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A REMARK ON GLOBAL EXISTENCE FOR SMALL INITIAL DATA OF THE MINIMAL SURFACE EQUATION IN MINKOWSKIAN SPACE TIME

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ABSTRACT. We show that the nonlinear wave equation corresponding to the minimal surface equation in Minkowski space time has a global solution for sufficiently small initial data.

1. INTRODUCTION

We show that the nonlinear wave equation corresponding to the minimal surface equation in Minkowski space time

$$\frac{\partial}{\partial t}\frac{\phi_t}{\sqrt{1+|\nabla_x\phi|^2-\phi_t^2}}-\sum_{i=1}^n\frac{\partial}{\partial x^i}\frac{\phi_i}{\sqrt{1+|\nabla_x\phi|^2-\phi_t^2}}=0 \tag{1.1}$$

where $\phi_i = \partial\phi/\partial x^i$, $\phi_t = \partial\phi/\partial t$, has global solutions for sufficiently small initial data

$$\phi\big|_{t=0}=\varepsilon f,\qquad \phi_t\big|_{t=0}=\varepsilon g, \tag{1.2}$$

$f \in C_0^\infty$ and $g \in C_0^\infty$, i.e., (1.1)-(1.2) has for fixed f and g a solution for all $t \geq 0$ if $\varepsilon > 0$ is sufficiently small. This is an interesting model in Lorentzian geometry proposed to me by Hamilton [Ha1]. It is also the equation for a membrane in field theory; see Hoppe [Ho1]. Also, Huisken and Struwe [HS1] have some recent results related to local existence for (1.1).

What makes the proof go through also in the physically important case of two space dimensions (ϕ itself corresponds to the third space dimension) is that the nonlinear terms satisfy the so-called "null condition" of Christodoulou [C1] and Klainerman [K2], [K4]. The purpose of this note is to present two simple proofs making use of the extra symmetries of the equation. The first proof uses a version of the method of [K2] that works also in two space dimensions. For equations in divergence form we can get a good L^2 estimate for the solution itself (see [L1]) that replaces the conformal energy estimate used in [K2]. The second proof uses a simplified version [C2] of the method of [C1], and it works also in one space dimension due to the fact that the equation satisfies a "double null condition"; see (1.5). The first proof does not work in the case of one space dimension but it has

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the advantage that it does not require compact support of initial data but merely some decay at infinity. Let

$$\Box = \partial_t^2 - \sum_{i=1}^{3} \partial_i^2 \qquad \text{and} \qquad Q_{00}(\phi,\psi) = \phi_t\psi_t - \sum_{i=1}^{n} \phi_i\psi_i \tag{1.3}$$

be a null form. We can write (1.1) as a wave equation with a right-hand side in divergence form

$$\Box\phi = \partial_t\big(\phi_t F(Q_{00}(\phi,\phi))\big) - \sum_{i=1}^{n} \partial_i\big(\phi_i F(Q_{00}(\phi,\phi))\big), \tag{1.4}$$

where $F(Q) = -1 + 1/\sqrt{1-Q}$. We can alternatively write it with null forms

$$\Box\phi = -\frac{Q_{00}\big(\phi, Q_{00}(\phi,\phi)\big)}{2(1-Q_{00}(\phi,\phi))}. \tag{1.5}$$

The idea behind the proofs is that we expect solutions of the nonlinear wave equation to decay like solutions of the linear homogeneous wave equation $\Box\phi = 0$; i.e., $|\partial^\alpha\phi| \le C\varepsilon(1+t)^{-(n-1)/2}$. This will make the right-hand sides of the nonlinear wave equations small for large t, and hence these equations will be close to the homogeneous case; so we can close the argument. What makes the proof go through in the lower-dimensional case is that for a null form there is an additional cancellation which leads to an additional decay of a factor of $(1+t)^{-1}$: $|\partial^\alpha Q(\phi,\phi)| \le C\varepsilon^2(1+t)^{-(n-1)-1}$.

2. The proof using vector fields

The idea of this argument is to exploit that solutions of linear wave equations $\Box v = 0$ satisfy the decay estimate; see, e.g., [Hö2],

$$|v(t,x)| \le C(f,g)\varepsilon(1+t+|x|)^{-(n-1)/2}(1+|t-|x||)^{-(n-1)/2}, \tag{2.1}$$

where $C(f,g)$ is a constant depending on some weighted Sobolev norm of initial data (f,g). We will use vector fields to obtain this kind of decay. For $(t,x) \in \mathbf{R}^{1+n}$ denote ∂_t by ∂_0 and ∂_{x_j} by ∂_j for $j = 1, ..., n$. Let

$$\Gamma_{jk} = \lambda_j x_j\partial_k - \lambda_k x_k\partial_j, \quad \text{for} \quad i \ne j, \qquad \text{and} \qquad \Gamma_{00} = \sum_{0}^{n} x_j\partial_j \tag{2.2}$$

where $\lambda = (1,-1,...,-1)$ and $x_0 = t$. Γ_{jk}, for $(j,k) \ne (0,0)$ are the vector fields associated with the Lorentz group that all commute with $\Box$, and Γ_{00} is the scaling vector fields whose commutator is $[\Gamma_{00}, \Box] = -2\Box$. Γ will symbolically stand for any of the vector fields Γ_{ij} or ∂_j , $i,j = 0,1,...,n$ and we will write Γ^I for a product of $|I|$ of such vector fields. We note that $[\partial_i, \Gamma_{jk}]$ is either 0 or else equal to $\pm\partial_l$ for some l. The operators $\{\Gamma_{jk}\}$ span the tangent space at every point where $t \ne |x|$. But when $t = |x|$ they only span the tangent space of the cone $t = |x|$; so we have (see [K1], [K2], [Ho1], [Hö2] or [L1])

$$|\partial\phi| \le C(|t-|x||)^{-1}\sum|\Gamma_{ij}\phi|, \qquad |\partial\phi| \le C(t+|x|)^{-1}\big(|x||\partial_t\phi| + \sum|\Gamma_{ij}\phi|\big), \tag{2.3}$$

where $|\partial w|^2 = \sum_0^n |\partial_j w|^2$. Now we also need to calculate the commutator of Γ with a null form Q. However, in these commutators null forms other than (1.3) will come up. For $0 \le i, j \le n$ and $i \neq j$, let

$$Q_{ij}(\phi,\psi) = (\partial_i\phi)\partial_j\psi - (\partial_j\phi)\partial_i\psi. \tag{2.4}$$

Let Q symbolically stand for any of the null forms (1.3) or (2.4). Then

$$\Gamma Q(\phi,\psi) = Q(\Gamma\phi,\psi) + Q(\phi,\Gamma\psi) + \sum a_{ij}Q_{ij}(\phi,\psi) \tag{2.5}$$

for some constants a_{ij}. For a null form we have better decay close to the light cone (see [K1], [K2])

$$|Q(\phi,\psi)| \le C(1+|t|+|x|)^{-1}\big(|\partial\phi||\Gamma\psi| + |\Gamma\phi||\partial\psi|\big) \tag{2.6}$$

where here $|\Gamma\phi|^2 = \sum_{|I|=1}|\Gamma^I\phi|^2$. The decay of ϕ will be obtained from the fact that $\Gamma^I\phi$ also satisfies nonlinear wave equations of the same form. Applying Γ^I to (1.4) we get

$$\begin{aligned}\Box\Gamma^I\phi = \sum_{j=0}^n \partial_j\Bigg(&\sum_{\substack{k\ge 3,\, |I_1|+\ldots+|I_k|\le|I|\\ |I_i|\le|I|/2,\, i<k,\ |I_k|\le|I|}} F_{jki_1\ldots i_kI_1\ldots I_k}(Q_{00}(\phi,\phi)) \\ &\times(\partial_{i_1}\Gamma^{I_1}\phi)(\partial_{i_2}\Gamma^{I_2}\phi)\cdots(\partial_{i_k}\Gamma^{I_k}\phi)\Bigg).\end{aligned} \tag{2.7}$$

Similarly, from (1.5) we get

$$\begin{aligned}&\Box\Gamma^I\phi + \frac{Q_{00}\big(\phi, Q_{00}(\phi,\Gamma^I\phi)\big)}{1+Q_{00}(\phi,\phi)} \\ &\quad= \sum_{\substack{k\ge 3,\, |I_1|+\ldots+|I_k|\le|I|+1,\\ |I_i|\le(|I|+1)/2,\, i<k,\ |I_k|\le|I|}} G_{ki_1\ldots i_kI_1\ldots I_k}(Q_{00}(\phi,\phi))(\partial_{i_1}\Gamma^{I_1}\phi) \\ &\qquad\times(\partial_{i_2}\Gamma^{I_2}\phi)\cdots(\partial_{i_k}\Gamma^{I_k}\phi)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\Box\Gamma^I\phi \\ &= \sum_{k\ge 3,\, |I_1|+\ldots+|I_k|\le|I|} H^{ki_0\ldots i_k}_{I_1\ldots I_k}(Q_{00}(\phi,\phi))Q_{i_0i_1}\Big(\Gamma^{I_1}\phi, Q_{i_2j_3}(\Gamma^{I_2}\phi,\Gamma^{I_3}\phi)\Big) \\ &\qquad\times Q_{i_4i_5}(\Gamma^{I_4}\phi,\Gamma^{I_5}\phi)\cdots Q_{i_{k-1}i_k}(\Gamma^{I_{k-1}}\phi,\Gamma^{I_k}\phi)\end{aligned} \tag{2.9}$$

where k is odd, and if $k=3$, then this is to be interpreted as the factor $Q_{i_4i_5}(\Gamma^{I_4}\phi, \Gamma^{I_5}\phi)$ being absent.

The proof will use the energy inequality applied to derivatives of the solution and some decay estimates. The energy inequality (see [K3] or [Hö2]) for a solution of

$$\Box\, w + \sum_{j,k=0}^n \gamma^{jk}(t,x)\partial_j\partial_k w = F, \qquad |\gamma| = \sum|\gamma^{jk}| \le \frac{1}{2} \tag{2.10}$$

says that

$$\begin{aligned}\|\partial w(t,\cdot)\|_{L^2} &\le 2\exp\big(\textstyle\int_0^t 2|\gamma'(\tau)|\,d\tau\big)\|\partial w(0,\cdot)\|_{L^2} \\ &\quad+ 2\int_0^t \exp\big(\textstyle\int_s^t 2|\gamma'(\tau)|\,d\tau\big)\|F(s,\cdot)\|_{L^2}\,ds,\end{aligned} \tag{2.11}$$

where $|\gamma'(t)| = \sum_{i,j,k} \sup |\partial_i \gamma^{jk}(t,\cdot)|$. It was not stated exactly like this in [Hö2], but (2.11) follows from the proof of the version there. As a consequence of the energy inequality, we also have the following estimate: If

$$\square\, w = \sum_0^n \partial_j F_j, \qquad w\big|_{t=0} = \varepsilon f, \quad w_t\big|_{t=0} = \varepsilon g, \tag{2.12}$$

then

$$||w(t,\cdot)||_{L^2} \le \sum_{j=0}^n \int_0^t ||F_j(s,\cdot)||_{L^2}\, ds + C(f,g,F_0(0,\cdot))m(t)\varepsilon \tag{2.13}$$

where $m(t) = 1$ if $n \ge 3$, $m(t) = \log(2+t)$ if $n = 2$ and $C(f,g,F_0(0,\cdot))$ stands for some constant depending on some weighted Sobolev norm of initial data f and g. The proof is a trick used in [L1]; if $\square v_j = F_j$, then $\square(w - \sum \partial_j v_j) = 0$. So $\|w(t,\cdot)\|_{L^2}$ is bounded by $\sum \|\partial v_j(t,\cdot)\|_{L^2}$, which can be estimated by the energy inequality (2.11), plus the norm for a solution of a linear homogeneous equation, which can be obtained from, e.g., (2.1). We will also need an $L^1 - L^\infty$ estimate of Hörmander [Hö1] (see also Klainerman [K1], [K2] for an earlier version and [L1] for a simple proof): The solution w of

$$\square\, w = F, \qquad w\big|_{t=0} = \varepsilon f, \quad w_t\big|_{t=0} = \varepsilon g \tag{2.14}$$

satisfies

(2.15)
$$|w(t,x)| \le C(1+t+|x|)^{-(n-1)/2}\Big(\sum_{|I|\le n-1} \int_0^t ||(\Gamma^I F)(s,\cdot)/ (1+s+|\cdot|)^{(n-1)/2}||_{L^1}\, ds + C(f,g)\varepsilon\Big).$$

Here the estimate for the linear homogeneous part, the second term, is (2.1). Whereas the proof of the energy inequality is merely integration by parts, the proofs of the decay estimates (2.15) and (2.1) require a detailed analysis of the fundamental solution or stationary phase.

Let $N \ge 2n+1$ and $\delta = 0$, if $n \ge 3$, and $0 < \delta < 1/2$ fixed, if $n = 2$. We will now prove that

$$M_1(t) = \sum_{|I|\le N} \|\partial \Gamma^I \phi(t,\cdot)\|_{L_2} \le K\varepsilon(1+t)^\delta, \tag{2.16}$$

$$M_2(t) = \sum_{|I|\le N} \|\Gamma^I \phi(t,\cdot)\|_{L_2} \le K\varepsilon(1+t)^\delta, \tag{2.17}$$

$$N_1(t) = \sum_{|J|\le (N+1)/2} \|\partial \Gamma^J \phi(t,\cdot)\|_{L^\infty} \le K\varepsilon(1+t)^{-(n-1)/2}, \tag{2.18}$$

$$N_2(t) = \sum_{|J|\le (N+1)/2+1} \|\Gamma^J \phi(t,\cdot)\|_{L^\infty} \le K\varepsilon(1+t)^{-(n-1)/2} \tag{2.19}$$

if K is sufficiently large and ε is sufficiently small. We observe that the bound for N_1 is a consequence of the bound for N_2 since, in particular, we can take one factor of $\Gamma = \partial$. (2.11) applied to (2.8) gives

$$\begin{aligned} M_1(t) &\le C\varepsilon \exp\big(\textstyle\int_0^t N_1(\tau)^2\, d\tau\big) \\ &\quad + \int_0^t \exp\big(\textstyle\int_s^t N_1(\tau)^2\, d\tau\big) C(N_1(s)) N_1(s)^2 M_1(s)\, ds \end{aligned} \tag{2.20}$$

and (2.13) applied to (2.7) gives

$$M_2(t) \le C\Big(C(f,g)m(t)\varepsilon + \int_0^t C(N_1(s))N_1(s)^2 M_1(s)\, ds\Big). \tag{2.21}$$

Finally, (2.15) applied to (2.9) using (2.6) and the Cauchy-Schwartz inequality gives

$$N_2(t) \le C(1+t)^{-(n-1)/2}\Big(C(f,g)\varepsilon + \int_0^t \frac{(N_1(s)+N_2(s))}{(1+s)^{(n-1)/2+1}}(M_1(s)+M_2(s))^2\, ds\Big) \tag{2.22}$$

if $(N+1)/2+1+n-1 \le N$, i.e., $N \ge 2n+1$. What will make the argument work also for $n=2$ is that the null condition gave an extra power of $(1+s)^{-1}$ in the integral (2.22). In fact, because we have a double null condition we actually have one more power, but we have no use of this here. The rest of the argument is now by continuity. We know that (2.16)–(2.19) are true for $t=0$ if K is large enough, and we know from the local existence theorem for hyperbolic equations (see, e.g., [Hö2]) that these quantities are continuous as long as they are bounded. We now assume that T_1 is the largest number such that (2.16)–(2.19) are true for $t \le T_1$ and show that these bounds together with (2.20)–(2.22) imply stronger bounds if K is sufficiently large and ε is sufficiently small. Hence by continuity we conclude that the bounds (2.16)–(2.19) must hold for $t \le T_2$ where $T_2 > T_1$, contradicting the maximality of T_1. To simplify notation, we now deal only with the most sensitive case $n=2$. Then

$$\begin{aligned} \exp\big(\textstyle\int_s^t N_1(\tau)^2\, d\tau\big) &\le \exp\big(K^2\varepsilon^2 \textstyle\int_s^t (1+\tau)^{-1}\, d\tau\big) \\ &= \exp\Big(K^2\varepsilon^2 \ln\big(\frac{1+t}{1+s}\big)\Big) = \Big(\frac{1+t}{1+s}\Big)^{K^2\varepsilon^2}. \end{aligned} \tag{2.23}$$

So it follows from (2.20) that

$$M_1(t) \le C\varepsilon(1+t)^{K^2\varepsilon^2} + \int_0^t C\varepsilon^3 K^3 \left(\frac{1+t}{1+s}\right)^{K^2\varepsilon^2}(1+s)^{\delta-1}\, ds \le K\varepsilon(1+t)^{\delta}/2 \tag{2.24}$$

if K is sufficiently large and ε is sufficiently small. Similarly, from (2.21) we get

$$M_2(t) \le C\varepsilon \log(2+t) + \int_0^t K^3\varepsilon^3(1+s)^{\delta-1}\, ds \le K\varepsilon(1+t)^{\delta}/2 \tag{2.25}$$

if K is sufficiently large and ε is sufficiently small. Finally, from (2.22) we get

$$N_2(t) \le C(1+t)^{-1/2}\Big(\varepsilon + \int_0^t K^3\varepsilon^3(1+s)^{2\delta-2}\, ds\Big) \le K\varepsilon(1+t)^{-1/2}/2 \tag{2.26}$$

if K is sufficiently large and ε is sufficiently small since $0 < \delta < 1/2$. This concludes the proof.

3. The proof using conformal inversion

We will reduce the global problem to a local problem, for which small data existence is known, using a conformal inversion or Kelvin transform. Let $\kappa : \mathbf{R}^{1+n} \ni (s,y) \to (t,x) \in \mathbf{R}^{1+n}$ and $\tilde{\phi}$ be defined by

$$\tilde{\phi} = \phi\circ\kappa\,\rho^{-\alpha}, \qquad x^i = \kappa^i(s,y) = y^i/\rho, \qquad \rho = s^2 - |y|^2, \qquad \alpha = \tfrac{n-1}{2}, \tag{3.1}$$

where $x^0 = t$ and $y^0 = s$. Let $m_{ij} = m^{ij}$ be the Minkowski metric $m_{00} = 1$, $m_{ii} = -1$ if $j \geq 1$ and $m_{ij} = 0$ if $i \neq j$. Then $\partial_i \kappa^k = \delta_i{}^k/\rho - 2y^j y^k m_{ij}/\rho^2$. If the metric in the x coordinates is m_{ij}, then the pull-back metric in the y coordinates is given by

$$g_{ij} = m_{kl}(\partial_i\kappa^k)\partial_j\kappa^l = m_{ij}/\rho^2, \qquad g^{ij} = m^{ij}\rho^2. \tag{3.2}$$

Then the norm is invariant: $g^{ij}(\partial_i\phi\circ\kappa)\partial_j\psi\circ\kappa = m^{ij}\phi_i\circ\kappa\,\psi_j\circ\kappa$ where $\partial_i\phi\circ\kappa = \partial\phi\circ\kappa/\partial y^i$ and $\phi_i = \partial\phi/\partial x^i$; i.e., if Q_{00} is the null form (1.3), then

$$Q_{00}(\phi,\psi)\circ\kappa = \rho^2 Q_{00}(\phi\circ\kappa\,,\psi\circ\kappa\,). \tag{3.3}$$

Expressing $\Box$ in the y coordinates we get

$$\begin{aligned}(\Box\phi)\circ\kappa &= \Box_g(\phi\circ\kappa) = (\det g)^{-1/2}\partial_i\big(g^{ij}(\det g)^{1/2}\partial_j\phi\circ\kappa\big)\\ &= \rho^{n+1}m^{ij}\partial_i\big(\rho^{-(n-1)}\partial_j\phi\circ\kappa\big).\end{aligned} \tag{3.4}$$

Since the operator $\Box_g - (n-1)R/(4n)$, where R is the scalar curvature, is conformally covariant and the scalar curvature in both cases vanishes, we obtain

$$\Box\tilde{\phi} = \rho^{-\alpha-2}(\Box\phi)\circ\kappa. \tag{3.5}$$

This also follows from (3.4) by an easy calculation using that $\Box\rho^{-\alpha} = 0$. Hence by (1.5), (3.5) and (3.3),

$$\Box\tilde{\phi} = -\rho^{-\alpha}\frac{Q_{00}\big(\rho^\alpha\tilde{\phi}, \rho^2 Q_{00}(\rho^\alpha\tilde{\phi},\rho^\alpha\tilde{\phi})\big)}{1-\rho^2 Q_{00}(\rho^\alpha\tilde{\phi},\rho^\alpha\tilde{\phi})}. \tag{3.6}$$

Since $m^{ij}(\partial_i\tilde{\psi})\partial_j\rho = 2(\Gamma_{00}\tilde{\psi})$, where $\Gamma_{00} = s\partial_s + y^i\partial_i$ and $\Gamma_{00}\rho = 2\rho$, we obtain

$$Q_{00}(\rho^\beta\tilde{\phi},\rho^\gamma\tilde{\psi}) = \rho^{\beta+\gamma-1}\Big(\rho Q_{00}(\tilde{\phi},\tilde{\psi}) + 2\beta\tilde{\phi}(\Gamma_{00}\tilde{\psi}) + 2\gamma\tilde{\psi}(\Gamma_{00}\tilde{\phi}) + 4\gamma\beta\tilde{\phi}\tilde{\psi}\Big). \tag{3.7}$$

Furthermore, $\Gamma_{00}Q_{00}(\tilde{\phi},\tilde{\phi}) = 2Q_{00}(\tilde{\phi},\Gamma_{00}\tilde{\phi}) - 2Q_{00}(\tilde{\phi},\tilde{\phi})$. Using (3.7) twice we see that we can factor out $\rho^{3\alpha}$ from $Q_{00}\big(\rho^\alpha\tilde{\phi},\rho^2 Q_{00}(\rho^\alpha\tilde{\phi},\rho^\alpha\tilde{\phi})\big)$. So (3.6) has the general form

$$\begin{aligned}\Box\tilde{\phi} = &-\rho^{2\alpha}\frac{\rho^2 Q_{00}\big(\tilde{\phi},Q_{00}(\tilde{\phi},\tilde{\phi})\big) + \rho c_1\tilde{\phi}Q_{00}(\tilde{\phi},\Gamma_{00}\tilde{\phi}) + \rho c_2\tilde{\phi}^2\Gamma_{00}^2\tilde{\phi}}{1-\rho^{2\alpha}\big(\rho^2 Q_{00}(\tilde{\phi},\tilde{\phi}) + 4\alpha\rho(\alpha\tilde{\phi}^2+\tilde{\phi}\Gamma_{00}\tilde{\phi})\big)}\\ &+\rho^{2\alpha}\frac{\rho(c_3\tilde{\phi}+c_4\Gamma_{00}\tilde{\phi})Q_{00}(\tilde{\phi},\tilde{\phi}) + \tilde{\phi}(c_5\tilde{\phi}^2 + c_6\tilde{\phi}\Gamma_{00}\tilde{\phi} + c_7(\Gamma_{00}\tilde{\phi})^2)}{1-\rho^{2\alpha}\big(\rho^2 Q_{00}(\tilde{\phi},\tilde{\phi}) + 4\alpha\rho(\alpha\tilde{\phi}^2+\tilde{\phi}\Gamma_{00}\tilde{\phi})\big)}\end{aligned} \tag{3.8}$$

for some constants $c_1, ..., c_7$. We can write this as

$$\Box\tilde{\phi} = \rho^{n-1}\big(\rho B^{ij}(s,y,\tilde{\phi},\partial\tilde{\phi})\partial_i\partial_j\tilde{\phi} + T(s,y,\tilde{\phi},\partial\tilde{\phi})\big), \qquad \rho = s^2 - |y|^2, \tag{3.9}$$

where B^{ij} and T are smooth functions of (s,y), $\tilde{\phi}$ and $\partial_i\tilde{\phi}$, for $i = 0,...,n$ vanishing to second respectively third order at $(\tilde{\phi},\partial\tilde{\phi}) = (0,0)$; $|B^{ij}(s,y,\tilde{\phi},\partial\tilde{\phi})| \leq C(|\tilde{\phi}| + |\partial\tilde{\phi}|)^2$ and $|T(s,y,\tilde{\phi},\partial\tilde{\phi})| \leq C(|\tilde{\phi}| + |\partial\tilde{\phi}|)^3$. The importance of (3.9) is that it is nonsingular at $\rho = 0$.

Having derived the transformation of the equation let us explain in what region it is applied. The transformation κ maps the interior of the forward light cone $\{(s,y);\, s>|y|\}$ onto itself $\{(t,x);\, t>|x|\}$ and the boundary $\{(s,y);\, s=|y|\}$ to infinity. Its inverse $\kappa^{-1}(t,x) = \big(t/(t^2-|x|^2), x/(t^2-|x|^2)\big) = (s,y)$ maps forward light cones $\Lambda_c = \{(t,x);\, t-|x|\ge c>0\}$ into backward light cones $\tilde{\Lambda}_c = \{(s,y);\, s+|y|\le 1/c,\, s-|y|\ge 0\}$ intersected with $\{(s,y);\, s\ge|y|\}$. The inverse is smooth on Λ_c since $t^2-|x|^2\ge c^2>0$ there. Note also that hyperplanes $\tilde{\mathcal{H}}_b = \{(s,y);\, s = 1/2b,\, |y|\le 1/2b\}$ are transformed to hyperboloids $\mathcal{H}_b = \{(t,x);\, (t-b)^2-|x|^2 = b^2\}$. We will now use this transformation in Λ_c which by the inverse is mapped to the compact region $\tilde{\Lambda}_c$ where we can use the standard local existence theorem for (3.9) to, after transforming back, obtain a global solution of our original equation (1.5) in Λ_c. Let us now explain how we can reduce it to a problem in Λ_c. First by scaling, $\phi_a(t,x) = \phi(at,ax)/a$ is a solution of (1.5) if ϕ is; so we may assume that data (1.2) are supported in the set $\{x; |x|\le 1\}$. Second, we can translate the solution in the time direction so that initial conditions are attained when $t=a>1$. By the local existence theorem we have, if initial data when $t=a$ are sufficiently small, a solution ϕ to (1.5), for $|t-a|\le a$, and it is as small as we wish there. Furthermore, by Huygens' principle the solution vanishes outside the forward light cone Λ_{a-1}, for $t\ge a$. We now want to show that the solution ϕ of (1.5) extends to a solution in all of Λ_{a-1} for $t\ge a$ by showing that we have a local solution $\tilde{\phi}$ of (3.9) in the compact set $\tilde{\Lambda}_{a-1}$. We will now describe how to obtain initial conditions for (3.9). We pick a particular hyperboloid $\mathcal{H}_b$, where $b=(a-1/a)/2>0$, that intersects with the plane $t=a$ exactly when $|x|=1$, and that is transformed by the inverse of κ to the plane $\tilde{\mathcal{H}}_b$. The intersection of $\mathcal{H}_b$ with the support of ϕ, $\mathcal{H}_b^1 = \{(t,x)\in\mathcal{H}_b;\, |x|\le 1\}$, is contained in the forward light cone Λ_{a-1} intersected with the set where $t\le a$. Hence we have a smooth solution ϕ on $\mathcal{H}_b^1$ which is as small as we wish there and which vanishes on $\mathcal{H}_b\setminus\mathcal{H}_b^1$. It follows that ϕ and its derivatives restricted to $\mathcal{H}_b$ are transformed onto smooth initial conditions for $\tilde{\phi}$ on $\tilde{\mathcal{H}}_b$, which are as small as we wish. The local existence theorem with these initial conditions on $\tilde{\mathcal{H}}_b$ gives us a smooth solution $\tilde{\phi}$ of (3.9) in $\{(s,y); s\le 1/(2b),\, s\ge|y|\}$ and hence in $\tilde{\Lambda}_{a-1}$ for $s\le 1/(2b)$, if initial conditions on $\tilde{\mathcal{H}}_b$ are sufficiently small. Transforming back gives us a global smooth solution ϕ of (1.5) in Λ_{a-1} and hence for all $t\ge a$. This concludes the proof.

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