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ON GROUP OPERATIONS ON HOMOGENEOUS SPACES

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ABSTRACT. It is proved that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

It is well known that not all infinite groups admit a non-discrete group topology (see, for example, [1, §9]). However, every infinite group admits a non-discrete zero-dimensional topology with continuous left shifts [3], [4], and every countably infinite group admits a non-discrete zero-dimensional topology with continuous shifts and inversion [6]. It is well known also that not all homogeneous spaces admit a structure of a topological group and even a structure of a group with continuous left shifts (see, for example, [1, §10]). The aim of this note is to prove that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

We begin with the Boolean version of this result.

Theorem 1. *Let X be a countably infinite homogeneous regular space. Then there is a Boolean group operation $+$ on X with continuous shifts.*

To prove Theorem 1 we need the following lemma.

Lemma. *Let X be a countably infinite homogeneous regular space and let $x, y \in X$, $x \neq y$. Then there are a clopen neighborhood U of x and a homeomorphism $h : X \to X$ such that $h(x) = y$, $h(U) = X \setminus U$ and $h^2 = id_X$.*

Proof. Enumerate X as $\{x_n : n < \omega\}$. Since X is homogeneous, there is a homeomorphism $g_0 : X \to X$ with $g_0(x) = y$. Since X is countable and regular, therefore zero-dimensional, one may choose a clopen neighborhood U_0 of x with $U_0 \cap g_0(U_0) = \emptyset$. Put $X_0 = U_0 \cup g_0(U_0)$ and define $h_0 : X_0 \to X_0$ by

$$h_0(x) = \begin{cases} g_0(x) & \text{if } x \in U_0, \\ g_0^{-1}(x) & \text{if } x \in g_0(U_0). \end{cases}$$

If $X_0 = X$, put $U = U_0$ and $h = h_0$. Otherwise, choose the first element x_{n_1} in the sequence $\{x_n : n < \omega\}$ with $x_{n_1} \notin X_0$ and pick any $y_{n_1} \in X \setminus (X_0 \cup \{x_{n_1}\})$. Let $g_1 : X \to X$ be a homeomorphism with $g_1(x_{n_1}) = y_{n_1}$ and let U_1 be a clopen

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neighborhood of x_{n_1} with $U_1 \cap g_1(U_1) = \emptyset$ and $U_1 \cup g_1(U_1) \subseteq X \setminus X_0$. Put $X_1 = X_0 \cup U_1 \cup g_1(U_1)$ and extend h_0 to $h_1 : X_1 \to X_1$ by

$$h_1(x) = \begin{cases} h_0(x) & \text{if } x \in X_0, \\ g_1(x) & \text{if } x \in U_1, \\ g_1^{-1}(x) & \text{if } x \in g_1(U_1). \end{cases}$$

If $X_1 = X$, put $U = U_0 \cup U_1$ and $h = h_1$. Otherwise, choose the first element x_{n_2} in the sequence $\{x_n : n < \omega\}$ with $x_{n_2} \notin X_1$, pick any $y_{n_2} \in X \setminus (X_1 \cup \{x_{n_2}\})$, and so forth.

If $X_n = X$ for some n, put $U = \bigcup_{i \le n} U_i$ and $h = h_n$. Otherwise, put $U = \bigcup_{n<\omega} U_n$ and $h = \bigcup_{n<\omega} h_n$. □

Proof of Theorem 1. We can assume that X is not discrete. Pick any element $u \in X$ and enumerate the set $X \setminus \{u\}$ as $\{x_n : n < \omega\}$. By the Lemma, we can construct a decreasing sequence $\{U_n : n < \omega\}$ of clopen neighborhoods of u with $U_0 = X$ and a sequence of homeomorphisms $h_n : U_n \to U_n$ such that $h_n(U_{n+1}) = U_n \setminus U_{n+1}$ and $h_n^2 = id_{U_n}$. By induction on n, it is easy to verify that for each $n < \omega$ the subsets $h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(U_{n+1})$, where $\varepsilon_i \in \{0,1\}$, form a partition of X and every $x \in X$ can be uniquely written in the form $x = h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(y)$, where $y \in U_{n+1}$. We construct sequences $\{U_n : n < \omega\}$ and $\{h_n : n < \omega\}$ satisfying, in addition, the condition $x_n \in B_n = \{h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(u) : \varepsilon_i \in \{0,1\}, i \le n\}$. To see that this can be done, assume that $x_n \notin B_{n-1}$. Then $x_n \in h_0^{\varepsilon_0} \cdots h_{n-1}^{\varepsilon_{n-1}}(y_n)$ for some $\varepsilon_i \in \{0,1\}$ and $y_n \in U_n$. Choose h_n with $h_n(u) = y_n$.

Now let $x, y \in X$. By the construction, there exists large enough n such that x and y can be uniquely written in the form $x = h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(u)$, $y = h_0^{\delta_0} \cdots h_n^{\delta_n}(u)$. Put $x + y = h_0^{\varepsilon_0+\delta_0} \cdots h_n^{\varepsilon_n+\delta_n}(u)$. It is clear that the operation is well defined and that $(X, +)$ is a Boolean group with zero u. We note that for every $z \in U_{n+1}$, $x + z = h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(z)$. To check that the left shifts in $(X, +)$ are continuous, let U be a neighborhood of $x+y$. Choose a neighborhood V of u such that $V \subseteq U_{n+1}$ and $h_0^{\varepsilon_0+\delta_0} \cdots h_n^{\varepsilon_n+\delta_n}(V) \subseteq U$. Then $W = h_0^{\delta_0} \cdots h_n^{\delta_n}(V)$ is a neighborhood of y and for every $z \in V$, $x + h_0^{\delta_0} \cdots h_n^{\delta_n}(z) = x + (y+z) = (x+y)+z = h_0^{\varepsilon_0+\delta_0} \cdots h_n^{\varepsilon_n+\delta_n}(z) \in U$; so $x + W \subseteq U$. □

Next we need the Local Isomorphism Theorem. It is close to [5, Theorem 2] (see also [2, Lemma 7.4]).

A space with a group operation is called a *left topological group* if all left shifts are continuous. A space X with a partial binary operation $\cdot$ and a distinguished element 1 is called a *local left topological group*, if for each element $x \in X$ there is an open neighborhood U_x of 1 such that

(1) for any $y \in U_x$, $x \cdot y$ is defined, $x \cdot 1 = x$, $x \cdot U_x$ is an open neighborhood of x, and a mapping $U_x \ni y \mapsto x \cdot y \in x \cdot U_x$ is a homeomorphism;
(2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ if $y \in U_x$, $z \in U_{x \cdot y} \cap U_y$, $y \cdot z \in U_x$.

For a local left topological group, from this point on, when we write $x \cdot y$ we mean $y \in U_x$ and when we write $x \cdot U$, where U is a neighborhood of 1, we mean $U \subseteq U_x$. A basic example of a local left topological group is an open neighborhood of the identity of a left topological group.

Let X and Y be local left topological groups. A map $f : X \to Y$ is called a *local homomorphism* if $f(1_X) = 1_Y$ and for every $x \in X$ there exists a neighborhood U_x

of 1_X such that $f(xz) = f(x)f(z)$ for all $z \in U_x$. A bijective local homomorphism f is called a *local isomorphism* if f^{-1} is also a local homomorphism. We observe that every open bijective local homomorphism is a local isomorphism.

Theorem 2. *All countably infinite non-discrete regular left topological groups are local isomorphic.*

Proof. Let X be the countably infinite Boolean group $\bigoplus_\omega \mathbb{Z}_2$ endowed with the direct sum topology, and let Y be an arbitrary countably infinite non-discrete regular left topological group. We shall define a local isomorphism $f : X \to Y$.

Let F be the semigroup of words on the letters 0 and 1 with empty word $\emptyset$, and let F' be the subsemigroup of F of words including $\emptyset$, in which the last letter is 1. We define a bijection $X \ni x \mapsto w(x) \in F'$ as follows. If $0 \neq x = (\varepsilon_n)_{n<\omega}$ and $m = \max\{n < \omega : \varepsilon_n = 1\}$, we put $w(x) = \varepsilon_0 \cdots \varepsilon_m$. If $x = 0$, we put $w(x) = \emptyset$.

For every $w \in F$, $|w|$ will denote the length of w. For every $n < \omega$, put $W_n = \{w \in F : |w| = n\}$. Each nonempty $w \in F$ has a unique representation in the form $w = w_1 w_2 \cdots w_k$, where $w_l = 0^{i_l} 1^{j_l}$, $1 \leq l \leq k$, $i_1, j_k \in \omega$, $j_1, i_2, j_2, \ldots, i_k \in \mathbb{N}$ (if $k = 1$, the requirement is $i_1, j_1 \in \omega$ and $i_1 + j_1 \in \mathbb{N}$). This representation will be called canonical. Words of the form $0^i 1^j$, where $i, j \in \omega$ and $i + j \in \mathbb{N}$, will be called basic. If a word w is basic or $w = \emptyset$, we put $w' = \emptyset$ and $w^* = w$. Otherwise, if $w = w_1 w_2 \cdots w_k$ is the canonical representation, we put $w' = w_1 w_2 \cdots w_{k-1}$ and $w^* = 0^{|w_1| + \cdots + |w_{k-1}|} w_k$.

Enumerate $Y \backslash \{1_Y\}$ as $\{y_n : 0 < n < \omega\}$. We shall construct a clopen $Y(w) \subseteq Y$ and $y(w) \in Y(w)$ for every $w \in F$ such that $Y(\emptyset) = Y$, $y(\emptyset) = 1_Y$ and the following conditions hold for all $n \in \mathbb{N}$:

(1) $Y(w0) \cup Y(w1) = Y(w)$ and $Y(w0) \cap Y(w1) = \emptyset$ for all $w \in W_{n-1}$;
(2) $y(w0) = y(w)$ for all $w \in W_{n-1}$;
(3) $Y(w) = y(w')Y(w^*)$ and $y(w) = y(w')y(w^*)$ for all $w \in W_n$;
(4) $y_n \in \{y(w) : w \in W_n\}$.

We take as $Y(0)$ a clopen neighborhood of 1_Y such that $y_1 \notin Y(0)$. Put $Y(1) = Y \setminus Y(0)$, $y(0) = 1_Y$ and $y(1) = y_1$.

Suppose that $Y(w)$ and $y(w)$ have already been constructed for all $w \in W_n$ such that conditions (1)–(4) hold.

It is obvious that the subsets $Y(w)$, where $w \in W_n$, form a partition of Y. So, one of them, say $Y(u)$, contains y_{n+1}. For some $z_{n+1} \in Y(u^*)$, $y_{n+1} = y(u')z_{n+1}$. If $z_{n+1} = y(u^*)$, we take as $Y(0^{n+1})$ any clopen neighborhood of 1_Y such that $Y(w) \backslash y(w)Y(0^{n+1}) \neq \emptyset$ for all basic $w \in W_n$. Then for every basic $w \in W_n$, put $y(w0) = y(w)$, $Y(w0) = y(w)Y(0^{n+1})$ and $Y(w1) = Y(w) \setminus Y(w0)$, and take as $y(w1)$ any element of $Y(w1)$. If $z_{n+1} \neq y(u^*)$, we take as $Y(0^{n+1})$ in addition such that $z_{n+1} \notin y(u^*)Y(0^{n+1})$ and put $y(u^*1) = z_{n+1}$. For all non-basic $v \in W_{n+1}$, we define $Y(v)$ and $y(v)$ by condition (3). Then $y(v) = y(v')y(v^*) \in y(v')Y(v^*) = Y(v)$ and $y_{n+1} = y(u')y(u^*1) = y((u1)')y((u1)^*) = y(u1)$. To check conditions (2) and (1), let $w \in W_n$. Then

$$y(w0) = y((w0)')y((w0)^*) = y(w)y(0^{n+1}) = y(w),$$
$$Y(w0) = y(w)Y(0^{n+1}) = y(w')y(w^*)Y(0^{n+1}) = y(w')Y(w^*0),$$
$$Y(w1) = y((w1)')Y((w1)^*) = y(w')Y(w^*1),$$

and so

$$Y(w0) \cup Y(w1) = y(w')[Y(w^*0) \cup Y(w^*1)] = y(w')Y(w^*) = Y(w),$$
$$Y(w0) \cap Y(w1) = \emptyset.$$

After this construction we have obtained the mapping $F' \ni w \mapsto y(w) \in Y$. It follows from (4), (2) and (1) that it is a bijection. Define the bijection $f : X \to Y$ by $f(x) = y(w(x))$. To verify that f is a local homomorphism, let $x \in X$, $w(x) = u$. Take any $z \in U_{|u|+1}$, where $U_n = \{(\varepsilon_i)_{i<\omega} \in X : \varepsilon_i = 0 \text{ for all } i < n\}$. Then $w(z) = 0^{|u|+1}v$ and $w(x+z) = u0v$. It is easy to show by induction on the length of the canonical representations using (3) that $y(u0v) = y(u)y(0^{|u|+1}v)$. Therefore, $f(x+z) = y(u0v) = y(u)y(0^{|u|+1}v) = f(x)f(z)$. To see that f is a local isomorphism, we note that $f(U_{n+1}) = Y(0^n)$; so f is open. □

Now we can prove our main result.

Theorem 3. *Let X be a countably infinite homogeneous regular space, and let G be a countably infinite group. Then there is a group operation $*$ on X with continuous left shifts such that $(X, *)$ is isomorphic to G.*

Proof. We may suppose that X is not discrete. By Theorem 1, there is a Boolean group operation $+$ on X with continuous shifts. We endow G with any non-discrete regular topology with continuous left shifts. By Theorem 2, there is a local isomorphism $f : (X, +) \to G$ (it even suffices that f be a bijective local homomorphism). For any $x, y \in X$, we define $x * y = f^{-1}(f(x)f(y))$. Obviously, $(X, *)$ is a group isomorphic to G. Next, given any $x \in X$, we can choose a neighborhood U of the identity such that $f(x+z) = f(x)f(z)$ for all $z \in U$, and then $x * z = f^{-1}(f(x)f(z)) = f^{-1}(f(x+z)) = x + z$. It follows from this that the left shifts of $(X, *)$ are continuous and open at the identity. Consequently, the left shifts of $(X, *)$ are continuous. □

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