THE SPLITTING PROBLEM FOR SUBSPACES
OF TENSOR PRODUCTS OF OPERATOR ALGEBRAS

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ABSTRACT. The main result of this paper is that if $N$ is a von Neumann algebra that is a factor and has the weak* operator approximation property (the weak* OAP), and if $R$ is a von Neumann algebra, then every $\sigma$-weakly closed subspace of $N \bar{\otimes} R$ that is an $N \bar{\otimes} C_1 R$-bimodule (under multiplication) splits, in the sense that there is a $\sigma$-weakly closed subspace $T$ of $R$ such that $S = N \bar{\otimes} T$. Note that if $S$ is a von Neumann subalgebra of $N \bar{\otimes} R$, then $S$ is an $N \bar{\otimes} C_1 R$-bimodule if and only if $N \bar{\otimes} C_1 R \subset S$. So this result is a generalization (in the case where $N$ has the weak* OAP) of the result of Ge and Kadison that if $N$ is a factor, then every von Neumann subalgebra $M$ of $N \bar{\otimes} R$ that contains $N \bar{\otimes} C_1 R$ splits. We also obtain other results concerning the splitting of $\sigma$-weakly closed subspaces of tensor products of von Neumann algebras and the splitting of normed closed subspaces of $C^*$-algebras that generalize results previously obtained for von Neumann subalgebras and $C^*$-subalgebras.

In this article we are concerned with the following question: if $N$ and $R$ are von Neumann algebras, and if $S$ is a $\sigma$-weakly closed $N \bar{\otimes} C_1 R$-bimodule of $N \bar{\otimes} R$, when do we have

$$S = N \bar{\otimes} T$$

for some $\sigma$-weakly closed subspace $T$ of $R$? If (1) holds for some $T$, then we say that $S$ splits. Note that if a $\sigma$-weakly closed subspace $S$ of $N \bar{\otimes} R$ splits, then it is an $N \bar{\otimes} C_1 R$-bimodule. So this requirement is necessary for splitting. This problem has been previously studied in the case when $S = M$ is a von Neumann subalgebra of $N \bar{\otimes} R$ containing $N \bar{\otimes} C_1 R$ (which is obviously equivalent to $M$ being an $N \bar{\otimes} C_1 R$-bimodule). Ge and Kadison showed in [4] that if $N$ is a factor, then every von Neumann algebra $M$ satisfying $N \bar{\otimes} C_1 R \subset M \subset N \bar{\otimes} R$ splits. In [10] Strătilă and Zsidó extended this result by showing that if $N$ is a von Neumann algebra with center $Z(N)$, and if $H$ is a Hilbert space, then a von Neumann algebra $M$ such that $N \bar{\otimes} C_1 B(H) \subset M \subset N \bar{\otimes} B(H)$ is of the form $N \bar{\otimes} P$ for some von Neumann subalgebra $P$ of $B(H)$ if and only if $M \cap (Z(N) \bar{\otimes} B(H)) = Z(N) \bar{\otimes} P$. By modifying the methods of [10], we are able to extend their result (and so the result of Ge and Kadison) to the case where $S$ is a $\sigma$-weakly closed subspace of $N \bar{\otimes} R$ that is an $N \bar{\otimes} C_1 R$-bimodule. However, we have to add a condition to $N$, namely that $N$ satisfies Property $S_\sigma$ (introduced by the author in [6]) or, equivalently, the weak* operator approximation property. We also observe that the proof of the main result

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in [10] (and Theorem 3.3 in [15]) can be easily modified to give a splitting result for certain norm closed subspaces of the spatial (= minimal) tensor product of C*-algebras. We would like to thank the referee for helpful comments and suggestions.

Our first result is an extension of Theorem 3.5 in [10]. Since the proof of Theorem 1 is essentially the same as the proof of Theorem 3.5 in [10], it is omitted. (The only difference between Theorem 1 below and Theorem 3.5 in [10] is that in [10] it is shown that if $N \subset M \subset N \vee N_0$ is an intermediate von Neumann algebra, then $\Phi(M) = M \cap N_0 = N' \cap M$ rather than $\Phi(S) = S \cap N_0 = N' \cap S$ if $S$ is a $\sigma$-weakly closed subspace of $N \vee N_0$ that is an $N$-bimodule. The only place in the proof of the equality $\Phi(M) = M \cap N_0$ in [10] where the fact that $M$ is a von Neumann algebra is used is that $N \subset M$ implies that $M$ is an $N$-bimodule, while the proof in [10] of the equality $M \cap N_0 = N' \cap M$ is valid for any subset $M$ of $N \vee N_0$.)

**Theorem 1.** Let $N, N_0 \subset B(H)$ be commuting von Neumann algebras with common center $Z$, and suppose $N_0$ is type I. Then

1. $\Phi \to \Phi|_N$ establishes a one-to-one correspondence between normal conditional expectations $\Phi : N \vee N_0 \to N_0$ and normal conditional expectations from $N$ to $Z$;
2. the normal conditional expectations $N \vee N_0 \to N_0$ separate the points of $N \vee N_0$;
3. for every normal conditional expectation $\Phi : N \vee N_0 \to N_0$ and every $\sigma$-weakly closed subspace $S$ of $N \vee N_0$ that is an $N$-bimodule, we have

$$\Phi(S) = S \cap N_0 = N' \cap S.$$ 

The proof of our main result (Theorem 2) uses Fubini products. So before proving the result we recall some of the basic definitions and results that we need. (See [6] and [7] for more details.) If $N$ and $R$ are von Neumann algebras, then for each $\phi \in N_*$, the right slice map $R_\phi : N \otimes R \to R$ is defined by $R_\phi = \phi \otimes id_R$. Note that $R_\phi$ is the unique $\sigma$-weakly continuous linear map from $N \otimes R \to R$ satisfying

$$R_\phi(a \otimes b) = \phi(a)b, \quad a \in N, b \in R.$$ 

Similarly, if $\psi \in R_*$, the left slice map $L_\psi : N \otimes R \to N$ is defined by $L_\psi = id_N \otimes \psi$. If $S$ and $T$ are $\sigma$-weakly closed subspaces of $N$ and $R$ respectively, then the Fubini product of $S$ and $T$ is the $\sigma$-weakly closed subspace of $N \otimes R$ defined by

$$F(S, T) = \{x \in N \otimes R : R_\phi(x) \in T \text{ and } L_\psi \in S \text{ for all } \phi \in N_* \text{ and } \psi \in R_*\}.$$

It is shown in [6] that $F(S, T)$ only depends on $S$ and $T$ and not on the containing von Neumann algebras. So if $N \subset B(H)$ and $R \subset B(K)$, we can replace $N$ by $B(H)$ and $R$ by $B(K)$ in the definition of $F(S, T)$. It is easy to see that we always have $S \otimes T \subset F(S, T)$. A $\sigma$-weakly closed subspace $S$ of $B(H)$ is said to have Property $S_\sigma$ if $F(S, T) = S \otimes T$ for all $\sigma$-weakly closed subspaces $T \subset B(K)$ for any Hilbert space $K$ (as shown in [6], it suffices to consider the case where $K$ is separable and infinite dimensional). If $1_{B(H)} \in S \subset B(H)$ and $1_{B(K)} \in T \subset B(K)$ are $\sigma$-weakly closed subspaces, then $(S \otimes T)' = F(S', T')$ (see [6]). In particular, if $N$ and $R$ are von Neumann algebras, then with $S = N'$ and $T = R'$, we get that

$$F(N, R) = (N' \otimes R')' = N'' \otimes R'' = N \otimes R$$

by Tomita’s Commutation Theorem, and, in fact, the above calculation shows that Tomita’s Commutation Theorem is equivalent to the statement that $F(N, R) =$
N \hat{\otimes} R$ whenever $N$ and $R$ are von Neumann algebras. However, not only are there subspaces $S$ and $T$ such that $F(S, T) \neq S \hat{\otimes} T$, but it is shown in [7] that for each of the types $II_1$, $II_\infty$ and $II_1 \lambda$, $0 \leq \lambda \leq 1$, there is a separably acting factor $N$ of that type and a unital $\sigma$-weakly closed subalgebra $T$ of $B(K)$ (where $K$ is a separable infinite-dimensional Hilbert space) such that $N \hat{\otimes} T \neq F(N, T)$. (We cannot choose $N$ to be type I, because all type I von Neumann algebras have Property $S_\sigma$ [6, Theorem 1.9].) It is also shown in [7] that for each of $S$ and $T$ such that $F(S, T) \neq S \hat{\otimes} T$, but it is shown in [7] that for each of the types $II_1$, $II_\infty$ and $II_1 \lambda$, $0 \leq \lambda \leq 1$, there is a separably acting factor $N$ of that type and a unital $\sigma$-weakly closed subalgebra $T$ of $B(K)$ (where $K$ is a separable infinite-dimensional Hilbert space) such that $N \hat{\otimes} T \neq F(N, T)$. (We cannot choose $N$ to be type I, because all type I von Neumann algebras have Property $S_\sigma$ [6, Theorem 1.9].) It is also shown in [7] that a subspace $S \subset B(H)$ has Property $S_\sigma$ if and only if it has the weak* OAP. (A subspace $S \subset B(H)$ has the weak* OAP if there is a net $\{\Phi_i\}$ of normal finite rank completely bounded maps from $S$ to $S$ such that for any Hilbert space $K$ and for any $x \in S \hat{\otimes} B(K)$, we have that $(\Phi_i \otimes id_{B(K)})(x) \to x$ $\sigma$-weakly. See [3] for a detailed treatment of the weak* OAP and other approximation properties of operator spaces.)

**Theorem 2.** Let $N$ be a von Neumann algebra with center $Z(N)$, let $R$ be a von Neumann algebra, and suppose $S$ is a $\sigma$-weakly closed subspace of $N \hat{\otimes} R$ that is a $N \hat{\otimes} C_1 R$-bimodule such that

$$S \cap (Z(N) \hat{\otimes} R) = Z(N) \hat{\otimes} T$$

for some $\sigma$-weakly closed subspace $T$ of $R$. Then

$$N \hat{\otimes} T \subset S \subset F(N, T).$$

In particular, if $N$ has the weak* OAP, or if $S$ is a von Neumann algebra, then

$$S = N \hat{\otimes} T.$$

**Proof.** We can assume that $R \subset B(H)$ for some Hilbert space $H$. Then $S \subset N \hat{\otimes} R \subset N \hat{\otimes} B(H)$, and $S$ is a $N \hat{\otimes} C_1(B(H))$-bimodule. Moreover, we have that $S \cap (Z(N) \hat{\otimes} B(H)) = (S \cap (N \hat{\otimes} R)) \cap (Z(N) \hat{\otimes} B(H)) = S \cap ((N \hat{\otimes} R) \cap (Z(N) \hat{\otimes} B(H))) = S \cap (Z(N) \hat{\otimes} R) = Z(N) \hat{\otimes} T$. Let $\phi \in N_*$ be a normal state, let $\psi = \phi|_{Z(N)}$, and let $e$ be the support projection of $\psi$. Then there exists a positive normal $Z(N)$-module map $\Phi_1$ from $N$ into $Z(N)$ such that $\phi = \psi \circ \Phi_1$ and such that $\Phi_1(1) = e$ (see Theorem 1 in [5] or Proposition 1.4 in [9]). It follows from this result (applied to all normal states of $N$) and a Zorn’s lemma argument that the normal conditional expectations from $N$ onto $Z(N)$ separate the points of $N$. Now let $\Phi_2$ be any normal conditional expectation from $N$ onto $Z(N)$, and define $\Phi$ on $N$ by $\Phi(x) = \Phi_1(x) + (1 - e)\Phi_2(x)$. Then $\Phi$ is a normal conditional expectation from $N$ onto $Z(N)$ such that $\phi = \psi \circ \Phi$. Since $\Phi \otimes id_{B(H)}$ is a normal conditional expectation from $(N \hat{\otimes} C_1(B(H)) \cap (Z(N) \hat{\otimes} B(H))) = N \hat{\otimes} B(H)$ to $Z(N) \hat{\otimes} B(H)$, it follows from Theorem 1 that

$$(\Phi \otimes id_{B(H)})(x) \in S \cap (Z(N) \hat{\otimes} B(H)) = Z(N) \hat{\otimes} T \text{ for all } x \in S,$$

and so $R_\psi((\Phi \otimes id_{B(H)})(x)) \in T$. But

$$R_\psi((\Phi \otimes id_{B(H)})(x)) = (\psi \circ id_{B(H)})(\Phi \otimes id_{B(H)})(x) = \phi \circ id_{B(H)}(x) = R_\phi(x),$$

and so $R_\phi(x) \in T$ whenever $x \in S$ and $\phi$ is a normal state of $N$. Since the map $\phi \to R_\phi$ is linear, and since every $\phi \in N_*$ is a linear combination of normal states, we also have that $R_\phi(x) \in T$ for all $\phi \in N_*$. Moreover, since $S \subset N \hat{\otimes} R$, $L_\psi(x) \in N$ for all $\psi \in R_*$. Hence $S \subset F(N, T)$. By assumption, $Z(N) \hat{\otimes} T \subset S$ and $S$ is an $N \hat{\otimes} C_1 R$-bimodule; so $(N \hat{\otimes} C_1 R)(Z(N) \hat{\otimes} T) = N \hat{\otimes} T \subset S$. Hence $Z(N) \hat{\otimes} T \subset S \subset F(N, T)$. As noted above, if $N$ has the weak* OAP, then $N$ has Property $S_\sigma$; so in this case we always have $N \hat{\otimes} T = F(N, T)$. Finally, if $S$ is a von Neumann algebra,
then \( S \cap (Z(N) \bar{\otimes} R) = Z(N) \bar{\otimes} T \) is also a von Neumann algebra, and so \( T \) is a von Neumann algebra. Hence we again have \( F(N,T) = N \bar{\otimes} T \).

If \( N \) is a factor, then
\[
S \cap (Z(N) \bar{\otimes} R) = S \cap (C1_N \bar{\otimes} R) \subset C1_N \bar{\otimes} R = \{ 1_N \otimes b : b \in R \}.
\]
So we always have \( S \cap (Z(N) \bar{\otimes} R) = C1_N \bar{\otimes} T = Z(N) \bar{\otimes} T \), where \( T \) is a \( \sigma \)-weakly closed subspace of \( R \). (In fact, \( T = \{ R_\phi(x) : x \in S \cap (Z(N) \bar{\otimes} R) \) and \( \phi \in N_* \} \).) Thus we get the following extension of the result of Ge and Kadison.

**Theorem 3.** Let \( N \) be a factor, let \( R \) be a von Neumann algebra, and suppose \( S \) is a \( \sigma \)-weakly closed subspace of \( N \bar{\otimes} R \) that is an \( N \bar{\otimes} C1_R \)-bimodule. Then there is a \( \sigma \)-weakly closed subspace \( T \) of \( R \) such that
\[
N \bar{\otimes} T \subset S \subset F(N,T).
\]
In particular, if \( N \) has the weak* OAP, or if \( S \) is a von Neumann algebra, then \( S = N \bar{\otimes} T \).

Our next result shows that the requirement in Theorem 3 that \( N \) have the weak* OAP is necessary for all \( \sigma \)-weakly closed \( N \bar{\otimes} C1_{B(H)} \)-bimodules of \( N \bar{\otimes} B(H) \) to split.

**Theorem 4.** Suppose \( N \) is a factor without the weak* OAP, and that \( H \) is a separable infinite-dimensional Hilbert space. Then there is a unital (but not self-adjoint) \( \sigma \)-weakly closed subalgebra \( S \) of \( N \bar{\otimes} B(H) \) that is an \( N \bar{\otimes} C1_{B(H)} \)-bimodule, but which does not split.

**Proof.** Since \( N \) is a von Neumann algebra without the weak* OAP, and so without Property \( S_\sigma \), there is an abelian subalgebra \( A \) of \( B(H) \) that is reflexive (and so \( \sigma \)-weakly closed and unital) such that \( F(N,A) \neq N \bar{\otimes} A \). (See Remark 1.1 in [4], noting that von Neumann algebras are always reflexive.) Let \( S = F(N,A) \). Then \( S \) is a \( \sigma \)-weakly closed unital subalgebra of \( N \bar{\otimes} B(H) \). (See Remark 3.16 in [7].) For \( \phi \in N_* \) and for \( a \in N \), define the normal linear functionals \( a\phi \) and \( \phi a \) on \( N \) by
\[
a\phi(b) = \phi(ab) \quad \text{and} \quad \phi a(b) = \phi(ba)
\]
for all \( b \in N \).

Then for all \( \phi \in N_* \), \( a \in N \), and \( x \in N \bar{\otimes} B(H) \), we have
\[
R_\phi((a \otimes 1_{B(H)}) x) = R_{a\phi}(x) \quad \text{and} \quad R_\phi(x(a \otimes 1_{B(H)})) = R_{\phi a}(x),
\]
as can be easily seen (first check the equalities in (3) for \( x \) of the form \( x = b \otimes c \)). It follows immediately from (3) that if \( x \in F(N,A) \), then \((a \otimes 1_{B(H)}) x \) and \( x(a \otimes 1_{B(H)}) \) are also in \( F(N,A) \) for all \( a \in N \); so \( S \) is an \( N \bar{\otimes} C1_{B(H)} \)-bimodule. Finally, note that if \( S = N \bar{\otimes} T \) for some \( \sigma \)-weakly closed subspace \( T \) of \( B(H) \), then \( T = A \), which contradicts \( F(N,A) \neq N \bar{\otimes} A \). Hence \( S \) does not split (and so is not selfadjoint). \( \square \)

Combining Theorems 3 and 4, we get the following result.

**Theorem 5.** A factor \( N \) has Property \( S_\sigma \) if and only if for every von Neumann algebra \( R \) (or just \( R = B(H) \)), every \( \sigma \)-weakly closed \( N \bar{\otimes} C1_R \)-bimodule of \( N \bar{\otimes} R \) splits.

The von Neumann subalgebra version of Theorem 3 in [4] is used to show the following result ([11, Theorem H]): if \( N \) and \( R \) are factors, if \( N \) is injective, and if \( P \) is a maximal injective von Neumann subalgebra of \( R \), then \( N \bar{\otimes} P \) is a maximal
injective von Neumann subalgebra of $N \hat{\otimes} R$. Injectivity makes sense for norm closed subspaces of $B(H)$, when these are viewed as operator spaces. By definition, an operator space $V$ is injective if for any operator space $W_0 \subset W$, every complete contraction from $W_0$ to $V$ has a completely contractive extension from $W$ to $V$. Since $B(H)$ is injective, it is not hard to show that an operator space $V \subset B(H)$ is injective if and only if there is a completely contractive projection from $B(H)$ onto $V$. (See [3] Proposition 4.1.6.) Using ideas from the proof of Theorem H, we obtain the next result.

**Theorem 6.** Let $N$ be an injective factor, and let $R$ be a von Neumann algebra. Then

1. if $T$ is a maximal $\sigma$-weakly closed injective subspace of $R$, then $N \hat{\otimes} T$ is a maximal $\sigma$-weakly closed injective $\hat{\otimes} C_1 R$-bimodule of $N \hat{\otimes} R$;

2. if $A$ is a maximal $\sigma$-weakly closed injective unital (but not necessarily self-adjoint) subalgebra of $R$, then $N \hat{\otimes} A$ is a maximal $\sigma$-weakly closed injective unital subalgebra of $N \hat{\otimes} R$.

**Proof.** (1) We first show that $N \hat{\otimes} T$ is injective. Since $T$ is $\sigma$-weakly closed, it is dual as a Banach space. So it follows from Theorem 1.3 in [1] that there is an injective von Neumann algebra $P$ and a projection $e$ in $P$ such that $T$ is completely isometric and weak* homeomorphic to $eP(1_P - e)$, and hence $N \hat{\otimes} T$ is completely isometric and weak* homeomorphic to $N \hat{\otimes} eP(1_P - e)$. Let $f = 1_N \otimes e$, and let $M = N \hat{\otimes} P$. Then $1_M - f = 1_N \otimes (1_P - e)$, and so $N \hat{\otimes} eP(1_P - e) = fM(1_M - f)$. Since $N$ and $P$ are injective von Neumann algebras, so is $M$ ([8, Proposition 10.24]). Thus another application of Theorem 1.3 in [1] shows that $N \hat{\otimes} T$ is injective. Now suppose that $S$ is a $\sigma$-weakly closed injective $\hat{\otimes} C_1 R$-bimodule of $N \hat{\otimes} R$, and that $N \hat{\otimes} T \subset S$. Since $N$ is injective, it has Property $S_\sigma$. (It is shown in [6] that semidiscrete von Neumann algebras have Property $S_\sigma$, and, as shown in [13], semidiscreteness is equivalent to injectivity.) Hence by Theorem 3, there is a $\sigma$-weakly closed subspace $T_0$ of $R$ such that $S = N \hat{\otimes} T_0$, and since $C_1 N \hat{\otimes} T \subset S = F(N, T_0)$, $T \subset T_0$. Suppose $N \subset B(K)$ for a Hilbert space $K$. Then $S \subset B(K) \hat{\otimes} B(H) = B(K \hat{\otimes} H)$ is injective. So there is a completely contractive projection $\Psi$ from $B(K \hat{\otimes} H)$ onto $S$. Let $\phi$ be a normal state of $N$, and let $\Phi$ be the unique normal linear map from $N \hat{\otimes} R$ onto $C_1 N \hat{\otimes} R$ such that $\Phi(a \otimes b) = \phi(a)1_N \otimes b$ for all $a \in N$ and $b \in R$. Then it is easily checked that $\Phi \circ \Psi$ is a completely contractive projection from $B(K \hat{\otimes} H)$ onto $C_1 N \hat{\otimes} T_0$. Hence $C_1 N \hat{\otimes} T_0$ is injective, from which it follows easily that $T_0$ is injective. By the maximality of $T$, $T = T_0$, and so $S = N \hat{\otimes} T$. Hence $N \hat{\otimes} T$ is a maximal $\sigma$-weakly closed injective $\hat{\otimes} C_1 R$-bimodule of $N \hat{\otimes} R$.

(2) Let $B$ be a $\sigma$-weakly closed injective unital subalgebra of $N \hat{\otimes} R$ such that $N \hat{\otimes} A \subset B$. Then since $A$ is unital, $N \hat{\otimes} C_1 R \subset B$, and so, since $B$ is an algebra, $B$ is an $N \hat{\otimes} C_1 R$-bimodule. Thus, by the same argument as in (1), $B = N \hat{\otimes} A_0$, where $A \subset A_0$ and $A_0$ is an injective $\sigma$-weakly closed subspace of $R$. Moreover, if $a$ and $b$ are elements of $A_0$, then $1_N \otimes a$ and $1_N \otimes b$ are elements of the algebra $B$; so $1_N \otimes ab = (1_N \otimes a)(1_N \otimes b) \in B = F(N, A_0)$, and so $ab \in A_0$. Hence $A_0$ is also a unital subalgebra of $R$, and so by the maximality of $A$, $A = A_0$, and so $B = N \hat{\otimes} A$. Thus $N \hat{\otimes} A$ is a maximal $\sigma$-weakly closed injective unital subalgebra of $N \hat{\otimes} R$. □

We conclude this paper with a result that generalizes the main result in [16] (and Theorem 3.3(1) in [15]). First we need some terminology. If $A$ and $B$ are C*-algebras, we denote by $A \otimes B$ the minimal or spatial tensor product of $A$ and
B. (If $A \subset B(H)$ and $B \subset B(K)$, then $A \otimes B$ is just the norm closed linear span in $B(H) \otimes B(K)$ of $\{a \otimes b : a \in A, b \in B\}$.) Slice maps and Fubini products can be defined as in the von Neumann algebra case, with the $\sigma$-weak topology replaced by the norm topology. (So, for example, if $\phi \in A^*$, then the right slice map $R_{\phi}$ is the unique norm bounded linear map from $A \otimes B$ to $B$ satisfying $R_{\phi}(a \otimes b) = \phi(a)b$ for all $a \in A$ and $b \in B$.) In contrast to the case of von Neumann algebras, the Fubini product depends on $A$ and $B$. (See section 5 of [7].) Since our first C*-algebra will be fixed, we use the following special notation: if $A$ and $B$ are C*-algebras, and if $T$ is a norm closed subspace of $B$, then

$$F_B(T) = \{x \in A \otimes B : R_{\phi}(x) \in T \text{ for all } \phi \in A^*\}.$$  

Note that since every bounded linear functional on $A$ is a linear combination of states, we also have that $F_B(T)$ is the norm closed linear span of $\{x \in A \otimes B : R_{\phi}(x) \in T \text{ for all states } \phi \text{ of } A\}$. A C*-algebra $A$ is said to have Property $S$ for subspaces (see [7]) if for every pair $(T, B)$, where $B$ is a C*-algebra and $T$ is a norm closed subspace of $B$, we have $F_B(T) = A \otimes T$. (A C*-algebra is said to have Property $S$ if $F_B(C) = A \otimes C$ whenever $C$ is a C*-subalgebra of a C*-algebra $B$. Property $S$ was introduced by Wassermann in [11], and was the inspiration for both Property $S_\alpha$ and Property $S$ for subspaces. In [12], Wassermann showed that there are C*-algebras that do not have Property $S$.) It is shown in [7] that a C*-algebra has Property $S$ for subspaces if and only if for any C*-algebra $B$ there is a net $\{\Phi_i\}$ of finite rank completely bounded maps from $A$ to $A$ such that $(\Phi_i \otimes id_B)(x) \to x$ in the point norm topology of $A \otimes B$. A C*-algebra with this approximation property is said to have the strong operator approximation property, or strong OAP (see p. 185 of [2] or Chapter 11 of [3]). Wassermann showed in [14] that nuclear C*-algebras have Property $S$, and his proof of this result (Proposition 10 in [14]) shows that nuclear C*-algebras have Property $S$ for subspaces (and also shows directly that nuclear C*-algebras have the strong OAP).

The main result (Theorem) in [16] is that if $A$, $D$ and $C$ are unital C*-algebras, if $A$ is simple and nuclear, and if $A \otimes C_{1_D} \subset C \subset A \otimes D$, then $C = A \otimes B$ for some C*-subalgebra $B$ of $D$. This result was obtained independently by Zacharias, using elementary maps (Theorem 3.3(1) of [15]), where it was explicitly observed that the only property of nuclearity that Zsido used was the fact that a nuclear C*-algebra has Property $S$. So the result remains valid if we just assume that $A$ is simple and has Property $S$. Our last theorem follows from a straightforward modification of the proof of [16, Theorem], or the proof of [15, Theorem 3.3(1)]; so the proof is omitted.

**Theorem 7.** Let $A$ and $B$ be unital C*-algebras, with $A$ simple, and suppose $S$ is a norm closed subspace of $A \otimes B$ that is also an $A \otimes C_{1_B}$-bimodule (under multiplication). Then

1. there is a norm closed subspace $T$ of $B$ such that $A \otimes T \subset S \subset F_B(T)$;
2. if $A$ has Property $S$ for subspaces, or, equivalently, if $A$ has the strong OAP (in particular, if $A$ is nuclear), then $S = A \otimes T$ for some norm closed subspace $T$ of $B$.

**Remark.** Theorem 3.3(2) in [15] can also be generalized to the subspace case, with the obvious modifications to the proof. The general result is: Suppose $A$ and $B$ are unital C*-algebras and that $A$ contains a unital abelian C*-subalgebra $D$ that has the pure state extension property (each pure state on $D$ extends uniquely to
Suppose \( C \) is a norm closed subspace of \( A \otimes B \) that is an \( A \otimes C \) \( B \)-bimodule, and that \( B_\omega = B_0 \) for each pure state \( \omega \) on \( A \) and some norm closed subspace \( B_0 \) of \( B \). (See [15] for the definition of \( B_\omega \). Note that if we just assume \( C \) is an \( A \otimes C \) \( B \)-bimodule, then the proofs of (i) and (iii) of Proposition 3.2 in [15] are still valid.) Then \( A \otimes B_0 \subset C \subset F_B(B_0) \), and thus \( C = A \otimes B_0 \) if \( A \) has Property S for subspaces.

References


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