PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 4, Pages 1029–1035
S 0002-9939(03)07291-5
Article electronically published on December 1, 2003

SEBESTYÉN MOMENT PROBLEM: THE MULTI-DIMENSIONAL CASE

DAN POPOVICI AND ZOLTÁN SEBESTYÉN

(Communicated by Joseph A. Ball)

To the memory of Gyula Farkas

ABSTRACT. Given a family $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ of vectors in a Hilbert space $\mathcal{H}$ we characterize the existence of a family of commuting contractions $\mathbf{T} = \{T_\omega\}_{w\in\Omega}$ on $\mathcal{H}$ having regular dilation and such that

$$h_{\mathbf{n}} = \mathbf{T}^{\mathbf{n}} h_{\mathbf{0}}, \quad \mathbf{n} \in \mathbb{Z}_+^\Omega.$$

The theorem is a multi-dimensional analogue for some well-known operator moment problems due to Sebestyén in case $|\Omega| = 1$ or, recently, to Găvruţă and Păunescu in case $|\Omega| = 2$.

Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{L}(\mathcal{H})$ the $\mathbf{C}^*$-algebra of bounded linear operators on $\mathcal{H}$.

Z. Sebestyén proposed in [5] the following operator moment problem:

Problem A. *Given a sequence $\{h_n\}_{n\geq 0}$ of vectors in $\mathcal{H}$, under what condition does there exist a contraction T on $\mathcal{H}$ such that*

$$h_n = T^n h_0, \quad n \geq 0\ ?$$

The two-dimensional analogue of this problem was proposed by Găvruţă and Păunescu [2]:

Problem B. *Given a doubly indexed sequence $\{h_m^n\}_{m,n\geq 0}$ of vectors in $\mathcal{H}$, under what condition does there exist a commuting pair of contractions (S, T) on $\mathcal{H}$ such that*

$$h_m^n = S^m T^n h_0^0, \quad m, n \geq 0\ ?$$

The solution given by Z. Sebestyén to the first problem was the following:

Theorem A ([5]). *Let $\{h_n\}_{n\geq 0} \subset \mathcal{H}$. Problem A has a solution if and only if*

$$\|\sum_{m,m'} c_{m,m'} h_{m+m'}\|^2 \leq \sum_{m,m',n,n'} c_{m,m'}\overline{c_{n,n'}}\langle h_{(m-n)^+ + m'}, h_{(m-n)^- + n'}\rangle,$$

for every finite double sequence $\{c_{n,n'}\}_{n,n'\geq 0}$ of complex numbers (we used, for any integer m, the notation $m^+ = \max\{m, 0\}$ and $m^- = -\min\{m, 0\}$).

Received by the editors October 22, 2002.
2000 *Mathematics Subject Classification.* Primary 47A57, 47A20.
Key words and phrases. Sebestyén operator moment problem, multi-contraction, regular dilation, positive definite function.

The solution to the second problem uses the theory of regular dilations:

Theorem B ([2])**.** *Let $\{h_m^n\}_{m,n\geq 0}$ be a sequence of vectors that spans $\mathcal{H}$. Problem B has a solution (S,T) having regular dilation if and only if*

$$\|\sum_{m,m',n,n'} c_{m,m'}^{n,n'} h_{m+m'}^{n+n'}\|^2 \leq \sum_{\substack{m,n,p,q\\ m',n',p',q'}} c_{m,m'}^{n,n'}\overline{c_{p,p'}^{q,q'}}\langle h_{(m-p)^++m'}^{(n-q)^++n'}, h_{(m-p)^-+p'}^{(n-q)^-+q'}\rangle,$$

for every finite double sequence $\{c_{m,m'}^{n,n'}\}_{m,m',n,n'\geq 0}$ of complex numbers and the following regularity condition holds:

$$\|h_{m+1}^n\|^2 + \|h_m^{n+1}\|^2 \leq \|h_m^n\|^2 + \|h_{m+1}^{n+1}\|^2, \quad m,n\geq 0.$$

Before we formulate our multi-dimensional analogue of the previous moment problems we need to introduce some notions.

Let Ω be a nonempty set. Introduce $\mathbb{Z}^\Omega$ (respectively $\mathbb{Z}_+^\Omega$) as the set of $\mathbb{Z}$-valued (respectively $\mathbb{Z}_+$-valued) functions on Ω having finite support. With pointwise defined operation, $\mathbb{Z}^\Omega$ becomes an additive abelian group with unit $\mathbf{0}$ (the null function) having $\mathbb{Z}_+^\Omega$ as a subsemigroup. We identify some particular elements of $\mathbb{Z}_+^\Omega$: for $\mathbf{m}=\{m_\omega\}_{\omega\in\Omega}$, let $\mathbf{m}^+=\{m_\omega^+\}_{\omega\in\Omega}$ and $\mathbf{m}^-=\{m_\omega^-\}_{\omega\in\Omega}$; for any finite subset $v\subset\Omega$, let $e(v)=\{\varphi_v(\omega)\}_{\omega\in\Omega}$, φ_v being the characteristic function of v on Ω.

In what follows the term *multi-contraction* will denote a family $\mathbf{T}=\{T_\omega\}_{\omega\in\Omega}$ of commuting contractions on $\mathcal{H}$. If $\mathbf{T}=\{T_\omega\}_{\omega\in\Omega}$ is a multi-contraction and $\mathbf{m}=\{m_\omega\}_{\omega\in\Omega}\in\mathbb{Z}_+^\Omega$ we define, as usual, $\mathbf{T}^\mathbf{m} := \prod_{\omega\in\Omega} T_\omega^{m_\omega}$. A multi-unitary operator $\mathbf{U}$ (multi-contraction consisting of unitary operators) acting on a Hilbert space $\mathcal{K}\supset\mathcal{H}$ is said to be a *unitary dilation* of a multi-contraction $\mathbf{T}$ if

$$\mathbf{T}^\mathbf{m} h = P_\mathcal{H}\mathbf{U}^\mathbf{m} h, \quad \mathbf{m}\in\mathbb{Z}_+^\Omega,\ h\in\mathcal{H},$$

$P_\mathcal{H}$ being the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. $\mathbf{U}$ is called *regular* if, moreover,

$$(\mathbf{T}^{\mathbf{m}^-})^*\mathbf{T}^{m^+}h = P_\mathcal{H}\mathbf{U}^\mathbf{m} h, \quad \mathbf{m}\in\mathbb{Z}^\Omega,\ h\in\mathcal{H}$$

holds. Here $\mathbf{U}$ is multi-unitary; so the definition of $\mathbf{U}^\mathbf{m}$ can be obviously extended to all of $\mathbf{m}\in\mathbb{Z}^\Omega$.

Remark. Regular dilations have been introduced and studied by S. Brehmer [1]. We should also mention here the paper of B. Sz.-Nagy [6], which brought more light on the subject.

By the classical theorem of M. A. Neumark [4] on positive definite functions on a group (see also [7]) we can state that a multi-contraction $\mathbf{T}$ has regular dilation if and only if the operator-valued function

$$\mathbb{Z}^\Omega \ni \mathbf{m} \mapsto T(\mathbf{m}) := (\mathbf{T}^{\mathbf{m}^-})^*\mathbf{T}^{\mathbf{m}^+} \in \mathcal{L}(\mathcal{H})$$

is positive definite on $\mathbb{Z}^\Omega$, i.e.,

$$\sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}_+^\Omega} \langle T(\mathbf{n}-\mathbf{m})h_\mathbf{n}, h_\mathbf{m}\rangle \geq 0, \tag{1}$$

for any finite family $\{h_\mathbf{n}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ of vectors in $\mathcal{H}$.

Equivalently ([1]; cf. also [3], [7]), $\mathbf{T}$ has regular dilation if and only if

$$\sum_{v\subset u}(-1)^{|v|}(T^{e(v)})^*T^{e(v)} \geq 0, \tag{2}$$

for any finite subset u of Ω.

Our terminology and approach in the following will be close to the one used in the monograph [7].

We are now ready to formulate the multi-dimensional analogue of the Sebestyén operator moment problem (Problem A):

Problem C. *Given a family $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ of vectors in $\mathcal{H}$, under what condition does there exist a multi-contraction $\mathbf{T}$ on $\mathcal{H}$ such that*

$$h_{\mathbf{n}} = \mathbf{T}^{\mathbf{n}} h_{\mathbf{0}}, \quad \mathbf{n} \in \mathbb{Z}_+^\Omega \ ?$$

Our solution applies to the theory of regular dilations (more precisely the results stated in the Remark) and also naturally extends the methods in [5].

Lemma. *Let $\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega}$ and $\{d_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega}$ be given families of complex numbers. The following conditions are equivalent:*

(i)

$$\sum_{\mathbf{n}'} c_{\mathbf{n},\mathbf{n}'} h_{\mathbf{n}'+\mathbf{p}} = \sum_{\substack{\mathbf{n}'\in\mathbb{Z}_+^\Omega \\ v\subset\Omega \ \textit{finite}}} (-1)^{|v|} d_{\mathbf{n}+e(v),\mathbf{n}'} h_{\mathbf{n}'+e(v)+\mathbf{p}}, \quad \mathbf{n},\mathbf{p}\in\mathbb{Z}_+^\Omega,$$

for any family $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega} \subset \mathcal{H}$ with finite support;

(ii)

$$\sum_{\mathbf{m}\geq\mathbf{n};\mathbf{m}'} c_{\mathbf{m},\mathbf{m}'} h_{\mathbf{m}+\mathbf{m}'-\mathbf{n}+\mathbf{p}} = \sum_{\mathbf{n}'} d_{\mathbf{n},\mathbf{n}'} h_{\mathbf{n}'+\mathbf{p}}, \quad \mathbf{n},\mathbf{p}\in\mathbb{Z}_+^\Omega,$$

for any family $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega} \subset \mathcal{H}$ with finite support;

(iii)

$$c_{\mathbf{n},\mathbf{n}'} = \sum_{v:e(v)\leq\mathbf{n}'} (-1)^{|v|} d_{\mathbf{n}+e(v),\mathbf{n}'-e(v)}, \quad \mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega;$$

(iv)

$$d_{\mathbf{n},\mathbf{n}'} = \sum_{\mathbf{q}\leq\mathbf{n}'} c_{\mathbf{n}+\mathbf{q},\mathbf{n}'-\mathbf{q}}, \quad \mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega.$$

Moreover, $\{c_{\mathbf{n},\mathbf{n}'}\}$ has finite support if and only if $\{d_{\mathbf{n},\mathbf{n}'}\}$ has finite support.

Proof. $(i) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iv)$ are easy observations (just compute the coefficient of each $h_{\mathbf{n}}$, $\mathbf{n}\in\mathbb{Z}_+^\Omega$).

For $(iii) \Leftrightarrow (iv)$ observe firstly that

$$\begin{aligned}\sum_{\mathbf{q}\leq\mathbf{n}'} \sum_{v:e(v)\leq\mathbf{n}'-\mathbf{q}} &(-1)^{|v|} d_{\mathbf{n}+\mathbf{q}+e(v),\mathbf{n}'-\mathbf{q}-e(v)} \\ &\overset{\mathbf{q}+e(v)=\mathbf{p}}{=} \sum_{\mathbf{p}:\mathbf{p}\leq\mathbf{n}'} \sum_{v:e(v)\leq\mathbf{p}} (-1)^{|v|} d_{\mathbf{n}+\mathbf{p},\mathbf{n}'-\mathbf{p}} = d_{\mathbf{n},\mathbf{n}'}, \quad \mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega,\end{aligned}$$

since $\sum_{v:e(v)\le\mathbf{p}}(-1)^{|v|}=0$ for $\mathbf{p}=\mathbf{0}$ or 1 for $\mathbf{p}\ne\mathbf{0}$. Conversely,

$$\begin{aligned}\sum_{v:e(v)\le\mathbf{n}'}(-1)^{|v|}\sum_{\mathbf{q}\le\mathbf{n}'-e(v)}c_{\mathbf{n}+e(v)+\mathbf{q},\mathbf{n}'-e(v)-\mathbf{q}}&\\ \overset{\mathbf{q}+e(v)=\mathbf{p}}{=}&\sum_{v:e(v)\le\mathbf{n}'}(-1)^{|v|}\sum_{\mathbf{p}:e(v)\le\mathbf{p}\le\mathbf{n}'}c_{\mathbf{n}+\mathbf{p},\mathbf{n}'-\mathbf{p}}\\ =&\sum_{\mathbf{p}\le\mathbf{n}'}c_{\mathbf{n}+\mathbf{p},\mathbf{n}'-\mathbf{p}}\sum_{v:e(v)\le\mathbf{p}}(-1)^{|v|}=c_{\mathbf{n},\mathbf{n}'},\quad \mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega,\end{aligned}$$

by the same argument as above. □

The solution for Problem C now follows:

Theorem C. *Let $\{h_\mathbf{n}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ be a given multi-indexed family of vectors in $\mathcal{H}$. The following conditions are equivalent:*
(a) *Problem C has a solution $\mathbf{T}=\{T_\omega\}_{\omega\in\Omega}$ having regular dilation;*
(b)

$$\|\sum_{\mathbf{m},\mathbf{m}'}c_{\mathbf{m},\mathbf{m}'}h_{\mathbf{m}+\mathbf{m}'}\|^2\le\sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'}c_{\mathbf{m},\mathbf{m}'}\overline{c_{\mathbf{n},\mathbf{n}'}}\langle h_{(\mathbf{m}-\mathbf{n})^++\mathbf{m}'},h_{(\mathbf{m}-\mathbf{n})^-+\mathbf{n}'}\rangle,$$

for every finite family $\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega}$ of complex numbers;
(c)

$$\sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'}c_{\mathbf{m},\mathbf{m}'}\overline{c_{\mathbf{n},\mathbf{n}'}}\langle h_{(\mathbf{m}-\mathbf{n})^++\mathbf{m}'},h_{(\mathbf{m}-\mathbf{n})^-+\mathbf{n}'}\rangle\ge 0,$$

for every finite family $\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^\Omega}$ of complex numbers;
(d)

$$\sum_{v\subset u}(-1)^{|v|}\|\sum_{\mathbf{n}}c_\mathbf{n}h_{\mathbf{n}+e(v)}\|^2\ge 0,$$

for every finite subset $u\subset\Omega$ and finite family $\{c_\mathbf{n}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ of complex numbers.

Proof. (b) ⇒ (c) is obvious. Replace $\{c_{\mathbf{n},\mathbf{n}'}\}$ in (c) by the finite family $\{d_{\mathbf{n},\mathbf{n}'}\}$ given by the previous Lemma (iii). The inequality in (c) becomes

$$\begin{aligned}\sum_{\mathbf{m},\mathbf{n}}\langle\sum_{v,\mathbf{m}'}(-1)^{|v|}d_{\mathbf{m}+e(v),\mathbf{m}'}h_{\mathbf{m}'+e(v)+(\mathbf{m}-\mathbf{n})^+},&\\ &\sum_{w,\mathbf{n}'}(-1)^{|w|}d_{\mathbf{n}+e(w),\mathbf{n}'}h_{\mathbf{n}'+e(w)+(\mathbf{m}-\mathbf{n})^-}\rangle\ge 0,\end{aligned}$$

by the same Lemma (i). Change variables $\mathbf{m}\leftrightarrow\mathbf{m}+e(v)$ and $\mathbf{n}\leftrightarrow\mathbf{n}+e(w)$ to obtain

$$\begin{aligned}(3)\qquad\sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'}\sum_{\substack{v:e(v)\le\mathbf{m}\\ w:e(w)\le\mathbf{n}}}(-1)^{|v|+|w|}d_{\mathbf{m},\mathbf{m}'}\overline{d_{\mathbf{n},\mathbf{n}'}}\langle h_{\mathbf{m}'+e(v)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^+},&\\ &h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^-}\rangle\ge 0.\end{aligned}$$

For $\mathbf{p}=\{p_\omega\}_{\omega\in\Omega}\in\mathbb{Z}^\Omega$, define $\pi(\mathbf{p})=\{\omega\in\Omega\mid p_\omega>0\}$. Any finite set $v\subset\Omega$ with $e(v)\le\mathbf{m}$ (or equivalently $v\subset\pi(\mathbf{m})$) can be partitioned by $v=v'\cup v''$ with $v'\subset\pi(\mathbf{m})\setminus\pi(\mathbf{m}-\mathbf{n}+e(w))$ and $v''\subset\pi(\mathbf{m})\cap\pi(\mathbf{m}-\mathbf{n}+e(w))$.

Observe that, for $\mathbf{m} \neq \mathbf{n}$ (suppose $(\mathbf{m}-\mathbf{n})^+ \neq \mathbf{0}$, i.e., $\pi(\mathbf{m}-\mathbf{n}) \neq \emptyset$) and fixed $\mathbf{m}', \mathbf{n}', w$:

$$\begin{aligned}
&\sum_{v\subset\pi(\mathbf{m})} (-1)^{|v|} \langle h_{\mathbf{m}'+e(v)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^+}, h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^-}\rangle \\
&= \sum_{v'\subset\pi(\mathbf{m})\setminus\pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v'|} \sum_{v''\subset\pi(\mathbf{m})\cap\pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v''|} \\
&\quad \times \langle h_{\mathbf{m}'+e(v')+e(v'')+(\mathbf{m}-e(v')-e(v'')-\mathbf{n}+e(w))^+}, h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v')-e(v'')-\mathbf{n}+e(w))^-}\rangle \\
&= \sum_{v'\subset\pi(\mathbf{m})\setminus\pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v'|} \left[\sum_{v''\subset\pi(\mathbf{m})\cap\pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v''|}\right] \\
&\quad \times \langle h_{\mathbf{m}'+e(v')+(m-e(v')-\mathbf{n}+e(w))^+}, h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v')-\mathbf{n}+e(w))^-}\rangle = 0,
\end{aligned}$$

the term in brackets being null ($\pi(\mathbf{m}) \cap \pi(\mathbf{m}-\mathbf{n}+e(w))$ contains $\pi(\mathbf{m}-\mathbf{n})$, which is non-void).

By this observation the positivity condition (3) becomes

$$\sum_{\mathbf{m},\mathbf{m}',\mathbf{n}'} \sum_{v,w\subset\pi(\mathbf{m})} (-1)^{|v|+|w|} d_{\mathbf{m},\mathbf{m}'} \overline{d_{\mathbf{m},\mathbf{n}'}} \langle h_{\mathbf{m}'+e(v\cup w)}, h_{\mathbf{n}'+e(v\cup w)}\rangle \geq 0$$

or, equivalently,

$$\sum_{\mathbf{m}} \left[\sum_{u\subset\pi(\mathbf{m})} (-1)^{|u|} \|\sum_{\mathbf{m}'} d_{\mathbf{m},\mathbf{m}'} h_{\mathbf{m}'+e(u)}\|^2\right] \geq 0$$

since $\sum_{v,w:v\cup w=u} (-1)^{|v|+|w|} = (-1)^{|u|}$.

The family $\{d_{\mathbf{m},\mathbf{m}'}\}$ with finite support being arbitrary, every sum in the brackets above must be positive or zero. The implication $(c) \Rightarrow (d)$ is proved.

$(d) \Rightarrow (b)$ Rewrite (d) in the form:

$$\sum_{\mathbf{m}\geq 0} \left[\sum_{u\subset\pi(\mathbf{m})} (-1)^{|u|} \|\sum_{\mathbf{m}'} d_{\mathbf{m},\mathbf{m}'} h_{\mathbf{m}'+e(u)}\|^2\right] \geq \|\sum_{\mathbf{m}'} d_{\mathbf{0},\mathbf{m}'} h_{\mathbf{m}'}\|^2,$$

for any finite subset $u \subset \Omega$ and finite family $\{d_{\mathbf{m},\mathbf{m}'}\}$ of complex numbers. Replace $\{d_{\mathbf{m},\mathbf{m}'}\}$ by the finite family $\{c_{\mathbf{m},\mathbf{m}'}\}$ given by Lemma (iv). Similar calculations as in the previous implication and further use of the Lemma lead us to condition (b).

$(a) \Rightarrow (d)$ By (a) the solution to Problem C is a multi-contraction $\mathbf{T}$ having regular dilation. Use the equalities $h_{\mathbf{n}} = \mathbf{T}^{\mathbf{n}} h_{\mathbf{0}}, \mathbf{n} \in \mathbb{Z}_+^{\Omega}$ in (2) to finally obtain (d).

$(b) \Rightarrow (a)$ Let $\mathcal{F}_0$ be the linear space of all finite families $\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^{\Omega}}$ of complex numbers. Endow $\mathcal{F}_0$ with the semi-inner product (in view of (b)):

$$\langle\{c_{\mathbf{n},\mathbf{n}'}\}, \{d_{\mathbf{n},\mathbf{n}'}\}\rangle_{\mathcal{F}_0} := \sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'} c_{\mathbf{m},\mathbf{m}'} \overline{d_{\mathbf{n},\mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'}\rangle.$$

Factorize $\mathcal{F}_0$ with respect to the null space of $\langle\cdot,\cdot\rangle_{\mathcal{F}_0}$. Denote by $\mathcal{F}$ its Hilbert space completion and by $\mathcal{H}'$ the span of $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^{\Omega}}$.

Define $V : \mathcal{F} \to \mathcal{H}'$,

$$V\{c_{\mathbf{n},\mathbf{n}'}\} := \sum_{\mathbf{n},\mathbf{n}'} c_{\mathbf{n},\mathbf{n}'} h_{\mathbf{n}+\mathbf{n}'}, \quad \{c_{\mathbf{n},\mathbf{n}'}\} \in \mathcal{F}_0.$$

By (a), V can be extended to a contraction on $\mathcal{F}$ into $\mathcal{H}'$. Moreover, by easy computations,

$$V^* h_{\mathbf{k}} = \{\delta^{(\mathbf{0},\mathbf{k})}_{(\mathbf{n},\mathbf{n}')}\}_{\mathbf{n},\mathbf{n}'} \in \mathbb{Z}_+^\Omega, \quad \mathbf{k} \in \mathbb{Z}_+^\Omega$$

(δ denotes here the Kronecker symbol).

For fixed $\omega \in \Omega$, define U_ω on $\mathcal{F}_0$ by

$$U_\omega\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'} := \{c_{\mathbf{n}+e(\{w\}),\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'}.$$

U_ω can be extended on $\mathcal{F}$, and it is not hard to see that its adjoint U_ω^* is isometric. Let

$$T'_\omega = VU_\omega^* V^* \in \mathcal{L}(\mathcal{H}').$$

Each T_ω is a contraction on $\mathcal{H}'$. Moreover,

$$T'_\omega h_{\mathbf{k}} = VU_\omega^* V^* h_{\mathbf{k}} = VU_\omega^*\{\delta^{(\mathbf{0},\mathbf{k})}_{(\mathbf{n},\mathbf{n}')}\}_{\mathbf{n},\mathbf{n}'} = V\{\delta^{(e(\{w\}),\mathbf{k})}_{(\mathbf{n},\mathbf{n}')}\}_{\mathbf{n},\mathbf{n}'} = h_{\mathbf{k}+e(\{w\})}, k \in \mathbb{Z}_+^\Omega$$

and, consequently,

$$T'_\omega T'_{\omega'} h_{\mathbf{n}} = h_{\mathbf{n}+e(\{\omega,\omega'\})} = T'_{\omega'} T'_\omega h_{\mathbf{n}}, \quad \{\omega,\omega'\} \subset \Omega,\ \mathbf{n} \in \mathbb{Z}_+^\Omega.$$

$\mathbf{T}' = \{T'_\omega\}_{\omega\in\Omega}$ is a multi-contraction on $\mathcal{H}'$ and, since

$$\mathbf{T}'^{\mathbf{n}} h_{\mathbf{0}} = h_{\sum_{\omega\in\Omega} n_\omega e(\{\omega\})} = h_{\mathbf{n}}, \quad \mathbf{n} \in \mathbb{Z}_+^\Omega,$$

$\mathbf{T}'$ is a solution for Problem C. In addition, by (c), for any $x_{\mathbf{m}} = \sum_{\mathbf{m}'} c_{\mathbf{m},\mathbf{m}'} h_{\mathbf{m}'} \in \mathcal{H}'$ $(\mathbf{m} \in \mathbb{Z}_+^\Omega)$,

$$\begin{aligned}
&\sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}_+^\Omega} \langle [\mathbf{T}'^{(\mathbf{m}-\mathbf{n})^-}]^* \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^+} x_{\mathbf{m}}, x_{\mathbf{n}}\rangle \\
&\qquad = \sum_{\mathbf{m},\mathbf{n}} \langle \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^+} x_{\mathbf{m}}, \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^-} x_{\mathbf{n}}\rangle \\
&\qquad = \sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'} c_{\mathbf{m},\mathbf{m}'} \overline{c_{\mathbf{n},\mathbf{n}'}} \langle \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^+} h_{\mathbf{m}'}, \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^-} h_{\mathbf{n}'}\rangle \\
&\qquad = \sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'} c_{\mathbf{m},\mathbf{m}'} \overline{c_{\mathbf{n},\mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'}\rangle \geq 0.
\end{aligned}$$

The positivity condition (1) is then verified and $\mathbf{T}'$ has a regular dilation. Extend $\mathbf{T}'$ on $\mathcal{H}$ by $T_\omega = T'_\omega \oplus 0_{\mathcal{H}''}$ $(\omega \in \Omega)$ corresponding to the orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$. $\mathbf{T} = \{T_\omega\}_{\omega\in\Omega}$ is a multi-contraction on $\mathcal{H}$ that extends $\mathbf{T}'$. $\mathbf{T}$ is still a solution of Problem C and, since

$$\begin{aligned}
&\langle \sum_{v\subset u} (-1)^{|v|} (\mathbf{T}^{e(v)})^* \mathbf{T}^{e(v)} (h' + h''), h' + h''\rangle \\
&= \sum_{v\subset u} (-1)^{|v|} \|\mathbf{T}'^{e(v)} h'\|^2 + \|h'\|^2 + \|h''\|^2 \geq 0, \quad h' \in \mathcal{H}', h'' \in \mathcal{H}'', u \subset \Omega \text{ finite},
\end{aligned}$$

$\mathbf{T}$ has regular dilation (following condition (2) for regularity). □

In particular, for Ω with $|\Omega| = 2$, we can improve the solution to Problem B. The condition that $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ spans $\mathcal{H}$ is no longer necessary, and the regularity condition is actually a consequence of the positivity condition. More precisely, we have the following theorem.

Theorem B+. *Let $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ be a given sequence of vectors in $\mathcal{H}$. The following conditions are equivalent:*

(*a*) *Problem B has a solution $\mathbf{T} = (T_1, T_2)$ having regular dilation;*

(*b*)

$$\|\sum_{\mathbf{m},\mathbf{m}'} c_{\mathbf{m},\mathbf{m}'} h_{\mathbf{m}+\mathbf{m}'}\|^2 \leq \sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'} c_{\mathbf{m},\mathbf{m}'} \overline{c_{\mathbf{n},\mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'} \rangle,$$

for every finite family $\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^2}$ of complex numbers;

(*c*)

$$\sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'} c_{\mathbf{m},\mathbf{m}'} \overline{c_{\mathbf{n},\mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'} \rangle \geq 0,$$

for every finite family $\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^2}$ of complex numbers;

(*d*)

$$\|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}+e_1}\|^2 + \|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}+e_2}\|^2 \leq \|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}}\|^2 + \|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}+e_1+e_2}\|^2,$$

$$\|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}+e_1}\| \leq \|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}}\| \quad \text{and} \quad \|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}+e_2}\| \leq \|\sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}}\|,$$

for every finite family $\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^2}$ of complex numbers (we use here $e_1 = (1,0)$ and $e_2 = (0,1)$).

References

[1] S. Brehmer, Über vertauschbare Kontraktionen des Hilbertschen Raumes, *Acta Sci. Math. Szeged* **22** (1961), 106-111. MR **24:**A1023a

[2] P. Găvruţă and D. Păunescu, Sebestyén's moment problem and regular dilations, *Acta Math. Hungar.* **94** (2002), 223-232. MR **2003g:**47014

[3] I. Halperin, Sz.-Nagy – Brehmer dilations, *Acta Sci. Math. Szeged* **23** (1962), 279-289. MR **27:**6127a

[4] M. A. Neumark, Positive definite operator functions on a commutative group, *Bulletin Acad. Sci. URSS Sér. Math.* **7** (1943), 237-244. MR **5:**272c

[5] Z. Sebestyén, Moment theorems for operators on Hilbert space, *Acta Sci. Math. Szeged* **44** (1982), 165-171. MR **84f:**47004

[6] B. Sz.-Nagy, Bemerkungen zur vorstehenden Arbeit des Herrn S. Brehmer, *Acta Sci. Math. Szeged* **22** (1961), 112-114. MR **24:**A1023b

[7] B. Sz.-Nagy and C. Foiaş, Harmonic analysis of operators on Hilbert space, *Akadémiai Kiadó*, Budapest, North-Holland, Amsterdam, London, 1970. MR **43:**947

Department of Mathematics, University of the West, Ro-1900 Timişoara, Bd. V. Pârvan 4, Romania

E-mail address: popovici@math.uvt.ro

Department of Applied Analysis, Loránd Eötvös University, H-1117 Budapest, Pázmány Péter sétány 1/C, Hungary

E-mail address: sebesty@cs.elte.hu