

SPECTRAL RADII AND EIGENVALUES OF SUBDIVISION OPERATORS

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ABSTRACT. This paper discusses the spectra of matrix subdivision operators. We establish some formulas for spectral radii of subdivision operators on various invariant subspaces in ℓ_p . A formula for the spectral radius of a subdivision operator, in terms of the moduli of eigenvalues, is derived under a mild condition. The results are even new in the scalar case. In this case, we show that the subdivision operator has no eigenvector in ℓ_p if the corresponding subdivision scheme converges for some $p \in [1, \infty)$.

§1. INTRODUCTION

This paper establishes some formulas for the spectral radii of subdivision operators with a matrix mask. Some relations among the spectral radii of the subdivision operator, the moduli of eigenvalues of the subdivision operator, and joint spectral radii of two finite matrices are discussed. Using a formula for the spectral radii, we prove that, in some cases, there exists no eigenvector of subdivision operators in ℓ_p .

Let $\mathbb{C}^{r \times m}$ be the space of $r \times m$ matrices with entries being complex numbers. Denote by $(\ell(\mathbb{Z}))^{r \times m}$ the set of all sequences $\lambda = (\lambda(\alpha))_{\alpha \in \mathbb{Z}}$, $\lambda(\alpha) \in \mathbb{C}^{r \times m}$. Suppose that N is a positive integer and that $a \in (\ell(\mathbb{Z}))^{r \times r}$ satisfying $a(\alpha) = 0 \forall \alpha \notin \{0, 1, \dots, N\}$. The (*matrix*) *subdivision operator* S_a , associated with *mask* a , is an operator defined on the space $(\ell(\mathbb{Z}^s))^{l \times r}$ as follows:

$$S_a c(\alpha) := \sum_{\beta \in \mathbb{Z}^s} c(\beta) a(\alpha - 2\beta), \quad \alpha \in \mathbb{Z}^s.$$

The *subdivision scheme with mask* a generates control points $c^n = (c^n(\alpha))_{\alpha \in \mathbb{Z}}$ at dyadic points $\alpha/2^n$, $\alpha \in \mathbb{Z}$, recursively by

$$c^n = S_a c^{n-1}, \quad n = 1, 2, \dots, \text{ and } c^0 = c.$$

Before moving further we need some notation. Assume that $\mathbb{C}^{r \times m}$ is equipped with a norm $\|\cdot\|$. Let $(\ell_p(\mathbb{Z}))^{r \times m}$ be the normed linear space of sequences $\lambda =$

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$(\lambda(\alpha))_{\alpha \in \mathbb{Z}}$ such that $\|\lambda\|_p < \infty$, where

$$\|\lambda\|_p := \left(\sum_{\alpha \in \mathbb{Z}} \|\lambda(\alpha)\|^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

and $\|\lambda\|_\infty := \sup_{\alpha \in \mathbb{Z}} \|\lambda(\alpha)\|$. Clearly, the set $(\ell_p(\mathbb{Z}))^{r \times m}$ is independent of the choice of norm $\|\cdot\|$ on $\mathbb{C}^{r \times m}$, and any two norms on $\mathbb{C}^{r \times m}$ induce two equivalent norms on $(\ell_p(\mathbb{Z}))^{r \times m}$. In what follows, no attention will be paid to the notation of norms on $\mathbb{C}^{r \times r}$.

Let $W_p(\mathbb{R})$ denote $L_p(\mathbb{R})$ for $1 \leq p < \infty$ and the space $C_u(\mathbb{R})$ of uniformly continuous and bounded functions on \mathbb{R} for $p = \infty$. $W_p(\mathbb{R})$ is a Banach space with the usual norm $\|\cdot\|_p$.

We are ready to define the convergence of a matrix subdivision scheme. Following [10] and [11], we say that *the subdivision scheme, associated with mask a , converges in L_p* if, for any $c \in (\ell_p(\mathbb{Z}))^{l \times r}$, there is a matrix-valued function $f_c \in (W_p(\mathbb{R}))^{l \times r}$ such that

$$(1.1) \quad \lim_{n \rightarrow \infty} 2^{-n/p} \|S_a^n c - \mu_p^n(f_c)\|_p = 0,$$

and $f_c \neq 0$ for some $c \in (\ell_p(\mathbb{Z}))^{l \times r}$, where $\mu_p^n(g) \in (\ell_p(\mathbb{Z}))^{l \times r}$, for a matrix-valued function $g \in (W_p(\mathbb{R}))^{l \times r}$ is given by

$$\mu_p^n(g) = \left(2^n \int_{\alpha/2^n}^{(\alpha+1)/2^n} g(x) dx \right)_{\alpha \in \mathbb{Z}}, \quad 1 \leq p < \infty$$

and $\mu_\infty^n(g) = (g(\alpha/2^n))_{\alpha \in \mathbb{Z}}$.

The vector subdivision schemes studied by [10] and [11], etc. correspond to the case $l = 1$. We note that either the convergence of the subdivision scheme S_a^n or the spectral radius $\rho_p(S_a)$ of S_a on $(\ell_p(\mathbb{Z}))^{l \times r}$ is independent of l . Nevertheless we find it convenient sometimes to consider S_a as an operator on $(\ell_p(\mathbb{Z}))^{l \times r}$ for general l instead of $l = 1$.

Subdivision schemes with matrix mask are a natural extension of subdivision schemes with scalar mask. They play an important role in computer graphics and wavelet analysis. The convergence of subdivision schemes has been studied extensively. There is a direct connection between the spectral radius of a subdivision operator S_a on the one hand, and the convergence of a subdivision scheme S_a^n on the other hand ([1], [10], [11], etc). When p is an even integer, $\rho_p(S_a)$ is realizable as the largest eigenvalue (in modulus) of an associated operator defined on a finite-dimensional space [14]. As for the spectrum of S_a , it was proved in [15] that for any $1 \leq p \leq \infty$, except for finitely many eigenvalues of S_a , any number in $\{z \in \mathbb{C} : |z| < \rho_p(S_a)\}$ lies in the residual spectrum of S_a .

It is known that the spectral radius of a subdivision operator may be expressed in terms of the p -joint spectral radius. An equality between the spectral radius $\rho_\infty(S_a)$ and the uniform joint spectral radius introduced in [12] was established in [5]. By use of the notion of p -joint spectral radius, Jia [7] extended this result to the general case $1 \leq p \leq \infty$.

In this paper, we are particularly interested in the following formula for the spectral radius S_a established for $l = r = 1$ in [5]:

$$(1.2) \quad \rho_p(S_a) = \lim_{n \rightarrow \infty} \|S_a^n \delta\|_p,$$

where $\delta = (\delta(\alpha))_{\alpha \in \mathbb{Z}} \in \ell(\mathbb{Z})$ is given by $\delta(0) = 1$ and $\delta(\alpha) = 0 \forall \alpha \neq 0$. One of the goals of this paper is to extend equality (1.2) by replacing δ with *any* $c \in (\ell_p(\mathbb{Z}))^{l \times r}$ satisfying a mild condition. The flexibility of c is used for the study of the spectral properties of S_a . For example, we prove easily that the largest modulus of l eigenvalues is equal to $\rho_p(S_a)$ provided that the corresponding eigenvectors satisfy that mild condition (see Theorem 2.7). As one of its corollaries, Theorem 3.1 generalizes a result proved by [6] to the case $1 \leq p \leq 2$. An application of the results to the studies of regularity of refinable vectors and the convergence of vector cascade algorithms is given in [3].

It is worth noting that the spectrum of the subdivision operator S_a , as an operator on $(\ell(\mathbb{Z}))^{l \times r}$, is also related with some properties of the corresponding refinable vector. Under a weak condition, the approximation order of the refinable vector can be characterized in terms of the eigenvectors of polynomial sequences (see [8] and [2], etc).

The paper is organized as follows. The notion of p -joint spectral radius will be given in the remaining part of this section. In Section 2, some formulas for the spectral radii of S_a on various invariant subspaces are given in terms of both p -joint spectral radii and some limits. As a corollary, under a mild condition on c , the equality (1.2) is established with δ being replaced by c . The existence of eigenvalues, associated with eigenvectors in ℓ_p , of subdivision operators is investigated in Section 3. For the scalar case, i.e., $r = 1$, we show that S_a has no eigenvalue in the open disc $\{z : |z| < \rho_p(S_a)\}$ in some cases. Moreover, if the subdivision operator generates a convergent subdivision scheme, we show that S_a has no eigenvector in $\ell_p(\mathbb{Z})$.

Although we present all results only in the univariate case, our methods apply for the multivariate case except for partial results of Theorem 3.1, Corollary 3.2 and Theorem 3.7.

At the end of this section we cite the notion of p -joint spectral radii for the convenience of the readers. The uniform joint spectral radius was introduced in [12] and was employed to investigate the regularity of the refinable function in [4]. The mean joint spectral radius was introduced in [13]. For $1 < p < \infty$, the p -joint spectral radius was introduced in [7] and applied to the study of L_p convergence of cascade algorithms. We recall from [7] the definition of p -joint spectral radius.

If V is a finite-dimensional space, let $\mathcal{B}(V)$ denote the collection of linear operators on V . Suppose that $\mathcal{A} \subseteq \mathcal{B}(V)$ is a finite set. For a positive integer n we denote by \mathcal{A}^n the Cartesian power of \mathcal{A} :

$$\mathcal{A}^n = \{(A_1, \dots, A_n) : A_1, \dots, A_n \in \mathcal{A}\}.$$

Suppose that $\mathcal{B}(V)$ is equipped with a norm $\|\cdot\|$, not necessarily the operator norm. For $1 \leq p < \infty$, let

$$\|\mathcal{A}^n\|_p = \left(\sum_{(A_1, \dots, A_n) \in \mathcal{A}^n} \|A_1 \cdots A_n\|^p \right)^{1/p},$$

and for $p = \infty$, define $\|\mathcal{A}^n\|_\infty = \max\{\|A_1 \cdots A_n\| : (A_1, \dots, A_n) \in \mathcal{A}^n\}$. The p -joint spectral radius of \mathcal{A} , for $1 \leq p \leq \infty$, is defined to be

$$(1.3) \quad \rho_p(\mathcal{A}) := \lim_{n \rightarrow \infty} \|\mathcal{A}^n\|_p^{\frac{1}{n}}.$$

This limit indeed exists and does not depend on the choice of norm on $\mathcal{B}(V)$ ([7]). In particular, if we choose the norm such that $\|A\| = \|A^*\|$ for any $A \in \mathcal{B}(V)$ with

A^* being its adjoint, then $\|\mathcal{A}^n\|_p = \|\mathcal{A}^{*n}\|$, where $\mathcal{A}^* := \{A^* : A \in \mathcal{A}\}$. It follows that

$$(1.4) \quad \rho_p(\mathcal{A}) = \rho_p(\mathcal{A}^*).$$

§2. SPECTRAL RADII OF SUBDIVISION OPERATORS

In this section we establish some formulas for spectral radii of subdivision operators. In particular, we generalize the formula (2.5) considerably.

In what follows, for $\varepsilon = 0, 1$, let $A_\varepsilon \in \mathcal{B}(\mathbb{C}^{rN \times l})$ be defined as follows:

$$(2.1) \quad A_\varepsilon c(\alpha) = \sum_{\beta=0}^{N-1} a(\varepsilon + 2\alpha - \beta)c(\beta) \quad \forall \alpha = 0, 1, \dots, N-1$$

for

$$c = \begin{pmatrix} c(0) \\ c(1) \\ \vdots \\ c(N-1) \end{pmatrix} \in \mathbb{C}^{rN \times l},$$

where $c(\beta) \in \mathbb{C}^{r \times r} \quad \forall \beta = 0, 1, \dots, N-1$. Clearly, the adjoint $A_\varepsilon^* \in \mathcal{B}(\mathbb{C}^{l \times rN})$ is given by

$$(2.2) \quad A_\varepsilon^* c(\beta) = \sum_{\alpha=0}^{N-1} c(\alpha)a(\varepsilon + 2\alpha - \beta) \quad \forall \beta = 0, 1, \dots, N-1,$$

for any $c = (c(0), \dots, c(N-1)) \in \mathbb{C}^{l \times rN}$. In [3] we pay our attention to $\mathcal{A} := \{A_0, A_1\}$ for deducing the convergence of cascade algorithms in Sobolev spaces to that in L_p space, while we are concerned with \mathcal{A}^* for spectra of subdivision operators in this paper.

Let $\alpha \in \mathbb{Z}$. For any nonnegative integer n , there are uniquely $\varepsilon_j \in \{0, 1\}, 1 \leq j \leq n$, and $\gamma \in \mathbb{Z}$ such that $\alpha = 2^n \gamma + 2^{n-1} \varepsilon_n + \dots + \varepsilon_1$. For any $c \in (\ell(\mathbb{Z}))^{l \times r}$, it is known (see [7]) that for $k \in K_N := \{0, 1, \dots, N-1\}$,

$$(2.3) \quad S_a^n c(\alpha - k) = A_{\varepsilon_1}^* \cdots A_{\varepsilon_n}^* \gamma c(k) \quad \forall \alpha \in \mathbb{Z}, k \in K_N,$$

where $\gamma c \in \mathbb{C}^{l \times rN}$, for $c = (c(\alpha))_{\alpha \in \mathbb{Z}} \in (\ell(\mathbb{Z}))^{l \times r}$ and for any $\gamma \in \mathbb{Z}$, is the restriction of $c(\gamma - \cdot)$ on K_N , i.e.,

$$\gamma c = (c(\gamma), c(\gamma - 1), \dots, c(\gamma - N + 1)) \in \mathbb{C}^{l \times rN}.$$

Summing over $\alpha \in \mathbb{Z}$ and $k \in K_N$ gives

$$(2.4) \quad N \|S_a^n c\|_p^p = \sum_{\varepsilon_n, \dots, \varepsilon_1} \sum_{\gamma \in \mathbb{Z}} \|A_{\varepsilon_1}^* \cdots A_{\varepsilon_n}^* \gamma c\|_p^p.$$

For any $c \in (\ell(\mathbb{Z}))^{l \times r}$, let $U(c) \subseteq \mathbb{C}^{l \times rN}$ be the subspace spanned by all vectors

$$A_{\eta_1}^* \cdots A_{\eta_j}^* \gamma c, \quad \gamma \in \mathbb{Z}, \eta_j = 0, 1, \quad j = 0, 1, 2, \dots$$

It is the smallest invariant subspace of both A_0^* and A_1^* , and is called the invariant subspace of both A_0^* and A_1^* generated by c .

Theorem 2.1. *Suppose that $c \in (\ell_p(\mathbb{Z}))^{l \times r}$. Then*

$$(2.5) \quad \lim_{n \rightarrow \infty} \|S_a^n c\|_p^{\frac{1}{n}} = \rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\}) = \rho_p(S_a|_{L(c)}),$$

where $L(c)$ is the smallest closed invariant subspace of S_a containing c .

Proof. Note that $U(c) \subseteq \mathbb{C}^{l \times rN}$ is of finite dimension. There is a nonnegative j_0 such that $U(c)$ is spanned by the set

$$\{A_{\eta_1}^* \cdots A_{\eta_j}^* \gamma c : \gamma \in \mathbb{Z}, \eta_j = 0, 1, j = 0, 1, \dots, j_0\}.$$

Consequently,

$$(2.6) \quad \|A\| := \left(\sum_{\gamma \in \mathbb{Z}} \sum_{\eta_1, \dots, \eta_{j_0}} \|AA_{\eta_1}^* \cdots A_{\eta_{j_0}}^* \gamma c\|^p \right)^{1/p}, \quad A \in \mathcal{B}(U(c)),$$

defines a norm on the set $\mathcal{B}(U(c))$. We note that the series in the right-hand side of the last equality converges for any $c \in (\ell_p(\mathbb{Z}))^{l \times r}$. Since $\mathcal{B}(U(c))$ is of finite dimension, there is a constant $\kappa > 0$ satisfying

$$(2.7) \quad \|Au\| \leq \kappa \|A\| \|u\| \quad \forall u \in U(c), A \in \mathcal{B}(U(c)).$$

Let $\mathcal{A} = \{A_0^*|_{U(c)}, A_1^*|_{U(c)}\} \subseteq \mathcal{B}(U(c))$. Recall the definition of $\|\mathcal{A}^n\|_p$ with norm (2.6) on $\mathcal{B}(U(c))$. It follows from (2.4) that $N \|S_a^{n+j_0} c\|_p^p = \|\mathcal{A}^n\|_p^p$. Therefore, the first equality in (2.5) is true by (1.3).

Suppose $d \in L(c)$. It is easily seen that $\gamma d \in U(c) \quad \forall \gamma \in \mathbb{Z}$. The inequality (2.7) gives

$$\left(\sum_{\varepsilon_n, \dots, \varepsilon_1} \sum_{\gamma \in \mathbb{Z}} \|A_{\varepsilon_1}^* \cdots A_{\varepsilon_n}^* \gamma d\|^p \right)^{1/p} \leq \kappa \|\mathcal{A}^n\|_p \left(\sum_{\gamma \in \mathbb{Z}} \|\gamma d\|^p \right)^{1/p}.$$

Note that $\sum_{\gamma \in \mathbb{Z}} \|\gamma d\|^p = N \|d\|_p^p$. We obtain by (2.7), $\|S_a^n d\|_p \leq \kappa \|\mathcal{A}^n\|_p \|d\|_p, n = 1, 2, \dots$. This implies that $\|(S_a|_{L(c)})^n\| \leq \kappa \|\mathcal{A}^n\|_p, n = 1, 2, \dots$. Therefore, $\rho_p(S_a|_{L(c)}) \leq \rho_p(\mathcal{A})$. The reverse inequality holds by the fact that $\lim_{n \rightarrow \infty} \|S_a^n c\|_p^{1/n} \leq \rho_p(S_a|_{L(c)})$ and the first equality in (2.5). This establishes the second equality in (2.5), thereby completing the proof of the theorem. \square

Corollary 2.2. *Suppose that $S \subseteq (\ell_p(\mathbb{Z}))^{l \times r}$ is an invariant subspace of S_a . Then*

$$\rho_p(S_a|_S) = \max_{c \in S} \lim_{n \rightarrow \infty} \|S_a^n c\|_p^{1/n}.$$

Proof. Since $S = \bigcup_{c \in S} L(c)$, we get $\rho_p(S_a|_S) = \sup_{c \in S} \rho_p(S_a|_{L(c)})$. Therefore Theorem 2.1 gives

$$\rho_p(S_a|_S) = \sup_{c \in S} \rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\}) = \rho_p(\{A_0^*|_{U(S)}, A_1^*|_{U(S)}\}),$$

where $U(S) := \bigcup_{c \in S} U(c)$. Note that $U(S)$ is also a finite-dimensional subspace and, thus, may be decomposed as a finite sum $U(S) = U(c^1) + \cdots + U(c^k)$ for some $c^j \in S, 1 \leq j \leq k$, where k is a positive integer. By the definition of joint spectral radius, it follows that

$$(2.8) \quad \rho_p(\{A_0^*|_{U(S)}, A_1^*|_{U(S)}\}) = \max_{1 \leq j \leq k} \rho_p(\{A_0^*|_{U(c^j)}, A_1^*|_{U(c^j)}\}).$$

The proof is complete by the first equality in (2.5). \square

In what follows we want to give a simple condition on c that ensures that

$$\rho_p(\{A_0^*, A_1^*\}) = \rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\}).$$

To this end, we consider the joint spectral radii concerning the following subspace:

$$(2.9) \quad U(*c) := \bigcup \{U(\lambda) : \lambda = \eta * c, \eta \in (\ell_0(\mathbb{Z}))^{1 \times l}\} \subseteq \mathbb{C}^{1 \times rN}.$$

Theorem 2.3. *For any $c \in (\ell_p(\mathbb{Z}))^{l \times r}$, we have*

$$(2.10) \quad \lim_{n \rightarrow \infty} \|S_a^n c\|_p^{\frac{1}{n}} = \max_{\lambda \in U(*c)} \rho_p(\{A_0^*|_{U(\lambda)}, A_1^*|_{U(\lambda)}\}) = \rho_p(\{A_0^*|_{U(*c)}, A_1^*|_{U(*c)}\}).$$

Proof. The second equality in (2.10) follows in the same way as (2.8). We prove now the first equality in (2.10).

If $\lambda \in U(*c)$, there is an $\eta \in (\ell_0(\mathbb{Z}))^{1 \times l}$ such that $\lambda = \eta * c$. Appealing to (2.3) we have

$$S_a^n \lambda(\alpha - k) = \sum_{\gamma' \in \mathbb{Z}} \eta(\gamma') A_{\varepsilon_1}^* \cdots A_{\varepsilon_n}^* c^{\gamma - \gamma'}(k), \quad 0 \leq k \leq N - 1,$$

where $\gamma \in \mathbb{Z}, \varepsilon_1, \dots$ and ε_n are determined by α in the same way as (2.3). Therefore $\|S_a^n \lambda\|_p \leq \kappa \|S_a^n c\|_p \|\eta\|_1$ for some constant κ independent of n and λ .

On the other hand, for $j = 1, \dots, l$, with a special choice of $\eta^j \in (\ell_0(\mathbb{Z}))^{1 \times l}$ and $\lambda^j = \eta^j * c$, $S_a^n \lambda^j(\alpha)$ specifies the j th row of $S_a^n c(\alpha)$, $\alpha \in \mathbb{Z}$. Thus,

$$\|S_a^n c\|_p \leq \max_{j=1, \dots, l} \|S_a^n \lambda^j\|_p \leq \kappa' \|S_a^n c\|_p.$$

Taking the $1/n$ power and letting $n \rightarrow \infty$ in the above inequalities we obtain by Theorem 2.1,

$$\max_{j=1, \dots, l} \lim_{n \rightarrow \infty} \|S_a^n \lambda^j\|_p^{\frac{1}{n}} = \rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\}).$$

Applying Theorem 2.1 to the left-hand side of the above equality for $c = \lambda^j, j = 1, \dots, l$, we obtain the first equality in (2.10). The proof is complete. \square

For any $u \in \mathbb{C}^{l \times N}$, we denote by $u_j \in \mathbb{C}^{1 \times rN}$ the j th row of u . If U is a subset of $\mathbb{C}^{l \times rN}$, let

$$\mathcal{R}(U) := \text{span}\{u_j : u \in U, 1 \leq j \leq l\} \subseteq \mathbb{C}^{1 \times rN}.$$

It is easily seen that $\mathcal{R}(U)$ is an invariant subspace of both A_0^* and A_1^* if U is. Moreover, $\rho_p(\{A_0^*|_U, A_1^*|_U\}) = \rho_p(\{A_0^*|_{\mathcal{R}(U)}, A_1^*|_{\mathcal{R}(U)}\})$ provided that U is invariant under both A_0^* and A_1^* . In particular, for any $c \in (\ell_p(\mathbb{Z}))^{l \times r}$ we have

$$(2.11) \quad \rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\}) = \rho_p(\{A_0^*|_{\mathcal{R}(U(c))}, A_1^*|_{\mathcal{R}(U(c))}\}).$$

Although the following auxiliary lemma is undoubtedly well known, we would like to present a proof for convenience. Recall that $K_N = \{0, 1, \dots, N - 1\}$. Let $(\ell(K_N))^{r \times 1}$ denote the subspace consisting of $c \in (\ell(\mathbb{Z}))^{r \times 1}$ with $\text{supp } c \in K_N$.

Lemma 2.4. *Let $c \in (\ell_p(\mathbb{Z}^s))^{l \times r}$ satisfy*

$$(2.12) \quad c * t \neq 0 \quad \forall t \in (\ell(K_N))^{r \times 1} \setminus \{0\}.$$

Then

$$(2.13) \quad \mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\}) = \mathbb{C}^{1 \times rN}.$$

Consequently, in this case,

$$(2.14) \quad \|A\| := \left(\sum_{\gamma \in \mathbb{Z}} \|A^\gamma c\|_p^p \right)^{1/p}$$

is a norm on $\mathcal{B}(\mathbb{C}^{1 \times rN})$.

Proof. The formula (2.14) follows easily from (2.14). The proof of (2.14) is elementary. Note that $\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\}) \subseteq \mathbb{C}^{1 \times rN}$ is spanned by the vectors

$$\gamma c_j := (c_j(\gamma), c_j(\gamma - 1), \dots, c_j(\gamma - N + 1)), \quad \gamma \in \mathbb{Z}, 1 \leq j \leq r,$$

where $c_j(\alpha) \in \mathbb{C}^{1 \times r}$ is the j th row of $c(\alpha)$, $\alpha \in \mathbb{Z}$. Assume to the contrary that (2.14) is not true. There would be a nonzero column vector $t = (t(0), \dots, t(N - 1))^T \in \mathbb{C}^{rN \times 1}$, with $t(\alpha) \in \mathbb{C}^{r \times 1}$, satisfying

$$\sum_{\alpha=0}^{N-1} c_j(\gamma - \alpha)t(\alpha) = 0 \quad \forall \gamma \in \mathbb{Z}, 1 \leq j \leq r.$$

We then extend t to a vector in $(\ell(K_N))^{r \times 1} \setminus \{0\}$, denoted by t again, by setting $t(\alpha) = 0$ for any $\alpha \notin K_N$, so that (2.12) holds. This contradicts our assumptions. The proof is complete. \square

Remark 2.5. We point out that (2.12) is satisfied for $c \in (\ell_p(\mathbb{Z}))^{l \times r} \setminus \{0\}$ in the following cases:

- (1) $l = r = 1$ and $1 \leq p \leq 2$;
- (2) $l = r = 1$ and the univariate case.

The following result extends (1.2) to a much more general setting.

Theorem 2.6. *Assume that $c \in (\ell_p(\mathbb{Z}))^{l \times r}$ satisfies the condition (2.12). Then*

$$\rho_p(S_a) = \rho_p(\{A_0, A_1\}) = \lim_{n \rightarrow \infty} \|S_a^n c\|_p^{\frac{1}{n}}.$$

Proof. The proof of the first equality is the same as [7]. To establish the second equality, by (1.4) and Theorem 2.1, it suffices to prove $\rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\}) = \rho_p(\{A_0^*, A_1^*\})$. We note first that $\rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\}) \leq \rho_p(\{A_0^*, A_1^*\})$.

Since $\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\}) \subseteq \mathcal{R}(U(c))$, we have

$$\rho_p(\{A_0^*|_{\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\})}, A_1^*|_{\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\})}\}) \leq \rho_p(\{A_0^*|_{\mathcal{R}(U(c))}, A_1^*|_{\mathcal{R}(U(c))}\}).$$

By (2.11), $\rho_p(\{A_0^*|_{\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\})}, A_1^*|_{\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\})}\}) \leq \rho_p(\{A_0^*|_{U(c)}, A_1^*|_{U(c)}\})$.

On the other hand, by the assumptions we know from Lemma 2.4 that $\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\}) = \mathbb{C}^{1 \times rN}$. Moreover, it is easily seen that $\rho_p\{A_0^*, A_1^*\}$ is independent of l , i.e., $\rho_p\{A_0^*, A_1^*\} = \rho_p\{A_0^*|_{\mathbb{C}^{1 \times rN}}, A_1^*|_{\mathbb{C}^{1 \times rN}}\}$. It follows that

$$\rho_p(\{A_0^*, A_1^*\}) = \rho_p(\{A_0^*|_{\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\})}, A_1^*|_{\mathcal{R}(\{\gamma c : \gamma \in \mathbb{Z}\})}\}).$$

The proof is complete. \square

With the help of Theorem 2.6, we may calculate the spectral radius of S_a in terms of the moduli of its eigenvalues.

Theorem 2.7. *Let $c^j \in (\ell_p(\mathbb{Z}^s))^{1 \times r}$ be an eigenvector of S_a with eigenvalues $\sigma_j, j = 1, \dots, l$, respectively. Suppose that $c := (c^1, \dots, c^l)^T \in (\ell_p(\mathbb{Z}))^{l \times r}$ satisfies the condition (2.12). Then $\rho_p(S_a) = \max\{|\sigma_1|, \dots, |\sigma_l|\}$.*

Proof. Since $S_a^n c^j = \sigma_j^n c^j$ and $c^j \neq 0, 1 \leq j \leq l$, we have $\lim_{n \rightarrow \infty} \|S_a^n c\|_p^{\frac{1}{n}} = \max\{|\sigma_1|, \dots, |\sigma_l|\}$. Therefore the proof is complete by Theorem 2.6. \square

§3. EIGENVECTORS OF SUBDIVISION OPERATORS

In this section we discuss the eigenvectors of a subdivision operator S_a . We prove that S_a has no eigenvalue in the open disc $\{z : |z| < \rho_p(S_a)\}$ in two cases as listed in Remark 2.5. Moreover, we have these cases, it is shown that, if S_a generates a convergent subdivision scheme, it has no an eigenvector in ℓ_p .

Theorem 3.1. *Suppose that either $l = r = 1$ and $1 \leq p \leq 2$ or $l = r = 1$ and we have the univariate case. Then S_a has no eigenvector in $\ell_p(\mathbb{Z})$ corresponding to an eigenvalue in the disc*

$$(3.1) \quad \{z \in \mathbb{C} : |z| < \rho_p(S_a)\}.$$

Proof. It follows immediately from Remark 2.5 and Theorem 2.7 and the discussion at the end of last section. \square

Under the assumption $\widehat{a}(\omega) \neq 0 \forall \omega \in \mathbb{R}$, Theorem 3.1 was established for $p = 2$ in [6] by a different method, which, on the other hand, applies to the case when a is not finitely supported.

By Theorem 2 of [15] and Theorem 3.1 we have

Corollary 3.2. *Under the hypotheses of Theorem 3.1, any point in the disc (3.1) lies in the residual spectrum of S_a .*

In the case $r > 1$, we can construct an example in which S_a has an eigenvalue σ with $|\sigma| < \rho_\infty(S_a)$. First we choose a sequence $a_1 \in \ell_0(\mathbb{Z})$ satisfying $\rho_\infty(S_{a_1}) > 1$. Let $a_2 \in \ell_0(\mathbb{Z})$ be given by $a_2(0) = a_2(1) = 1$ and $a_2(\alpha) = 0 \forall \alpha \notin \{0, 1\}$. Clearly, $e := (\dots, 1, 1, 1, \dots)$ is an eigenvector of S_{a_2} with eigenvalue 1. Then we define $a \in (\ell_0(\mathbb{Z}))^{2 \times 2}$ by $a(\alpha) = \text{diag}(a_1(\alpha), a_2(\alpha)) \forall \alpha \in \mathbb{Z}$. It is easy to check that

$$1 = \rho_\infty(S_{a_2}) < \rho_\infty(S_{a_1}) = \rho_\infty(S_a).$$

On the other hand, $(0, e)$ is an eigenvector of S_a with eigenvalue 1, where 0 stands for the zero element of $\ell(\mathbb{Z})$.

We now restrict ourselves to the subdivision operators that generate convergent subdivision schemes.

Lemma 3.3. *If the subdivision scheme associated with a converges in L_p for some p , then the spectral radius of S_a satisfies $\rho_p(S_a) = 2^{1/p}$.*

Proof. We note first that, for any $g \in (W_p(\mathbb{R}))^{l \times r}$ and $\mu_p^n(g)$ given in Section 1, it follows that $\lim_{n \rightarrow \infty} 2^{-n/p} \|\mu_p^n(g)\|_p = \|g\|_p$. This together with the equality (1.1) yields that, for any $c \in (\ell_p(\mathbb{Z}))^{l \times r}$, $\lim_{n \rightarrow \infty} 2^{-n/p} \|S_a^n c\|_p = \|f_c\|_p$, where f_c is given by (1.1). Consequently, for any $c \in (\ell_p(\mathbb{Z}))^{l \times r}$,

$$\lim_{n \rightarrow \infty} \|S_a^n c\|_p^{1/n} = \begin{cases} 2^{1/p}, & f_c \neq 0, \\ 0, & f_c = 0. \end{cases}$$

But, the notion of convergence of the subdivision scheme implies that there is at least a $c \in (\ell_p(\mathbb{Z}))^{l \times r}$ such that $f_c \neq 0$. Therefore, $\rho_p(S_a) \geq 2^{1/p}$.

In particular, we may choose a $c \in (\ell_p(\mathbb{Z}))^{l \times r}$ satisfying (2.12). Then Theorem 2.6 gives $\lim_{n \rightarrow \infty} \|S_a^n c\|_p^{1/n} = \rho_p(S_a)$. As is known, $\lim_{n \rightarrow \infty} \|S_a^n c\|_p^{1/n} = 0$ or $2^{1/p}$. It follows that $\rho_p(S_a) \leq 2^{1/p}$. Therefore, the result of the lemma follows. \square

Lemma 3.4. *Suppose $c^1 \in (\ell_p(\mathbb{Z}))^{1 \times r}$ satisfies*

$$(3.2) \quad {}^\alpha c^1 A_\varepsilon = {}^\alpha c^1 \quad \forall \alpha \in \mathbb{Z}, \quad \varepsilon = 0, 1.$$

Then the following conclusions are true.

- (1) *For any constant $\sigma \neq 1$, c^1 is not an eigenvector of S_a with eigenvalue σ .*
- (2) *If c^1 is an eigenvector of S_a with eigenvalue σ , then $p = \infty$ and*

$$(3.3) \quad c^1(\alpha) = c^1(0) \quad \forall \alpha \in \mathbb{Z}.$$

Proof. We establish (1) first. Suppose to the contrary that c^1 is an eigenvector of S_a with eigenvalue $\sigma \neq 1$. This means that $S_a c^1 = \sigma c^1$ and $c^1 \neq 0$. But we are able to prove $c^1 = 0$ as follows.

In fact, it follows from (2.3) and (3.2) that

$$(3.4) \quad {}^\alpha c = \sigma^{(2\alpha + \varepsilon)} c \quad \forall \alpha \in \mathbb{Z}, \quad \varepsilon = 0, 1.$$

If $\sigma = 0$, it follows from the above equality that $c^1(\alpha) = \sigma c^1(2\alpha + \varepsilon) = 0, \alpha \in \mathbb{Z}$.

If $\sigma \neq 0$, setting $\alpha = \varepsilon = 0$ in (3.4) we get ${}^0 c = 0$. Recursively we may deduce that ${}^\alpha c = 0$ for any $\alpha \geq 0$. For example, ${}^1 c = 0$ is true by letting $\alpha = 0$ and $\varepsilon = 1$ in (3.4).

For $\alpha < 0$, the proof for ${}^\alpha c = 0$ is similar as above. We only note that ${}^{-1} c = 0$ by setting $\alpha = -1$ and $\varepsilon = 1$ in (3.4). We have $c^1 = 0$, as desired. This proves (1).

Suppose now that c^1 is an eigenvector of S_a with eigenvalue 1. As in the proof of (1), we may deduce that c^1 satisfies (3.3). Note $c^1 \in (\ell_p(\mathbb{Z}))^{1 \times r}$. We conclude that $p = \infty$ and, therefore, complete the proof. \square

Theorem 3.5. *Suppose that the subdivision scheme associated with a converges in L_p for some $1 \leq p \leq \infty$. If σ is an eigenvalue of S_a on $(\ell_p(\mathbb{Z}))^{1 \times r}$, then the following statements are true.*

- (1) *If $1 \leq p < \infty$, then $|\sigma| < 2^{1/p}$.*
- (2) *If $p = \infty$, then either $|\sigma| < 1$ or $\sigma = 1$. Furthermore, if $\sigma = 1$, the corresponding eigenvector c^1 satisfies (3.3).*

Proof. Suppose that $1 \leq p < \infty$. We claim that if the subdivision scheme associated with a converges in L_p then, for any $d \in (\ell_p(\mathbb{Z}))^{1 \times r}$, there is a positive constant $\tau < 1$, dependent on d , satisfying

$$(3.5) \quad 2^{-n} \sum_{\gamma \in \mathbb{Z}} \sum_{\varepsilon_n, \dots, \varepsilon_1} \|{}^\gamma d A_{\varepsilon_n} \cdots A_{\varepsilon_1} (A_\varepsilon - I)\|^p \leq \tau^n \quad \forall \varepsilon = 0, 1, \quad n = 1, 2, \dots,$$

where I is the identity matrix of the appropriate size.

Recall that, for any $d \in (\ell_p(\mathbb{Z}))^{1 \times r}$, the function f_d is given by (1.1). Since $f_d \in (L_p(\mathbb{R}))^{1 \times r}$, it is easy to check that

$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{k \in K_N} \sum_{\alpha \in \mathbb{Z}} \|\mu_p^{n+1}(f_d)(2\alpha + \varepsilon - k) - \mu_p^n(f_d)(\alpha - k)\|^p = 0, \quad \varepsilon = 0, 1.$$

It follows from (1.1) that, for $d \in (\ell_p(\mathbb{Z}))^{1 \times r}$,

$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{k \in K_N} \sum_{\alpha \in \mathbb{Z}} \|S_a^{n+1} d(2\alpha + \varepsilon - k) - S_a^n d(\alpha - k)\|^p = 0, \quad \varepsilon = 0, 1.$$

Substituting (2.3) into the above equality we obtain for $d \in (\ell_p(\mathbb{Z}))^{1 \times r}$ that

$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{\gamma \in \mathbb{Z}} \sum_{\varepsilon_n, \dots, \varepsilon_1} \|{}^\gamma d A_{\varepsilon_n} \cdots A_{\varepsilon_1} (A_\varepsilon - I)\|^p = 0, \quad \varepsilon = 0, 1.$$

Since $c^{1 \times r N}$ is of finite dimension, it is not difficult to derive (3.5) from the last relation (see, e.g., the proof of Theorem 2.6 of [9]). Therefore (3.5) is true, as claimed.

Suppose that $c^1 \in (\ell_p(\mathbb{Z}))^{1 \times r}$ is an eigenvector of S_a with eigenvalue σ . Let $\alpha = 2^n \gamma + 2^{n-1} \varepsilon_n + \dots + \varepsilon_1$. Then

$$(3.6) \quad \gamma c A_{\varepsilon_n} \dots A_{\varepsilon_1} (A_\varepsilon - I) = \sigma^n ({}^\alpha c^1 (A_\varepsilon - I)), \quad \varepsilon = 0, 1.$$

Meanwhile, appealing to Lemma 3.4 we know that (3.2) is not true. Therefore, $\sum_{\alpha \in \mathbb{Z}} \|{}^\alpha c^1 (A_\varepsilon - I)\|^p \neq 0$ for some ε .

Setting $d = c^1$ in the left-hand side of (3.5) and taking (3.6) into account we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(2^{-n} \sum_{\gamma \in \mathbb{Z}} \sum_{\varepsilon_n, \dots, \varepsilon_1} \|{}^\gamma c A_{\varepsilon_n} \dots A_{\varepsilon_1} (A_\varepsilon - I)\|^p \right)^{\frac{1}{np}} \\ &= \lim_{n \rightarrow \infty} \left(2^{-n} |\sigma|^{pn} \sum_{\alpha \in \mathbb{Z}} \|{}^\alpha c^1 (A_\varepsilon - I)\|^p \right)^{\frac{1}{np}} = 2^{-1/p} |\sigma|. \end{aligned}$$

Therefore, the inequality (3.5) implies that $2^{-1/p} |\sigma| \leq \tau < 1$. This establishes (1).

The analogue of (3.5) for $p = \infty$ is, for a constant $\tau \in (0, 1)$,

$$2^{-n} \|{}^\gamma d A_{\varepsilon_n} \dots A_{\varepsilon_1} (A_\varepsilon - I)\| \leq \tau^n \quad \forall \varepsilon_n, \dots, \varepsilon_1, \varepsilon, \gamma \in \mathbb{Z}, \quad n = 1, 2, \dots$$

If $\sigma \neq 1$, we know that (3.2) is not true. Similar to the proof of (1) we can conclude that $|\sigma| < 1$. The conclusion for $\sigma = 1$ follows from Lemma 3.4. The proof is complete. \square

Corollary 3.6. *Suppose that the subdivision scheme associated with a converges in L_p for some $p \in [1, \infty)$. Then there are not eigenvectors $c^j \in (\ell_p(\mathbb{Z}))^{1 \times r}$ of $S_a, 1 \leq j \leq l$, such that $c := (c^1, \dots, c^l)^T \in (\ell_p(\mathbb{Z}))^{l \times r}$ satisfies condition (2.12).*

Proof. Assume that $c^j \in (\ell_p(\mathbb{Z}))^{1 \times r}$ is an eigenvector of S_a with eigenvalues $\sigma_j, 1 \leq j \leq r$, respectively. Then by Theorem 3.3 we have $|\sigma_j| < 2^{1/p}$ for $1 \leq j \leq l$. Recall from Lemma 3.3 that $\rho_p(S_a) = 2^{1/p}$. This contradicts Theorem 2.7 if c satisfies (2.12). The proof is complete. \square

Let $l = r = 1$. As is known, for either (i) $1 \leq p \leq 2$ or (ii) $1 \leq p < \infty$ in the univariate case, any $c \in \ell_p(\mathbb{Z})$ satisfies (2.12). Combining Theorem 3.5 and Corollary 3.6 we have an interesting result as follows.

Theorem 3.7. *Let $l = r = 1$. Suppose that the subdivision scheme associated with a converges in L_p . Then for either (i) $1 \leq p \leq 2$ or (ii) $1 \leq p < \infty$ in the univariate case, S_a has no eigenvector in $\ell_p(\mathbb{Z})$.*

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