CONVERGENCE OF SEQUENCES
OF PAIRWISE INDEPENDENT RANDOM VARIABLES

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Abstract. In spite of the fact that the tail $\sigma$-algebra of a sequence of pairwise independent random variables may not be trivial, we have discovered that if such a sequence converges in probability or almost everywhere, then the limit has to be a constant. This enables us to provide necessary and sufficient conditions for its convergence, in terms of its marginal distribution functions.

Let $\{X_n: n \geq 1\}$ be a sequence of pairwise independent random variables. It is known (see Robertson and Womack (1985)), that the elements of the tail $\sigma$-algebra induced by such a sequence may not satisfy the 0-1 law. Therefore, it came to us as a pleasant surprise when we discovered that the limit, in probability or almost everywhere, of such a sequence is a constant in all cases, and it is even more interesting that the almost everywhere convergence of such sequences depends only on the distribution function of individual $X_n$’s. These can all be described in the following theorem.

Theorem. Let $\{X_n: n \geq 1\}$ be a sequence of pairwise independent random variables and let $m_{X_n}$ be a median of $X_n$. Then

(a) $X_n$ converges in probability iff for all $\epsilon > 0$, $P\{|X_n - m_{X_n}| > \epsilon\} \to 0$ and $m_{X_n}$ converges,

(b) $X_n$ converges almost everywhere iff for all $\epsilon > 0$, $\sum_{n=1}^{\infty} P\{|X_n - m_{X_n}| > \epsilon\} < \infty$ and $m_{X_n}$ converges.

Proof. Only the necessity parts in (a) and (b) need justification. Assume that $X_n$ converges in probability. Construct a sequence of independent random variables $\{Y_n: n \geq 1\}$ such that $Y_n$ has the same distribution as $X_n$ for each $n \geq 1$. Since convergence in probability involves the joint distribution of two random variables at a time and $(X_i, X_j)$ has the same distribution as $(Y_i, Y_j)$ for all $i, j \geq 1$, $Y_n$ also converges in probability. Thus it converges almost everywhere to the same limit on a subsequence and the Kolmogorov 0-1 law tells us that the limit is a constant, say $c$. But given $\epsilon > 0$, $P\{|X_n - c| > \epsilon\} = P\{|Y_n - c| > \epsilon\}$. Therefore, $X_n$ converges in probability (almost everywhere) iff $P\{|X_n - c| > \epsilon\} \to 0$ ($\sum_{n=1}^{\infty} P\{|X_n - c| > \epsilon\} < \infty$), where we used the fact that the Borel-Cantelli lemma for independent random variable remains intact for pairwise independent random variables; see Theorem 4.2.5 in Chung (1974). Also $X_n \overset{P}{\to} c$ implies that $m_{X_n} \to c$. As a result we can
replace the constant $c$ by $m_{X_n}$, if we wish, at the expense of $\epsilon$, and since $\epsilon > 0$ is arbitrary, we are through.

References
