ON GROUP OPERATIONS ON HOMOGENEOUS SPACES

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(Communicated by Alan Dow)

Abstract. It is proved that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

It is well known that not all infinite groups admit a non-discrete group topology (see, for example, [1, §9]). However, every infinite group admits a non-discrete zero-dimensional topology with continuous left shifts [3], [4], and every countably infinite group admits a non-discrete zero-dimensional topology with continuous shifts and inversion [6]. It is well known also that not all homogeneous spaces admit a structure of a topological group and even a structure of a group with continuous left shifts (see, for example, [1, §10]). The aim of this note is to prove that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

We begin with the Boolean version of this result.

Theorem 1. Let $X$ be a countably infinite homogeneous regular space. Then there is a Boolean group operation $+$ on $X$ with continuous shifts.

To prove Theorem 1 we need the following lemma.

Lemma. Let $X$ be a countably infinite homogeneous regular space and let $x, y \in X$, $x \neq y$. Then there is a clopen neighborhood $U$ of $x$ and a homeomorphism $h : X \to X$ such that $h(x) = y$, $h(U) = X \setminus U$ and $h^2 = id_X$.

Proof. Enumerate $X$ as $\{x_n : n < \omega\}$. Since $X$ is homogeneous, there is a homeomorphism $g_0 : X \to X$ with $g_0(x) = y$. Since $X$ is countable and regular, therefore zero-dimensional, one may choose a clopen neighborhood $U_0$ of $x$ with $U_0 \cap g_0(U_0) = \emptyset$. Put $X_0 = U_0 \cup g_0(U_0)$ and define $h_0 : X_0 \to X_0$ by

$$h_0(x) =
\begin{cases}
g_0(x) & \text{if } x \in U_0, \\
g_0^{-1}(x) & \text{if } x \in g_0(U_0).
\end{cases}$$

If $X_0 = X$, put $U = U_0$ and $h = h_0$. Otherwise, choose the first element $x_{n_1}$ in the sequence $\{x_n : n < \omega\}$ with $x_{n_1} \notin X_0$ and pick any $y_{n_1} \in X \setminus (X_0 \cup \{x_{n_1}\})$. Let $g_1 : X \to X$ be a homeomorphism with $g_1(x_{n_1}) = y_{n_1}$ and let $U_1$ be a clopen neighborhood of $y_{n_1}$ with $U_1 \cap g_1(U_1) = \emptyset$.

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neighborhood of \( x_{n_1} \) with \( U_1 \cap g_1(U_1) = \emptyset \) and \( U_1 \cup g_1(U_1) \subseteq X \setminus X_0 \). Put \( X_1 = X_0 \cup U_1 \cup g_1(U_1) \) and extend \( h_0 \) to \( h_1 : X_1 \to X_1 \) by

\[
h_1(x) = \begin{cases} 
  h_0(x) & \text{if } x \in X_0, \\
  g_1(x) & \text{if } x \in U_1, \\
  g_1^{-1}(x) & \text{if } x \in g_1(U_1).
\end{cases}
\]

If \( X_1 = X \), put \( U = U_0 \cup U_1 \) and \( h = h_1 \). Otherwise, choose the first element \( x_{n_2} \) in the sequence \( \{x_n : n < \omega \} \) with \( x_{n_2} \notin X_1 \), pick any \( y_{n_2} \in X \setminus (X_1 \cup \{x_{n_2}\}) \), and so forth.

If \( X_n = X \) for some \( n \), put \( U = \bigcup_{i \leq n} U_i \) and \( h = h_n \). Otherwise, put \( U = \bigcup_{n < \omega} U_n \) and \( h = \bigcup_{n < \omega} h_n \). \( \square \)

Proof of Theorem 1. We can assume that \( X \) is not discrete. Pick any element \( u \in X \) and enumerate the set \( X \setminus \{u\} \) as \( \{x_n : n < \omega \} \). By the Lemma, we can construct a decreasing sequence \( \{U_n : n < \omega \} \) of clopen neighborhoods of \( u \) with \( U_0 = X \) and a sequence of homeomorphisms \( h_0 : U_n \to U_n \) such that \( h_n(U_{n+1}) = U_n \setminus U_{n+1} \) and \( h_n^2 = \text{id}_{U_n} \). By induction on \( n \), it is easy to verify that for each \( n < \omega \) the subsets \( h_n^0 \cdots h_n^\varepsilon(U_{n+1}) \), where \( \varepsilon \in \{0,1\} \), form a partition of \( X \) and every \( x \in X \) can be uniquely written in the form \( x = h_n^0 \cdots h_n^\varepsilon(y) \), where \( y \in U_{n+1} \). We construct sequences \( \{U_n : n < \omega \} \) and \( \{h_n : n < \omega \} \) satisfying, in addition, the condition \( x_n \in B_n = \{h_n^0 \cdots h_n^\varepsilon(u) : \varepsilon \in \{0,1\}, i \leq n\} \). To see that this can be done, assume that \( x_n \notin B_{n-1} \). Then \( x_n \in h_n^0 \cdots h_n^{\varepsilon_n-1}(y_n) \) for some \( \varepsilon_n \in \{0,1\} \) and \( y_n \in U_n \). Choose \( h_n \) with \( h_n(u) = y_n \).

Now let \( x, y \in X \). By the construction, there exists large enough \( n \) such that \( x \) and \( y \) can be uniquely written in the form \( x = h_n^0 \cdots h_n^\varepsilon(u) \), \( y = h_n^0 \cdots h_n^\delta(u) \). Put \( x + y = h_n^{\varepsilon + \delta} \cdots h_n^\delta(u) \). It is clear that the operation is well defined and that \( (X, +) \) is a Boolean group with zero \( u \). We note that for every \( z \in U_{n+1} \), \( x + z = h_n^{\varepsilon + \delta} \cdots h_n^\delta(z) \). To check that the left shifts in \( (X, +) \) are continuous, let \( U \) be a neighborhood of \( x \). Choose a neighborhood \( V \) of \( y \) such that \( V \subseteq U_{n+1} \) and \( h_n^{\varepsilon + \delta} \cdots h_n^\delta(V) \subseteq U \). Then \( W = h_n^{\varepsilon + \delta} \cdots h_n^\delta(V) \) is a neighborhood of \( y \) and for every \( z \in V \), \( x + h_n^{\varepsilon + \delta} \cdots h_n^\delta(z) = x + y + z = h_n^{\varepsilon + \delta} \cdots h_n^\delta(z) \in U \); so \( x + W \subseteq U \). \( \square \)

Next we need the Local Isomorphism Theorem. It is close to [5, Theorem 2] (see also [2, Lemma 7.4]).

A space with a group operation is called a left topological group if all left shifts are continuous. A space \( X \) with a partial binary operation \( \cdot \) and a distinguished element \( 1 \) is called a local left topological group, if for each element \( x \in X \) there is an open neighborhood \( U_x \) of \( 1 \) such that

1. For any \( y \in U_x \), \( x \cdot y \) is defined, \( x \cdot 1 = x \), \( x \cdot U_x \) is an open neighborhood of \( x \), and a mapping \( U_x \ni y \mapsto x \cdot y \in x \cdot U_x \) is a homeomorphism;
2. \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \) if \( y \in U_x \), \( z \in U_{x \cdot y} \cap U_y \), \( y \cdot z \in U_x \).

For a local left topological group, from this point on, we write \( x \cdot y \) we mean \( y \in U_x \) and when we write \( x \cdot U \), where \( U \) is a neighborhood of \( 1 \), we mean \( U \subseteq U_x \).

A basic example of a local left topological group is an open neighborhood of the identity of a left topological group.

Let \( X \) and \( Y \) be local left topological groups. A map \( f : X \to Y \) is called a local homomorphism if \( f(1_X) = 1_Y \) and for every \( x \in X \) there exists a neighborhood \( U_x \)
of $I_X$ such that $f(zx) = f(x)f(z)$ for all $z \in U_x$. A bijective local homomorphism $f$ is called a local isomorphism if $f^{-1}$ is also a local homomorphism. We observe that every open bijective local homomorphism is a local isomorphism.

**Theorem 2.** All countably infinite non-discrete regular left topological groups are local isomorphic.

**Proof.** Let $X$ be the countably infinite Boolean group $\bigoplus \omega \mathbb{Z}_2$ endowed with the direct sum topology, and let $Y$ be an arbitrary countably infinite non-discrete regular left topological group. We shall define a local isomorphism $f : X \to Y$.

Let $F$ be the semigroup of words on the letters 0 and 1 with empty word $\emptyset$, and let $F'$ be the subsemigroup of $F$ of words including $\emptyset$, in which the last letter is 1. We define a bijection $X \ni x \mapsto w(x) \in F'$ as follows. If $0 \neq x = (\varepsilon_n)_{n<\omega}$ and $m = \max\{n < \omega : \varepsilon_n = 1\}$, we put $w(x) = \varepsilon_0 \cdots \varepsilon_m$. If $x = 0$, we put $w(x) = \emptyset$.

For every $w \in F$, $|w|$ denotes the length of $w$. For every $n < \omega$, put $W_n = \{w \in F : |w| = n\}$. Each nonempty $w \in F$ has a unique representation in the form $w = w_1w_2 \cdots w_k$, where $w_l = 0^li^j$, $1 \leq l \leq k$, $i_1, j_k \in \omega$, $j_1, j_2, j_2, \ldots, i_k \in \mathbb{N}$ (if $k = 1$, the requirement is $i_1, j_1 \in \omega$ and $i_1 + j_1 \in \mathbb{N}$). This representation will be called canonical. Words of the form $0^i1^j$, where $i, j \in \omega$ and $i + j \in \mathbb{N}$, will be called basic. If a word $w$ is basic or $w = \emptyset$, we put $w' = \emptyset$ and $w^* = w$. Otherwise, if $w = w_1w_2 \cdots w_k$ is the canonical representation, we put $w' = w_1w_2 \cdots w_{k-1}$ and $w^* = 0w_1 + 1w_2 + \cdots + (w_k-1)w_k$.

Enumerate $Y \setminus \{1_Y\}$ as $\{y_n : 0 < n < \omega\}$. We shall construct a clopen $Y(w) \subseteq Y$ and $y(w) \in Y(w)$ for every $w \in F$ such that $Y(\emptyset) = Y$, $y(\emptyset) = 1_Y$ and the following conditions hold for all $n \in \mathbb{N}$:

1. $Y(w_0) \cup Y(w_1) = Y(w)$ and $Y(w_0) \cap Y(w_1) = \emptyset$ for all $w \in W_{n-1}$;
2. $y(w_0) = y(w)$ for all $w \in W_{n-1}$;
3. $Y(w) = y(w')Y(w^*)$ and $y(w) = y(w')y(w^*)$ for all $w \in W_n$;
4. $y_n \in \{y(w) : w \in W_n\}$.

We take as $Y(0)$ a clopen neighborhood of $1_Y$ such that $y_1 \notin Y(0)$. Put $Y(1) = Y \setminus Y(0)$, $y(0) = 1_Y$ and $y(1) = y_1$.

Suppose that $Y(w)$ and $y(w)$ have already been constructed for all $w \in W_n$ such that conditions (1)–(4) hold.

It is obvious that the subsets $Y(w)$, where $w \in W_n$, form a partition of $Y$. So, one of them, say $Y(u)$, contains $y_{n+1}$. For some $z_{n+1} \in Y(u^*)$, $y_{n+1} = y(u^*)z_{n+1}$. If $z_{n+1} = y(u^*)$, we take as $Y(u)^{n+1}$ any clopen neighborhood of $1_Y$ such that $Y(u)\setminus y(u)Y(0^n+1) \neq \emptyset$ for all basic $w \in W_n$. Then for every basic $w \in W_n$, put $y(w_0) = y(w)$, $Y(w_0) = y(w)Y(0^n+1)$ and $Y(w_1) = Y(w) \setminus Y(w_0)$, and take as $y(w_1)$ any element of $Y(w_0)$. If $z_{n+1} \neq y(u^*)$, we take as $Y(u)^{n+1}$ in addition such that $z_{n+1} \notin y(u^*)Y(0^n+1)$ and put $y(u^*) = z_{n+1}$. For all non-basic $v \in W_{n+1}$, we define $Y(v)$ and $y(v)$ by condition (3). Then $y(v) = y(v')y(v^*) \in y(v')Y(v^*) = Y(v)$ and $y_{n+1} = y(u^*)y(u^*) = y((u^1)^*) = y(1)$. To check conditions (2) and (1), let $w \in W_n$. Then

$$y(w_0) = y((w_0)^*)y((w_0)^*) = y(w)g(0^n+1) = y(w),$$

$$Y(w_0) = y(w)Y(0^n+1) = y(w')y(w^*)Y(0^n+1) = y(w')Y(w^*0),$$

$$Y(w_1) = y((w_1)^*)Y((w_1)^*) = y(w')Y(w^*1),$$
and so
\[
Y(w0) \cup Y(w1) = y(w')[Y(w*0) \cup Y(w*1)] = y(w')Y(w*) = Y(w),
\]
\[
Y(w0) \cap Y(w1) = \emptyset.
\]

After this construction we have obtained the mapping \(F' \ni w \mapsto y(w) \in Y\). It follows from (4), (2) and (1) that it is a bijection. Define the bijection \(f : X \to Y\) by \(f(x) = y(w(x))\). To verify that \(f\) is a local homomorphism, let \(x \in X\), \(w(x) = u\). Take any \(z \in U_{|u|+1}\), where \(U_n = \{(\varepsilon_i)_{i<\omega} \in X : \varepsilon_i = 0\text{ for all } i < n\}\). Then \(w(z) = 0^{|u|+1}v\) and \(w(x + z) = u0v\). It is easy to show by induction on the length of the canonical representations using (3) that \(y(u0v) = y(u)y(0^{|u|+1}v)\).

Therefore, \(f(x + z) = y(u0v) = y(u)y(0^{|u|+1}v) = f(x)f(z)\). To see that \(f\) is a local isomorphism, we note that \(f(U_{n+1}) = Y(0^n)\); so \(f\) is open. \(\square\)

Now we can prove our main result.

**Theorem 3.** Let \(X\) be a countably infinite homogeneous regular space, and let \(G\) be a countably infinite group. Then there is a group operation \(\ast\) on \(X\) with continuous left shifts such that \((X, \ast)\) is isomorphic to \(G\).

**Proof.** We may suppose that \(X\) is not discrete. By Theorem 1, there is a Boolean group operation \(+\) on \(X\) with continuous shifts. We endow \(G\) with any non-discrete regular topology with continuous left shifts. By Theorem 2, there is a local isomorphism \(f : (X, +) \to G\) (it even suffices that \(f\) be a bijective local homomorphism). For any \(x, y \in X\), we define \(x \ast y = f^{-1}(f(x)f(y))\). Obviously, \((X, \ast)\) is a group isomorphic to \(G\). Next, given any \(x \in X\), we can choose a neighborhood \(U\) of the identity such that \(f(x + z) = f(x)f(z)\) for all \(z \in U\), and then \(x \ast z = f^{-1}(f(x)f(z)) = f^{-1}(f(x + z)) = x + z\). It follows from this that the left shifts of \((X, \ast)\) are continuous and open at the identity. Consequently, the left shifts of \((X, \ast)\) are continuous. \(\square\)

**References**


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