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COMMON BOREL DIRECTIONS OF A MEROMORPHIC FUNCTION WITH ZERO ORDER AND ITS DERIVATIVE

TIEN-YU PETER CHERN

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Abstract. There is a meromorphic function of zero order for which the function and its derivative have no common Borel direction.

1. Introduction

We ask a general question: Is it true that a meromorphic function and its derivative always have a common Borel direction? In 1928, G. Valiron [7] posed such a problem for meromorphic functions of finite positive order. A. Rauch [4] gave a sufficient condition such that a Borel direction of an entire function $f(z)$ with finite positive order is also a Borel direction of $f'(z)$. Chia-tai Chuang [2] extended Rauch's result to meromorphic functions under the condition that $f(z)$ has two distinct Borel exceptional values in an angle containing the Borel direction of $f(z)$. On the other hand, in 1951, H. Milloux [3] proved that if $f(z)$ is an entire function of finite positive order λ, then each Borel direction with order λ for $f'(z)$ is also a Borel direction with order λ for $f(z)$.

Guang-hou Zhang [9] generalized Milloux's result to meromorphic functions under the condition that ∞ is a Borel exceptional value of $f(z)$ in an angle containing the Borel direction of $f'(z)$. However, the Common Borel Direction Problem due to Valiron is still open for meromorphic functions of finite positive order.

In this paper, for meromorphic functions of zero order, we answer the Common Borel Direction Problem negatively, as follows.

Theorem 1. *There is a meromorphic function f of zero order such that f and f' have no common Borel direction.*

To prove Theorem 1, we need the following results as preliminaries.

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Theorem 2. *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ with finite log-order λ $(=\limsup_{r\to+\infty} \log T(r,f)/\log\log r)$. If $f(z)$ satisfies the growth condition*

$$\limsup_{r\to+\infty} \frac{T(r,f)}{(\log r)^2} = \infty, \tag{1.1}$$

then there exists a direction $\Delta(\theta) = \{z : \arg z = \theta\}$, $0 \le \theta < 2\pi$, such that for every small positive number ϵ $(< \frac{\pi}{2})$ and every $a \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the equation

$$\limsup_{r\to+\infty} \frac{\log n(r,\theta,\epsilon, f=a)}{\log\log r} = \lambda - 1 \tag{1.2}$$

holds with at most two possible exceptional values of a, where $n(r,\theta,\epsilon,f=a)$ denotes the number of roots, counting multiplicities, of the equation $f(z) = a$ in $\{z : |\arg z - \theta| < \epsilon,\ |z| < r\}$.

The ray $\Delta(\theta)$ in Theorem 2 is called a *Borel direction of log-order $\lambda - 1$ for $f(z)$*.

Theorem 3. *If $f(z)$ is a transcendental function meromorphic in $\mathbb{C}$ with zero order, then $f(z)$ and its derivative $f'(z)$ have the same log-order.*

2. Proofs

Proof of Theorem 1. Let

$$f(z) = \prod_{n=1}^{+\infty} \frac{z + e^{\sqrt{n}}}{z - e^{\sqrt{n}}}.$$

Then $T(r,f) \cong (\frac{1}{3} + o(1))(\log r)^3$. So, $f(z)$ has log-order 3. Given any $0 < \epsilon < \frac{\pi}{2}$, let $\Omega_1 = \{z : \frac{\pi}{2} - \epsilon < \arg z < \frac{\pi}{2} + \epsilon\}$, $\Omega_2 = \{z : \frac{3\pi}{2} - \epsilon < \arg z < \frac{3\pi}{2} + \epsilon\}$, $\Omega_3 = \{z : \frac{\pi}{2} + \epsilon \le \arg z \le \frac{3\pi}{2} - \epsilon\}$, $\Omega_4 = \mathbb{C} - \bigcup_{i=1}^{3} \Omega_i$. It is well known (see [5]) that $f(z) \to 0$ uniformly for $z \in \Omega_3$; $f(z) \to \infty$ uniformly for $z \in \Omega_4$. In $\Omega_1 \cup \Omega_2$, $f(z)$ omits zero and ∞ and $f'(z)$ is bounded. Since $T(r,f)$ has log-order 3, by Theorem 2, $f(z)$ has a Borel direction $\Delta(\theta)$ of log-order 2, $\Delta(\theta) \subset \Omega_1 \cup \Omega_2$. On the other hand, by Theorem 3, $f'(z)$ has the same log-order as $f(z)$, and by Theorem 2, $f'(z)$ has a Borel direction $\Delta(\theta')$ of log-order 2, for f' is bounded in $\Omega_1 \cup \Omega_2$ and $f \to 0$ uniformly in Ω_3, $\Delta(\theta') \subset \Omega_4$. This complete the proof of Theorem 1. ☐

To prove Theorem 2, we employ the following result.

Lemma 1 (A Fundamental Theorem of Valiron-Milloux) (see [8, p. 69, Theorem 3.4])**.** *Let $f(z)$ be meromorphic in $|z| < R$, and let*

$$N = n(R, f = \alpha) + n(R, f = \beta) + n(R, f = \gamma), \tag{2.1}$$

where α, β and γ are three distinct complex numbers with their spherical distances larger than a positive number d. Then, there exists a point z_0, with $|z_0| < R$, such that for every $r \in (0, R)$, and every complex number a,

$$n(r, f = a) < \frac{CR^2}{(R-r)^2} \left\{ (N+1) \log \frac{2R}{R-r} + \log^+(1/d) + \log \frac{1}{|f(z_0), a|} \right\}, \tag{2.2}$$

where $|f(z_0), a|$ denotes the spherical distance between $f(z_0)$ and a, and C is a positive constant.

The method that we use to prove Theorem 2 is due to H. H. Chen [1].

Proof of Theorem 2. It suffices to prove that the left-hand side of the expression (1.2) $\geq \lambda - 1$, since f is of finite log-order λ, and it is well known that for each $a \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $n(r, \theta, \epsilon, f = a)$ has log-order $\leq \lambda - 1$.

Since $\limsup_{r\to+\infty} \log T(r, f)/\log\log r = \lambda$, by (1.1), there exists a direction $\Delta(\theta)$ with $0 \leq \theta < 2\pi$ such that for every positive number ϵ, $0 < \epsilon < \pi/2$, we have

$$\limsup_{r\to+\infty} \frac{\log T(r,\theta,\epsilon)}{\log\log r} = \lambda \qquad (\text{if } \lambda > 2); \tag{2.3}$$

$$\limsup_{r\to+\infty} \frac{T(r,\theta,\epsilon)}{(\log r)^2} = +\infty \qquad (\text{if } \lambda = 2) \tag{2.3$'$}$$

where $T(r, \theta, \epsilon)$ denotes the Ahlfors-Shimizu characteristic of $f(z)$ over the angular region $\{z : |\arg z - \theta| < \epsilon\}$ (for the definition of $T(r, \theta, \epsilon)$, see [6, p. 272]).

The direction $\Delta(\theta)$ is a Borel direction of log-order $\lambda - 1$ for $f(z)$; for otherwise, then there are three distinct values a_i $(i = 1, 2, 3)$ in $\widehat{\mathbb{C}}$ and two positive numbers ϵ_1 and δ such that

$$\sum_{i=1}^{3} n(r, \theta, \epsilon_1, f = a_i) < (\log r)^{\lambda-1-\delta} \tag{2.4}$$

for sufficiently large r. Without loss of generality, let $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Set

$$\eta = \tfrac{1}{8}\sin\epsilon_1 \tag{2.5}$$

and

$$z_j = (1+\eta)^{j-1}(1+\frac{\eta}{2})e^{i\theta_0}, \tag{2.6}$$

for each $j = 1, 2, \ldots$, applying Lemma 1 on the function $f(z_j + (8\eta|z_j|)z)$, where $|z| < 1$. For each $a \in \widehat{\mathbb{C}}$, if we denote $n(r; w; f = a)$ by the number of roots of the equation $f(z) = a$ in the disk $|z - w| < r$, then we have

$$n(\eta|z_j|; z_j; f = a) < C\left\{1 + \sum_{i=1}^{3} n(8\eta|z_j|; z_j; f = a_i) + \log\frac{1}{|f(z_j'), a|}\right\}, \tag{2.7}$$

where z_j' is a point in $|z - z_j| < 8\eta|z_j|$, and C is a positive number. Since

$$\int\int_{\widehat{\mathbb{C}}} \log\frac{1}{|f(z_j'), a|} d\sigma_a = \text{constant}, \tag{2.8}$$

so we have

$$\begin{aligned} \frac{1}{\pi}\int\int_{|z-z_j|\leq\eta|z_j|} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} dxdy &= \frac{1}{\pi}\int\int_{\widehat{\mathbb{C}}} n(\eta|z_j|; z_j; f = a) d\sigma_a \\ &\leq C\left\{1 + \sum_{i=1}^{3} n(8\eta|z_j|; z_j; f = a_i)\right\} \end{aligned} \tag{2.9}$$

where $d\sigma_a$ is a spherical area element in terms of a.

If r is sufficiently large with $(1+\eta)^J \leq r \leq (1+\eta)^{J+1}$, applying (2.9) and noticing that the region $\{z : |\arg z - \theta| \leq \eta/2,\ (1+\eta)^j \leq r \leq (1+\eta)^{j+1}\}$ is contained in the disk $\{z : |z - z_{j+1}| \leq \eta |z_{j+1}|\}$, then we deduce

$$\begin{aligned} S(r,\theta,\frac{\eta}{2}) &= \int\int_{\substack{|\arg z-\theta|\leq\frac{\eta}{2}\\ |z|\leq r}} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}dxdy \\ &\leq S_0 + \sum_{j=j_0}^{J+1} \frac{1}{\pi}\int\int_{|z-z_0|\leq\eta|z_j|} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}dxdy \\ &\leq S_0 + C\sum_{j=j_0}^{J+1}\left\{1+\sum_{i=1}^{3} n(8\eta|z_j|;z_j;f=a_i)\right\}, \end{aligned} \tag{2.10}$$

where S_0 is a positive constant.

For any $\alpha \in \widehat{\mathbb{C}}$, the number of the disks $|z - z_j| \leq 8\eta|z_j|$ that contain α is bounded by an absolute constant. Hence one can deduce from (2.9) that

$$S(r,\theta,\frac{\eta}{2}) \leq S_0 + C\left\{J+1+\sum_{i=1}^{3} n((1+8\eta)|z_j|;z_j;f=a_i)\right\}. \tag{2.11}$$

By (2.11) and noticing that $J = O(\log r)$ and $|z_{j+1}| \leq (1 + \eta/2)r$, we obtain

$$S(r,\theta,\frac{\eta}{2}) = O(\log r) + O((\log r)^{\lambda-1-\delta}), \tag{2.12}$$

and hence

$$T(r,\theta,\frac{\eta}{2}) = O((\log r)^2) + O((\log r)^{\lambda-\delta}). \tag{2.13}$$

(2.13) contradicts (2.3) together with (2.3′). This completes the proof of Theorem 2. □

Proof of Theorem 3. Let $\lambda_{\log}(f)$ denote the log-order of $T(r, f)$. Since

$$T(r,f') = m(r,f') + N(r,f') \leq 2T(r,f) + m(r,f'/f)$$

and

$$m(r,f'/f) = O(\log r),$$

hence we have $\lambda_{\log}(f') \leq \lambda_{\log}(f)$.

On the other hand, applying an inequality of Chia-Tai Chuang [8, p. 95, Theorem 4.1], we have $\lambda_{\log}(f) \leq \lambda_{\log}(f')$. This completes the proof of Theorem 3. □

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Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

E-mail address: `pchern@math.msu.edu`

Current address: Department of Applied Mathematics, I-Shou University, Kaohsiung 840, Taiwan

E-mail address: `tychern@isu.edu.tw`