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AN ASYMPTOTIC STABILITY RESULT FOR SCALAR DELAYED POPULATION MODELS

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ABSTRACT. We give sufficient conditions for the global asymptotic stability of the scalar delay differential equation $\dot{x}(t) = (1 + x(t))F(t, x_t)$, without assuming that zero is a solution. A result of Yorke (1970) is revisited.

1. INTRODUCTION

Let $C := C([-r, 0]; \mathbb{R})$ be the space of continuous functions from $[-r, 0]$ to $\mathbb{R}$, $r > 0$, equipped with the sup norm $|\varphi|_C = \max_{-r\le\theta\le0} |\varphi(\theta)|$. Consider a scalar functional differential equation (FDE)

$$\dot{x}(t) = (1 + x(t))F(t, x_t), \quad t \ge 0, \tag{1.1}$$

where $F : [0,\infty) \times C \to \mathbb{R}$ is continuous. As usual, $x_t \in C$ is defined by $x_t(\theta) = x(t + \theta), -r \le \theta \le 0$. Due to the biological interpretation of the model, we only consider solutions of (1.1) with *admissible* initial conditions

$$x_0 = \varphi, \quad \varphi \in C_{-1}, \tag{1.2}$$

where

$$C_\alpha := \{\varphi \in C : \varphi(\theta) \ge \alpha \text{ for } \theta \in [-r, 0) \text{ and } \varphi(0) > \alpha\},$$

for $\alpha \in \mathbb{R}$. Uniqueness of solutions for the initial value problems (1.1) and (1.2) is assumed.

Our purpose is to establish sufficient conditions for boundedness of solutions of (1.1), and for the global asymptotic stability of (1.1) in the set of admissible solutions. Recall that an FDE in C is said to be *globally asymptotically stable* (in the set of admissible solutions) if any two solutions $x_1, x_2 : [0,\infty) \to \mathbb{R}$ with admissible initial conditions satisfy

$$\lim_{t\to\infty} (x_1(t) - x_2(t)) = 0. \tag{1.3}$$

Equation (1.1) usually appears in population dynamics. In fact, for a general scalar delayed population model

$$\dot{y}(t) = y(t)f(t, y_t), \quad t \ge 0, \tag{1.4}$$

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where $f : [0,\infty) \times C \to \mathbb{R}$ is a continuous function, consider only positive solutions of (1.4), or, in other words, solutions with initial conditions $y_0 = \psi \in C_0$. Through the change of variables $x(t) = y(t) - 1$, (1.4) reads as (1.1), where F is defined by $F(t,\varphi) = f(t, 1+\varphi)$. Clearly, initial conditions $y_0 \in C_0$ for (1.4) become $x_0 \in C_{-1}$ for (1.1).

When zero is an equilibrium of (1.1), sufficient conditions for its global asymptotic stability have been recently established:

Theorem 1.1 ([1]). *For F as in* (1.1)*, assume that*

(H1) *there exists a piecewise continuous function $a : [0,\infty) \to [0,\infty)$ and $T_0 \geq 0$, such that for each $\varepsilon > 0$ there is $\eta = \eta(\varepsilon) > 0$ and such that for $t \geq T_0$ and $\varphi \in C_{-1}$, we have*

$$F(t,\varphi) \leq -\eta a(t) \text{ if } \varphi \geq \varepsilon \quad \text{and} \quad F(t,\varphi) \geq \eta a(t) \text{ if } \varphi \leq -\varepsilon;$$

(H2) *for a as in* (H1)*, $\int_0^\infty a(t)dt = \infty$;*

(H3) *there exists a piecewise continuous function $b : [0,\infty) \to [0,\infty)$ such that*

$$-b(t)M(\varphi) \leq F(t,\varphi) \leq b(t)M(-\varphi), \quad \text{for} \quad t \geq 0, \varphi \in C_{-1},$$

where

$$M(\varphi) = \max\{0, \sup_{\theta\in[-r,0]} \varphi(\theta)\};$$

(H4) *for b as in* (H3)*, there is $T \geq 0$ such that $\int_{t-r}^t b(s)ds \leq \frac{3}{2}$, for $t \geq T$.*

Let $x(t) = x(\varphi)(t)$ be the solution of (1.1)-(1.2)*. Then, $x(t)$ is defined on $[0,\infty)$, bounded away from -1, and $x(t) \to 0$ as $t \to \infty$.*

Some explanation for the notation in (H1) is in order. For $c \in \mathbb{R}$, we use c to denote both the real constant and the constant function $\varphi(\theta) = c$ in C. In C, we consider the usual partial order $\varphi \geq \psi$ if and only if $\varphi(\theta) \geq \psi(\theta),\ \theta \in [-r,0]$. Also, we say that a solution $x(t)$ on $[0,\infty)$ is bounded away from -1 if there is $\varepsilon > 0$ such that $x(t) \geq -1+\varepsilon$ for $t \geq 0$.

In [1], hypotheses (H1) and (H2), respectively (H3) and (H4), were used to force nonoscillatory, respectively oscillatory, solutions to zero, as t tends to infinity. Observe that, in particular, (H3) implies that zero is a solution of (1.1).

A motivation for the present work was to establish a global asymptotic stability result for (1.1) when zero is not an equilibrium of (1.1). Another motivation came from the well-known work of Yorke [9]. For a general scalar FDE

$$\dot{x}(t) = F(t, x_t), \quad t \geq 0, \tag{1.5}$$

with $F : [0,\infty) \times C \to \mathbb{R}$ continuous and such that $F(t,0) = 0, t \geq 0$, the uniform and asymptotic stability of its zero solution has been extensively studied since [9]. For significant improvements on [9] in the framework of (1.5), see [5], [7] and [8]. In [9], it was proven that if (H3) holds with $b(t) \equiv b$ a positive constant and $br < 3/2$, then all the oscillatory solutions of (1.5) with initial conditions sufficiently small tend to zero as $t \to \infty$. Yoneyama [7] generalized this result, replacing the constant b by a nonnegative continuous function $b(t)$, i.e., assuming (H3), and further imposing

$$\sup_{t\geq 0} \int_t^{t+r} b(s)ds < \frac{3}{2}, \qquad \inf_{t\geq 0} \int_t^{t+r} b(s)ds > 0.$$

To deduce that eventually monotone solutions of (1.5) tend to zero as t tends to ∞, in [7] and [9] the following condition was assumed:

$$\text{for all sequences } t_n \to \infty \text{ and } \varphi_n \in C, |\varphi_n|_C \le \beta, \text{ if } \varphi_n \to c \ne 0, \text{ we have that } F(t_n, \varphi_n) \text{ does not converge to zero.} \tag{1.6}$$

Assumption (H1) first appeared in [8], in the context of (1.5) with possible unbounded delay. This shows that the requirements in Theorem 1.1 or similar ones have been extensively considered for (1.5) (see [1] for additional references). For stability results on particular equations that can be written as (1.1), see [2], [3] and [4]. The general situation of (1.1) was addressed in [1].

Yorke also stated the following [9, Corollary 3.4]. If there is $\alpha > 0$ with $\alpha r < 3/2$ such that F satisfies

$$F(t,\varphi_1) - F(t,\varphi_2) \le \alpha M(\varphi_2 - \varphi_1), \quad t \ge 0, \quad \varphi_1, \varphi_2 \in C, \tag{1.7}$$

then the solutions of (1.5) with initial conditions $x_{t_0} = \varphi \in C$ are defined for $t \ge t_0$, and any two solutions $x_1(t), x_2(t)$ satisfy $x_1(t) - x_2(t) \to 0$ as $t \to \infty$. However, since (1.6) was not assumed, nor any other condition controlling the behavior of nonoscillatory solutions, condition (1.7) with $0 < \alpha r < 3/2$ is not sufficient to conclude that any two solutions of (1.5) satisfy (1.3), as the following counter-example shows.

Example 1.2. Consider (1.5), with $F(t,\varphi) = g(\varphi(-1))$ for $t \ge 0$ and $\varphi \in C = C([-1,0];\mathbb{R})$, where $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = -x$ if $x \le 0$, $g(x) = 0$ if $x > 0$:

$$\dot{x}(t) = g(x(t-1)), \quad t \ge 0. \tag{1.8}$$

Condition (1.7) holds with $\alpha = 1$. On the other hand, all nonnegative constants are equilibria of (1.8); therefore the zero solution is not attracting.

2. Global asymptotic stability of (1.1)

We now establish criteria for boundedness of solutions of (1.1), and for the global asymptotic stability of the equation.

Theorem 2.1. *Assume that:* (i)

$$F(t,0) \to 0 \quad \text{as} \quad t \to \infty; \tag{2.1}$$

(ii) *there exists* $\beta : [0,\infty) \to [0,\infty)$ *continuous such that*

$$F(t,\varphi_1) - F(t,\varphi_2) \le \beta(t) \max_{\theta\in[-r,0]} (\varphi_2(\theta) - \varphi_1(\theta)), \quad t \ge 0, \quad \varphi_1, \varphi_2 \in C_{-1}; \tag{2.2}$$

(iii) *for* β *as in* (2.2), *there is* $T \ge 0$ *such that*

$$\lambda := \inf_{t\ge T} \int_{t-r}^t \beta(s)ds > 0, \tag{2.3}$$

$$\mu := \sup_{t\ge T} \int_{t-r}^t \beta(s)ds < \infty. \tag{2.4}$$

Then the solutions of the initial value problems (1.1)-(1.2) *are defined, bounded and bounded away from* -1 *on* $[0,\infty)$. *Furthermore, if* $\mu e^\mu < 3/2$, *then any two solutions* $x_1(t), x_2(t)$ *satisfy* (1.3).

Proof. Let $x(t)$ be a solution of (1.1) with initial condition $x_0 = \varphi \in C_{-1}$.

Step 1. $x(t)$ is defined on $[0,\infty)$.

From (2.2), we get

$$F(t,\varphi) - F(t,0) \le -\beta(t) \min_{\theta\in[-r,0]} \varphi(\theta),$$
$$F(t,0) - F(t,\varphi) \le \beta(t) \max_{\theta\in[-r,0]} \varphi(\theta), \quad t \ge 0,\ \varphi \in C_{-1};$$

hence

(2.5)
$$-\beta(t) \max_{\theta\in[-r,0]} \varphi(\theta) \le F(t,\varphi) - F(t,0) \le -\beta(t) \min_{\theta\in[-r,0]} \varphi(\theta), \quad t \ge 0,\ \varphi \in C_{-1}.$$

In particular,

$$F(t,\varphi) - F(t,0) \le \beta(t), \quad t \ge 0,\ \varphi \in C_{-1}, \tag{2.6}$$

and from (2.1) we derive that $F(t,\varphi) \le \beta(t) + F(t,0)$ is bounded from above on $[0,a] \times C_{-1}$, for all $a > 0$. Using standard arguments (see, e.g., [1]), we conclude that $x(t)$ is defined for $t \ge 0$, with $x(t) > -1$ on $[0,\infty)$.

Step 2. $x(t)$ is bounded on $[0,\infty)$.

Fix any $\delta > 0$. From (2.1), let $T_0 \ge T$ be such that $|F(t,0)| \le \delta$ for $t \ge T_0 - r$. From (2.6), it follows that $\dot{x}(t) \le (1 + x(t))(\beta(t) + \delta)$; thus $(1 + x(t)) \le (1 + x(s)) \exp\left(\int_s^t (\delta + \beta(\tau))d\tau\right), t \ge s \ge T_0 - r$. For $t \ge T_0$ and $\theta \in [-r,0]$, we obtain

$$x_t(\theta) \ge -1 + (1 + x(t))e^{-K_0},$$

where $K_0 = K_0(\delta) := \delta r + \mu$ and μ is as in (2.4). From (2.5), we now derive that

$$\dot{x}(t) \le (1 + x(t))[\delta + \beta(t) - \beta(t)(1 + x(t))e^{-K_0}], \qquad t \ge T_0. \tag{2.7}$$

Effect the change of variables $y(t) = 1 + x(t)$. The above inequality becomes

$$\dot{y}(t) \le y(t) f(t, y(t)), \quad t \ge T_0,$$

where $f(t,y) = \delta + \beta(t)(1 - e^{-K_0} y), t \ge T_0, y \ge 0$. We consider now the ODE

$$\dot{y}(t) = y(t) f(t, y(t)), \quad t \ge T_0, \tag{2.8}$$

and use a comparison result. Since $y = 0$ is a solution of the Ricatti equation (2.8), it can actually be solved. However, in order to derive bounds for solutions of (2.8), it is more useful to apply results in [6]. Clearly, $f, \frac{\partial f}{\partial y}$ are continuous and $\frac{\partial f}{\partial y} \le 0$ for $t \ge T_0, y \ge 0$. On the other hand, with

$$c = c(\delta) := e^{K_0(\delta)}(1 + \delta r/\lambda),$$

where $\lambda > 0$ is as in (2.3), we have $c > 0, f(t,c) \le \delta$ and

$$\int_t^{t+r} f(s,c)ds \le \delta r + (1 - e^{-K_0} c)\lambda = 0, \quad t \ge T_0.$$

Thus, assumptions A1, A2 and A3 of [6] hold, after a scaling in time $t \mapsto rt$. Invoking [6, Theorem 1], we conclude that the solutions of the ODE (2.8) with initial conditions $y(T_0) > 0$ are bounded on $[T_0,\infty)$ and satisfy

$$y(t) \le \max(y(T_0), c)e^{r\delta}, \quad t \ge T_0. \tag{2.9}$$

From (2.7) and a comparison result, we conclude that $x(t)$ is bounded for $t \ge 0$.

Step 3. $x(t)$ is bounded away from -1 on $[0,\infty)$.

The proof follows along the lines in step 2. From step 2, let $x(t) \le K$ on $[-r,\infty)$, for some $K > 0$. Using (2.5), we get

$$\dot{x}(t) \ge -(1 + x(t))[\delta + K\beta(t)], \quad t \ge T_0 - r.$$

By integrating the above inequality, we have

$$x_t(\theta) \leq -1 + (1 + x(t))e^{K_1}, \quad t \geq T_0, \quad \theta \in [-r, 0],$$

where $K_1 = \delta r + K\mu$. Using again (2.5), we obtain

$$\dot{x}(t) \geq (1 + x(t))[-\delta + \beta(t) - \beta(t)(1 + x(t))e^{K_1}], \quad t \geq T_0. \tag{2.10}$$

The change of variables $y(t) = 1 + x(t)$ leads to

$$\dot{y}(t) \geq y(t)g(t, y(t)), \quad t \geq T_0,$$

where $g(t, y) = -\delta + \beta(t)(1 - e^{K_1}y), t \geq T_0, y \geq 0$. Consider the ODE

$$\dot{y}(t) = y(t)g(t, y(t)), \quad t \geq T_0. \tag{2.11}$$

Suppose that δ was chosen so that $\delta r < \lambda$, and define $d = e^{-K_1}(1 - \delta r/\lambda) > 0$. Then $g(t, d) \geq -\delta$ and

$$\int_t^{t+r} g(s, d)ds \geq -\delta r + (1 - e^{K_1}d)\lambda = 0, \quad t \geq T_0.$$

Hence, assumptions A1-A4 in [6, Theorem 2] hold, and it follows that the solutions of (2.11) with $y(T_0) > 0$ are bounded away from 0. A comparison result applied to (2.10) ends the proof of this step.

Step 4. If $\mu e^{\mu} < 3/2$, then $\lim_{t\to\infty}(x(t) - u(t)) = 0$ for any other admissible solution $u(t)$ of (2.1).

Fix a solution $u(t)$ and effect the change of variables $z(t) = \dfrac{x(t) - u(t)}{1 + u(t)}$. Then (2.1) is transformed into

$$\dot{z}(t) = (1 + z(t))F_0(t, z_t),$$

with

$$F_0(t, \varphi) = F(t, (u_t + 1)\varphi + u_t) - F(t, u_t), \quad t \geq 0, \varphi \in C.$$

Condition (2.2) implies that F_0 satisfies

$$-\beta(t) \max_{\theta\in[-r,0]}[(u(t + \theta) + 1)\varphi(\theta)] \leq F_0(t, \varphi) \leq -\beta(t) \min_{\theta\in[-r,0]}[(u(t + \theta) + 1)\varphi(\theta)],$$

from which (H1) and (H3) follow, with $a(t) = m_0\beta(t)$ and $b(t) = M_0\beta(t)$ for $t \geq T_0$, where $0 < m_0 \leq u(t) + 1 \leq M_0, t \geq T_0$. Condition (2.3) implies (H2). On the other hand, choosing $u(t)$ so that $u(T_0) + 1 \leq c = c(\delta)$, from (2.9) we obtain $1 + u(t) \leq c(\delta)e^{r\delta},\ t \geq T_0$. Since $c(\delta)e^{r\delta} \to e^{\mu}$ as $\delta \to 0^+$, if $\mu e^{\mu} < 3/2$, it follows that $\delta > 0$ can be chosen so that $M_0\mu \leq 3/2$, and consequently hypothesis (H4) is satisfied. From Theorem 1.1 we conclude that $z(t) \to 0$ as $t \to \infty$. □

The proof of step 4 leads to the following corollary:

Corollary 2.2. *Let* $u(t)$ *be an admissible solution of* (1.1)*, and assume* (2.1)–(2.4)*. If* $u(t) + 1 \leq M_0$ *for* $t \geq 0$ *large and* $M_0\mu \leq 3/2$*, then* $x(t) - u(t) \to 0$ *as* $t \to \infty$*, for every admissible solution* $x(t)$ *of* (1.1)*.*

Remark 2.3. We point out that condition (2.3) cannot be replaced by the weaker assumption

$$\int^{\infty} \beta(s)ds = \infty. \tag{2.12}$$

In fact, consider the scalar FDE on $C = C([-1,0];\mathbb{R})$:

$$\dot{x}(t) = -\beta(t)(1+x(t))\Big[x(t-1) + 3\frac{1+t}{2+t}\Big], \quad t \geq 0, \tag{2.13}$$

where

$$\beta(t) = \frac{2+t}{2(1+t)(3+t)}, \qquad t \geq 0.$$

For β as above and $F : I \times C \to \mathbb{R}$ defined by $F(t,\varphi) = -\beta(t)\Big(\varphi(-1) + 3\frac{1+t}{2+t}\Big)$, clearly (2.1), (2.2) and (2.12) are fulfilled. Since

$$\int_{t-1}^{t} \beta(s)ds \to 0 \quad \text{as} \quad t \to \infty,$$

then (2.3) fails and (2.4) holds for any $\mu > 0$. On the other hand, (2.13) is not globally asymptotically stable, nor are all of its admissible solutions bounded away from -1. In fact, the function

$$x(t) = \frac{1}{3+t} - 1$$

is a solution of (2.13) with initial condition $x_0 \in C_{-1}$, and $x(t) \to -1$ as $t \to \infty$.

Remark 2.4. Also in Theorem 2.1, condition (2.2) cannot be replaced by (cf. (1.7))

$$F(t,\varphi_1) - F(t,\varphi_2) \leq \beta(t)M(\varphi_2 - \varphi_1), \quad t \geq 0, \quad \varphi_1, \varphi_2 \in C_{-1},$$

for $\beta : [0,\infty) \to [0,\infty)$ continuous and satisfying (2.3) and (2.4). In fact, consider

$$\dot{x}(t) = \alpha(1+x(t))g(x(t-1)), \quad t \geq 0, \tag{2.14}$$

for g as in (1.8), and $\alpha e^{\alpha} \in (0, 3/2)$. Again every $c \geq 0$ is an equilibrium of (2.14); thus (1.3) fails.

Example 2.5. Consider the logistic model

$$\dot{y}(t) = b(t)y(t)[a(t) - L(t, y_t)], \quad t \geq 0, \tag{2.15}$$

where $b : [0,\infty) \to [0,\infty), a : [0,\infty) \to \mathbb{R}$ are continuous functions, $L : [0,\infty) \times C \to \mathbb{R}$ is continuous with $L(t,\cdot)$ nonzero linear operators. Through the change $y(t) = x(t) + 1$, (2.15) becomes (1.1), with

$$F(t,\varphi) = b(t)[a(t) - L(t,1) - L(t,\varphi)].$$

Assume that $L(t,\cdot)$ are positive operators, i.e., $L(t,\varphi) \geq 0$ whenever $\varphi \geq 0$, for all $t \geq 0$. Then (2.2) is fulfilled with $\beta(t) = b(t)L(t,1) = b(t)\|L(t,\cdot)\|$. If

$$b(t)[a(t) - L(t,1)] \to 0 \quad \text{as} \quad t \to \infty$$

and

$$0 < \lambda \leq \int_{t-r}^{t} a(s)b(s)ds \leq \mu \quad \text{for } t \geq 0 \text{ large},$$

with $\mu e^{\mu} < 3/2$, then any two solutions $y_1(t), y_2(t)$ of (2.15) with initial conditions in C_0 satisfy $y_1(t) - y_2(t) \to 0$ as $t \to \infty$.

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