$ depend only on $A$, $i$, and $\alpha$ but not on $R$ or $\pi$.

Collecting these multiplicities by the dimension of the corresponding irreducible varieties we define the following invariants:

Definition 3.6. Let $I \subseteq R$ be an ideal. If $CC(H^n_{\pi}^{-i}(R)) = \sum m_{i, \alpha} V_{\alpha}$ is the characteristic cycle of the local cohomology modules $H^n_{\pi}^{-i}(R)$, then we define

$$\gamma_{p, i}(R/I) := \left\{ \sum m_{i, \alpha} \mid \dim(\pi(V_{\alpha})) = p \right\}.$$}

One may prove that these invariants have the same properties as Lyubeznik numbers (see [1, Section 4]). Namely, let $d = \dim(R/I)$. Then $\gamma_{p, i}(R/I) = 0$ if $i > d$, $\gamma_{p, i}(R/I) = 0$ if $p > i$ and $\gamma_{d, d}(R/I) \neq 0$. In particular, we can collect them in a triangular matrix that we will denote by $\Gamma(R/I)$. We point out that these invariants are finer than the Lyubeznik numbers.

Example 3.7. Let $R = k[[x_1, x_2, x_3, x_4, x_5]]$. Consider the ideals

- $I_1 = (x_1, x_2, x_5) \cap (x_3, x_4, x_5)$;
- $I_2 = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4)$.

The characteristic cycle of the corresponding local cohomology modules can be computed by means of [1, Theorem 3.8]. Collecting the multiplicities we obtain the triangular matrices

$$\Gamma(R/I_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad \Gamma(R/I_2) = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$
Computing the Lyubeznik numbers (see [1, Theorem 4.4]), we obtain the triangular matrix
\[
\Lambda(R/I_1) = \Lambda(R/I_2) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
2 & & \\
\end{pmatrix}.
\]
We have to point out that the quotient ring \(R/I_1\) is Buchsbaum, but \(R/I_2\) is not.

**Remark 3.8.** In order to compute the Lyubeznik numbers \(\Lambda_{p,i}(R/I)\) for a given ideal \(I \subseteq R\) and arbitrary \(i,p\), we have to refer to U. Walther’s algorithm [11]. When \(I\) is a squarefree monomial ideal, a description of these invariants is given in [1] and [13]. Some other particular computations may also be found in [9], [7], [8] and [12]. The multiplicities of the characteristic cycle of \(H^n_I(H^{n-1}_p(R))\), where \(I\) is a squarefree monomial ideal and \(p\) is any homogeneous prime ideal, have been computed in [2].

When \(I\) is a squarefree monomial ideal (resp. the defining ideal of an arrangement of linear varieties), the multiplicities of the characteristic cycle of \(H^n_I(R)\) have been computed in [1] (resp. [3]).

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**References**


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