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UNIQUE CONTINUATION FOR SECOND-ORDER PARABOLIC OPERATORS AT THE INITIAL TIME

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Abstract. We consider second-order parabolic equations with time independent coefficients. Under reasonable assumptions, it is known that the fundamental solution satisfies certain Gaussian bounds related to the associated geodesic distance. In this article we prove a sharp unique continuation property at the initial time which matches exactly the above-mentioned kernel bounds.

1. Introduction and statement of the result

Let $T > 0$ be a positive number and $n \geq 1$. We consider a second-order parabolic operator P in $[0, T] \times \mathbb{R}^n$,

$$P = \partial_t + A$$

where A is a second-order elliptic operator of the form

$$A = -\sum_{i,j=1}^{n} \partial_i a^{ij}(t,x)\partial_j + \sum_{i=1}^{n} b^i(t,x)\partial_i + c(t,x)\,, \tag{1.1}$$

with $a^{ij} = a^{ji}$ and, for some $\lambda > 0$,

$$\begin{cases} \lambda|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n\,, \\ \\ a^{ij} \in \mathrm{Lip}\,, \qquad b^i, c \in L^\infty\,. \end{cases} \tag{1.2}$$

Let $\{a_{ij}\}$ be the inverse matrix of $\{a^{ij}\}$. We denote by $d(x_0, x)$ the geodesic distance function between x_0 and x associated to the Riemannian metric $\{a_{ij}\}$ at time 0. The geodesic balls with respect to this distance are

$$B_r(x_0) := \{x \in \mathbb{R}^n \,:\, d(x_0, x) < r\}.$$

For such a parabolic operator a strong unique continuation property was established by Lin [2] under the additional assumption that coefficients are time independent:

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Theorem 1.1. *Assume that the coefficients of the operator P are time independent and satisfy* (1.2). *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and $(t_0, x_0) \in (0, T] \times \Omega$. Let $u \in L^2([0,T]; H^1(\Omega))$ be a solution of the equation*

$$Pu = 0 \qquad \text{in} \qquad]0, T[\ \times \Omega$$

vanishing of infinite order at $(t_0, x_0) \in\ ]0, T[\times\Omega$; i.e.,

$$\lim_{r \to 0} r^{-N} \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |u(t,x)|^2 \, dx \, dt = 0$$

for all positive integers N. Then $u(t_0, x) = 0$ for every $x \in \Omega$.

The assumption that the coefficients are time independent turned out not to be necessary. Indeed, under slightly stronger assumptions, strong unique continuation for operators with first and zero-order time-dependent coefficients was established by Poon [3]. In effect one can use Carleman estimates to prove in a relatively simple manner that the same result holds assuming the coefficients only satisfy (1.2).

It is essential in the above results that $t_0 > 0$; in other words, that one can use backward information about solutions to the parabolic equation. In this paper we consider the following question. What happens if t_0 is the initial time, $t_0 = 0$? In this case we can only use forward information about the heat flow, which is much weaker due to the parabolic regularizing effect. As one can see in the following example, it is no longer enough to require that u vanish of infinite order at (t_0, x_0); instead, one needs some exponential decay.

Example 1.2. Let $P = \partial_t - \Delta$ be the constant coefficients heat operator and let u_0 be a smooth compactly supported function. The function

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) \, dy$$

is the solution for the constant coefficients heat equation $Pu = 0$ in $[0, \infty) \times \mathbb{R}^n$ with initial data u_0. If u_0 is supported outside the Euclidean ball $B_r(x_0)$, then the following pointwise bound holds:

$$|u(t,x)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(r-|x-x_0|)^2}{4t}} \int_{\mathbb{R}^n} |u_0(y)| \, dy, \qquad x \in B_r(x_0)\,.$$

This shows that any decay rate that is weaker than $e^{-\frac{C}{t}}$ for all $C > 0$ says nothing about the support of the initial data.

On the positive side, the anisotropic unique continuation results obtained in Tataru [6] show that the above type of exponential decay suffices for a unique continuation result at the initial time. More precisely, the following is a consequence of the results in [6]:

Theorem 1.3. *Under assumption* (*1.2*), *given $0 < r < R$, there exists $C(r,R) > 0$ so that if $u \in L^2(0,T; H^1(B_R(x_0)))$ solves $Pu = 0$ and satisfies*

$$\|u\|_{L^2(0,t;H^1(B_r(x_0)))} \leq e^{-C(r,R)/4t}, \qquad t \in [0,T],$$

then $u(0) = 0$ in $B_R(x_0)$. Moreover, if the radii are comparable to $r \approx R$, then one can take $C(r,R) = O((R-r)^2)$.

The example above shows that the best constant $C(R,r)$ one can hope for is $C(R,r) = (R-r)^2$.

Open Problem 1.4. Is the result in Theorem 1.3 true for all $C(r,R) > (R-r)^2$?

We conjecture that the answer is affirmative at least in dimension $n = 1$. In higher dimensions it appears that the problem may be more delicate. In this paper we prove the result in the special case when the coefficients of the operator P are time independent.

Theorem 1.5. *Assume that P has time-independent coefficients satisfying (1.2). Let $0 < r < R$. If $u \in L^2(0,T;H^1(B_R(x_0)))$ solves*

$$(\partial_t + A)u = 0 \qquad \text{in }]0,T[\times B_R(x_0),$$

$$u(0,x) = u_0(x) \qquad \text{in } B_R(x_0),$$

and satisfies

$$\|u\|_{L^2(0,t;H^1(B_r(x_0)))} < e^{-R^2/4t}, \tag{1.3}$$

then $u_0(x) = 0$ if $d(x_0,x) < R$.

The idea is to reduce the proof of the above theorem to a seemingly unrelated result for the corresponding wave equation with time-independent coefficients, proved in [5] and [7].

Theorem 1.6. *Let $v \in \mathcal{D}'(]-R,R[;H^1(\mathbb{R}^n))$ be a solution of the wave equation*

$$\partial_t^2 v + Av = 0 \qquad \text{in }]-R,R[\times\mathbb{R}^n$$

with time-independent coefficients satisfying (1.2). *If v vanishes near $[-R,R]\times\{x_0\}$, then $v = 0$ in $\{|t| + d(x_0,x) \le R\}$.*

2. Proof of Theorem 1.5

The proof of Theorem 1.5 proceeds as follows. We first reduce the problem to a similar problem for solutions to the homogeneous parabolic equation in all of $\mathbb{R}^n$. Next, in order to make the ideas more transparent, we prove Theorem 1.5 in the case when A is selfadjoint. Finally, we show how these arguments can be modified if A is not selfadjoint.

2.1. A reduction argument.

We first extend the coefficients a^{ij} to all of $\mathbb{R}^n$ so that condition (1.2) remains uniformly satisfied. We also extend b^i and c by 0 outside $B_R(x_0)$. The corresponding operators are still denoted by A and P.

Remark 2.1. We observe that it suffices to consider the case of A with a nonnegative zero-order term $c(x)$ (one can reduce the proof to this case, possibly conjugating the operator P with $e^{-t\tau}$ for $\tau > \|c\|_{L^\infty}$).

Then the following result is well known.

Theorem 2.2. *There exists a unique solution $u \in L^2(0,T;H^1(\mathbb{R}^n))$ of the problem*

$$(\partial_t + A)u = 0, \qquad u(0) = u_0 \in L^2(\mathbb{R}^n)\,.$$

Moreover,

$$u(t,x) = \int_{\mathbb{R}^n} K(t,x,y)\,u_0(y)\,dy$$

and there exists $c > 0$ such that, for every $\delta \in]0,1]$,

$$0 \le K(t,x,y) \le \frac{c}{(\delta t)^{\frac{n}{2}}} e^{\frac{c}{\delta}} e^{-\frac{d(x,y)^2}{4(1+\delta)t}}\,. \tag{2.1}$$

The part of the above result concerning existence, uniqueness and representation of u can be proved using classical semigroup theory while for the estimate (2.1) the reader is referred to [4].

Lemma 2.3. *It suffices to prove Theorem 1.5 in the case when* $u \in L^2(0,T;H^1(\mathbb{R}^n))$ *solves the homogeneous equation*

$$(\partial_t + A)u = 0, \qquad u(0) = u_0 \in L^2(\mathbb{R}^n) \tag{2.2}$$

in $[0,T]\times\mathbb{R}^n$.

Proof. Let u be as in Theorem 1.5 with initial data $u_0 \in L^2(B_R(x_0))$. For $\epsilon > 0$ let χ be a smooth cutoff function that equals 1 in $B_{R-\epsilon}(x_0)$ and 0 outside $B_R(x_0)$. Set $v_0 = \chi u_0$, extended by 0 outside $B_R(x_0)$. We denote by v the solution to (2.2) with initial data v_0. Then the difference $w = \chi u - v$ solves

$$(\partial_t + A)w = [A,\chi]u, \qquad w(0) = 0\,.$$

The commutator $[A,\chi]u$ is an L^2 function supported in $]0,T[\times B_R(x_0)\setminus B_{R-\epsilon}(x_0)$. Then we can use bounds (2.1) for the heat kernel $K(t,x,y)$ to estimate w near x_0.

Since $[A,\chi]u$ is supported outside the geodesic ball $B_{R-\epsilon}(x_0)$, the Duhamel's formula and the bound (2.1) yield that w decays exponentially near x_0, say

$$|w(t,x)| \lesssim e^{-\frac{(R-2\epsilon)^2}{4t}}, \qquad x \in B_r(x_0), \quad r<\epsilon.$$

Here and later, the notation $A \lesssim B$ means $A \le cB$ with a universal constant c. We combine this with a trivial energy estimate for w in $B_r(x_0)$ and with the bound we have for u to conclude that

$$\|v\|_{L^2(0,t;H^1(B_r(x_0)))} \lesssim e^{-\frac{(R-2\epsilon)^2}{4t}}.$$

Then we can apply Theorem 1.5 to v to conclude that $v_0 = 0$ in $B_{R-2\epsilon}(x_0)$. Hence $u_0 = 0$ in $B_{R-2\epsilon}(x_0)$. But ϵ is arbitrary; therefore $u_0 = 0$ in $B_R(x_0)$. This completes our proof. ☐

2.2. **The selfadjoint case.** Since u is a global solution to (2.2) it follows that

$$u(t) = e^{-tA}u_0\,, \tag{2.3}$$

where $t \mapsto e^{-tA}u_0$ extends to a bounded function for $\Re t \ge 0$ holomorphic on $\Re t > 0$. For $t \ge 0$ we define

$$v(t,x) := \frac{e^{-i\pi/4}}{4i\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{ist^2}(s-i0)^{-3/2}e^{-\frac{A}{4is}}u_0\,ds\,. \tag{2.4}$$

The singularity at 0 can be avoided by moving the contour of integration away from 0 into the lower half-space. This makes the integral absolutely convergent in L^2 for all $u_0 \in L^2$. We claim that v solves the Cauchy problem

$$\begin{cases} v_{tt} + Av = 0, \\ v(0) = 0, \qquad v_t(0) = u_0\,. \end{cases} \tag{2.5}$$

By density it suffices to prove the claim for $u_0 \in H^2(\mathbb{R}^n)$. Pushing the contour of integration into the lower half-space it is easy to see that

$$\lim_{t\to 0} v(t,x) = \frac{e^{-i\pi/4}}{4i\sqrt{\pi}} \int_{-\infty}^{+\infty} (s-i0)^{-3/2}e^{-\frac{A}{4is}}u_0\,ds\; = 0.$$

To compute the time derivative of v we observe that

$$\frac{e^{-i\pi/4}}{4i\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{ist^2}(s-i0)^{-3/2}\,ds \,= t.$$

Then we can write

$$v(t) = \frac{e^{-i\pi/4}}{4i\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{ist^2}(s-i0)^{-3/2}(e^{-\frac{A}{4is}}-1)u_0\,ds \,+ tu_0.$$

Using the fact that $u_0 \in H^2(\mathbb{R}^n)$, for large s we have

$$\|(e^{-\frac{A}{4is}}-1)u_0\|_{L^2} \lesssim \frac{1}{s}\|u_0\|_{H^2}.$$

Then we differentiate with respect to t to obtain

$$v_t(t) = \frac{te^{-i\pi/4}}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{ist^2}(s-i0)^{-1/2}(e^{-\frac{A}{4is}}-1)u_0\,ds \,+ u_0$$

with the integral absolutely convergent. If we pass to the limit, we conclude that

$$\lim_{t\to 0} v_t(t,x) = u_0.$$

To compute the second time derivative of v we make a linear change of variable to get

$$v_t = \frac{e^{-i\pi/4}}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{is}(s-i0)^{-\frac{1}{2}}(e^{-\frac{At^2}{4is}}-1)u_0 ds \,+ u_0.$$

This we can differentiate with respect to t to obtain another absolutely convergent integral, namely

$$v_{tt} = \frac{te^{-i\pi/4}}{4i\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{is}(s-i0)^{-\frac{3}{2}} e^{-\frac{At^2}{4is}} Au_0 ds.$$

Reversing the above change of variable we obtain the last relation

$$v_{tt} = Av.$$

Now, we claim that v vanishes in $[-R,R]\times B_{\frac{r}{2}}(x_0)$. Indeed, for $x \in B_r(x_0)$ we consider the holomorphic function $G^x(z) = e^{-\frac{A}{4iz}}u_0(x)$ in the lower half-plane. From (1.3) combined with energy estimates for the heat equation we obtain

$$\|G^x(z)\|_{L^2(B_{\frac{r}{2}}(x_0))} \le ce^{R^2\Im z} \qquad \text{for } z \in i\mathbb{R}^-\,,$$

for some positive constant c. Also, we trivially get, for some positive constant still denoted by c,

$$\|G^x(z)\|_{L^2(B_{\frac{r}{2}}(x_0))} \le c \qquad \text{for } z \in \mathbb{R}\,.$$

Then, by the Phragmén-Lindelöf theorem in complex analysis we get

$$\|G^x(z)\|_{L^2(B_{\frac{r}{2}}(x_0))} \le ce^{R^2\Im z} \qquad \Im z \le 0.$$

Hence, the Cauchy formula yields that v vanishes in $[-R,R]\times B_{\frac{r}{2}}(x_0)$. Then the conclusion follows by Theorem 1.6.

2.3. **Proof of Theorem 1.5 for A non-selfadjoint.** The difficulty in this case is that we can no longer use directly the formula (2.4) to produce a solution for the wave equation because the operator e^{-tA} might be undefined in the full half-plane $\Re t \geq 0$. To understand the range of t for which this is well defined we begin with an estimate for the resolvent $R_z = (z - A)^{-1}$. Here we think of A as an unbounded, closed, densely defined operator in $L^2(\mathbb{R}^n)$, whose domain is $D(A) = H^2(\mathbb{R}^n)$.

Lemma 2.4. *Suppose (1.2) holds in $\mathbb{R}^n$. Then for large enough C we have*

$$\|R_z(A)\|_{L^2 \to L^2} \leq (C + |\Im z|)^{-\frac{1}{2}}, \qquad C\Re z \leq |\Im z|^2 - C^2. \tag{2.6}$$

Proof. We need to show that for z as above the operator $z - A$ is surjective and

$$\|(A - z)u\|_{L^2}^2 \geq (C + |\Im z|)\|u\|_{L^2}^2\,, \qquad \forall u \in L^2\,. \tag{2.7}$$

Since A is closed, (2.7) implies that $A - z$ has closed range. On the other hand, the Lax-Milgram lemma shows that $A - z$ is surjective for $\Re z < -C$, with large enough C. Then (2.7) combined with a continuity argument implies that $A - z$ is surjective for all z in the above range. Indeed, denoting

$$D = \{z \in \mathbb{C};\ C\Re z \leq |\Im z|^2 - C^2\},$$
$$\Gamma = \{z \in D : A - z \quad \text{is surjective with a bounded inverse}\}$$

is open in D. If we also show that Γ is closed, then we can conclude that $\Gamma = D$. Let $z_j \in \Gamma$ be a sequence converging to $z \in D$. For $f \in L^2$ we can find $u_j \in L^2$ such that $(A - z_j)u_j = f$. The estimate (2.7) shows that u_j is bounded in L^2, and using elliptic regularity we conclude that in effect u_j is bounded in H^2. On a subsequence, u_n converges weakly to some $u \in H^2$, and passing to the limit we conclude that $(A - z)u = f$, i.e., $A - z$ is surjective. Hence $z \in \Gamma$.

It remains to prove (2.7). For this we set $w = C - 2i\Im z$ and evaluate the inner product

$$\begin{aligned}
\Re\langle (A-z)u, wu\rangle &= C\langle a^{ij}\partial_i u, \partial_j u\rangle + \langle (2|\Im z|^2 - C\Re z)u, u\rangle + \Re\langle (b^i\partial_i + c)u, wu\rangle \\
&\geq \lambda C\|\nabla u\|_{L^2}^2 + (|\Im z|^2 + C^2)\|u\|_{L^2}^2 \\
&\quad -\|b\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^2}\sqrt{4|\Im z|^2 + C^2} \\
&\geq (|\Im z|^2 + C^2 - \alpha(4|\Im z|^2 + C^2))\|u\|_{L^2}^2 \\
&\quad +\Big(\lambda C - \frac{1}{\alpha}\|b\|_{L^\infty}^2\Big)\|\nabla u\|_{L^2}^2
\end{aligned}$$

where $\|b\|_{L^\infty} := n\sup_{1\leq i\leq n}\|b_i\|_{L^\infty}$ and α is a small fixed constant. Then for C large enough we get

$$\Re\langle (A-z)u, wu\rangle \geq \frac{1}{2}(|\Im z|^2 + C^2)\|u\|_{L^2}^2$$

which in turn implies (2.7). □

Now we turn our attention to the operator e^{-At}. A priori this is well defined for $t \in \mathbb{R}^+$, continuous in t for $t > 0$ and analytic for $t > 0$. We seek to extend it to the half-space $\Re t > 0$.

Lemma 2.5. *Suppose (1.2) holds in $\mathbb{R}^n$. Then the operator e^{-At} is well defined and holomorphic in the region $\Re t > 0$. Furthermore, the following bound holds for large enough C:*

$$\|e^{-At}\|_{L^2 \to L^2} \lesssim (\Re t)^{-1} e^{C\frac{|t|^2}{\Re t}}(\Re t + C|t|^2)^{\frac{1}{4}}. \tag{2.8}$$

Proof. Let γ be the curve $C\Re z = \frac{1}{4}|\Im z|^2 - C^2$. For $\Re t > 0$ we define the exponential of A as the path integral

$$e^{-At} = \int_\gamma e^{-zt} R_z(A)\, dz.$$

By Lemma 2.4 the integral is absolutely convergent. It is also easy to verify that $e^{-At} : L^2 \to D(A)$ and

$$\frac{d}{dt} e^{-At} = Ae^{-At}.$$

To bound it we parametrize γ by $z = C^{-1}(x^2 - C^2) + 2ix$ with $x \in \mathbb{R}$. Then, easy computations yield

$$\begin{aligned}\|e^{-At}\|_{L^2 \to L^2} &\leq C^{-1} \int_{-\infty}^{\infty} e^{2x\Im t - C^{-1}(x^2 - C^2)\Re t} \frac{2}{C}(x^2 + C^2)^{\frac{1}{4}}\, dx \\ &\lesssim (\Re t)^{-1} e^{C\frac{|t|^2}{\Re t}} (\Re t + C|t|^2)^{\frac{1}{4}}.\end{aligned}$$

It remains to verify that the initial data is indeed u_0, i.e., that

$$\lim_{t \to 0} e^{-At} u_0 = u_0, \qquad u_0 \in L^2,\ t \in \mathbb{R}^+.$$

It suffices to prove this for u_0 in a dense subset of L^2. Indeed, if $u_0 \in D(A)$ and $t > 0$, then

$$(e^{-At} - 1)u_0 = \int_\gamma e^{-zt} \left(R_z(A) - \frac{1}{z} \right) dz u_0 = \int_\gamma e^{-zt} \frac{1}{z} R_z(A) dz A u_0.$$

We can pass to the limit,

$$\lim_{t \to 0} (e^{-At} - 1)u_0 = \int_\gamma \frac{1}{z} R_z(A) dz A u_0.$$

But the last integral is 0, because the contour of integration can be shifted towards $-\infty$. This concludes the proof of the lemma. $\square$

Now we can conclude the proof of the theorem. We modify (2.4) and set

$$v(t,x) := \frac{e^{-i\pi/4}}{4i\sqrt{\pi}} \int_{\gamma_1} e^{ist^2} (s - i0)^{-3/2} e^{-\frac{A}{4is}} u_0\, ds \tag{2.9}$$

where $\gamma_1 = \mathbb{R} - \frac{1}{2}iC$. If $s \in \gamma_1$, then $z = (4is)^{-1}$ is on the circle

$$\mathcal{C} = \left\{ z \in \mathbb{C} \,:\, |z|^2 = \frac{\Re z}{2C} \right\}$$

and satisfies $\Re z = C(2C^2 + 8|\Re s|^2)^{-1}$. Hence by Lemma 2.5 we get

$$\|e^{-\frac{A}{4is}}\|_{L^2 \to L^2} \lesssim (C^2 + |\Re s|^2)^{\frac{3}{4}},$$

which makes the integral in (2.9) well defined as an L^2-valued distribution in $(0, \infty)$. We want to argue as in the proof for A selfadjoint to show that v is the unique solution of the equation (2.5). It suffices to prove this for u_0 in a dense subset of L^2. However, in this case $D(A)$ no longer seems to suffice; therefore, we use

$$D(A^2) = \{u \in H^2(\mathbb{R}^n) \,:\, Au \in H^2(\mathbb{R}^n)\}$$

instead. If the coefficients are more regular, then $D(A^2) = H^4(\mathbb{R}^n)$, but in our case we can only say that it is a dense subspace of $H^2(\mathbb{R}^n)$. This is because for $u_0 \in H^2$ we have

$$\lim_{\epsilon \to 0} (I + i\epsilon A)^{-1} u_0 = u_0 \qquad \text{in } H^2.$$

Let B be the ball whose boundary is the circle $\mathcal{C}$,

$$B = \left\{ z \in \mathbb{C} \,:\, |z|^2 \leq \frac{\Re z}{2C} \right\}.$$

Then (2.8) implies that

$$\|e^{-At}\|_{L^2 \to L^2} \lesssim (\Re t)^{-\frac{3}{4}}, \qquad t \in B.$$

For $u_0 \in D(A)$ we have

$$\frac{d}{dz} e^{-zA} u_0 = e^{-zA} A u_0.$$

Integrating this along the lines $0 \to |t| \to t$ we obtain

$$\|(e^{-At} - 1) u_0\|_{L^2} \lesssim |t|^{\frac{1}{4}} \|A u_0\|_{L^2}, \qquad t \in B.$$

We also have

$$\frac{d}{dz}(e^{-zA} - 1 + zA) u_0 = (1 - e^{-zA}) A u_0.$$

Integrating this on the straight line from 0 to t we obtain

$$\|(e^{-At} - 1 + tA) u_0\| \lesssim |t|^{\frac{5}{4}} \|A^2 u_0\|_{L^2}, \qquad t \in B. \tag{2.10}$$

Now the proof of the fact that v solves (2.5) proceeds as in the selfadjoint case, but with $u_0 \in D(A^2)$ (which allows us to use (2.10)).

The Phragmén-Lindelöf Theorem applies without any modifications in this case and shows that

$$\|s^{-\frac{3}{2}} e^{-\frac{A}{4is}} u_0\|_{L^2(B_r(x_0))} \lesssim e^{-R^2 \Im s}, \qquad \Im s \leq -\frac{1}{2} C.$$

Then the Paley-Wiener theorem for distributions shows that the inverse Fourier transform of $s^{-\frac{3}{2}} e^{-\frac{A}{4is}} u_{0|B_r(x_0)}$ is supported in $[R^2, \infty)$, which implies that $v(t,x) = 0$ for $|t| < R$, $x \in B_r(x_0)$. By Theorem 1.6, this shows that $u_0 = v_t(0)$ vanishes in $B_R(x_0)$.

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