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ASYMPTOTIC ESTIMATES FOR A TWO-DIMENSIONAL PROBLEM WITH POLYNOMIAL NONLINEARITY

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ABSTRACT. In this paper we give asymptotic estimates of the least energy solution u_p of the functional

$$J(u)=\int_\Omega |\nabla u|^2 \quad \text{constrained on the manifold } \int_\Omega |u|^{p+1}=1$$

as p goes to infinity. Here Ω is a smooth bounded domain of $\mathbb{R}^2$. Among other results we give a positive answer to a question raised by Chen, Ni, and Zhou (2000) by showing that $\lim_{p\to\infty}||u_p||_\infty=\sqrt{e}$.

1. INTRODUCTION

In this paper we consider the following elliptic minimization problem. Let us define a C^2-functional on $H^1_0(\Omega)$:

$$J(u)=\int_\Omega |\nabla u|^2 \quad \text{constrained on the manifold } \int_\Omega |u|^{p+1}=1 \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and p is a real number greater than 1. Then we define

$$S_p=\inf_{u\in H^1_0(\Omega)} J(u). \tag{1.2}$$

By standard results it is easy to see that S_p is achieved at a function $u_p\in H^1_0(\Omega)$ that satisfies

$$\begin{cases} -\Delta u_p = S_p u_p^p & \text{in } \Omega,\\ u_p>0 & \text{in } \Omega,\\ u_p=0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

By Lemma 2.1 we get $S_p=O\left(\frac{1}{p}\right)$ for p large. Setting $v_p=S_p^{\frac{1}{p-1}}u_p$ we are in the framework of [8], [9] and [6] where some asymptotic results about this problem were obtained.

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In particular, it was proved in [8] and [9] that the minimizer u_p looks like a sharp "spike". More precisely it was shown that, for a suitable sequence $p_n \to \infty$,

$$||u_{p_n}||_\infty \le C, \tag{1.4}$$

and in the sense of distribution,

$$\frac{u_{p_n}^{p_n}}{||u_{p_n}||_{p_n}^{p_n}} \to \delta_{x_0}. \tag{1.5}$$

Moreover, the point x_{p_n} where the minimizer u_{p_n} achieves its maximum converges to a critical point of the Robin function.

In this paper we obtain estimates of a different nature, greatly improving some partial results obtained in [5], where uniqueness and qualitative properties of the least energy solution were proved.

Here we use the two-dimensional blow-up technique introduced in [1]. The blow-up function is obtained by linearizing the nonlinear term plog u_p around the point of maximum of u_p. More precisely let us define the function $z_p(x) : \Omega_p = \frac{\Omega - x_p}{\varepsilon_p} \mapsto \mathbb{R}$,

$$z_p(x) = \frac{p}{u_p(x_p)}\big(u_p(\varepsilon_p x + x_p) - u_p(x_p)\big) \tag{1.6}$$

where x_p is the point where u_p achieves its maximum and $\varepsilon_p^2 = \frac{1}{pS_p u_p(x_p)^{p-1}}$. Then we obtain the following.

Theorem 1.1. *For any sequence z_{p_n} with $p_n \to \infty$, there exists a subsequence of z_{p_n}, still denoted by z_{p_n}, such that $z_{p_n} \to z$ in $C^2_{loc}(\mathbb{R}^2)$, where $z(x) = \log \frac{1}{\left(1+\frac{|x|^2}{8}\right)^2}$.*

The main result of the paper is a consequence of the above theorem.

Theorem 1.2. *Let u_p be a solution to* (1.3). *Then*

$$\lim_{p\to\infty} ||u_p||_\infty = \sqrt{e}. \tag{1.7}$$

Note that the estimate $\limsup_{p\to\infty} ||u_p||_\infty \le \sqrt{e}$ was proved in [9]. Here we give a positive answer to a question raised in [4], where some numerical computations suggested the validity of (1.7).

2. Proof of Theorem 1.1

In this section we recall some results about the solution u_p, and then we give the proof of Theorem 1.1.

We start by recalling the following estimates on S_p, due to Ren and Wei ([8]).

Lemma 2.1. *Let S_p be defined as in* (1.2). *Then*

$$\lim_{p\to\infty} pS_p = 8\pi e. \tag{2.1}$$

Proof. Setting $v_p = (S_p)^{\frac{1}{p-1}} u_p$ we have that v_p also achieves S_p and satisfies

$$\begin{cases} -\Delta v_p = v_p^p & \text{in } \Omega, \\ v_p > 0 & \text{in } \Omega, \\ v_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

From Corollary 2.3 of [8] we get $\lim_{p\to\infty} p\int_\Omega v_p^{p+1} = 8\pi e$. Hence, recalling that $\int_\Omega u_p^{p+1} = 1$ we derive that $\lim_{p\to\infty} p(S_p)^{\frac{p+1}{p-1}} = 8\pi e$, which implies (2.1). □

Let us denote by x_p the point where $u_p(x_p) = ||u_p||_\infty$. By Lemma 4.1 of [8] we know that x_p is far away from the boundary of Ω. The next lemma provides additional information on the rate of $u_p(x_p)$.

Lemma 2.2. *We have that*

$$\lim_{p\to\infty} u_p(x_p)^{p-1} = +\infty. \tag{2.3}$$

Proof. Let us denote by $\lambda_1(\Omega)$ the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Then we have

$$1 = \int_\Omega |u_p|^{p+1} \le u_p(x_p)^{p-1} \int_\Omega |u_p|^2 \le u_p(x_p)^{p-1}\lambda_1(\Omega)^{-1} \int_\Omega |\nabla u_p|^2. \tag{2.4}$$

Recalling that $\int_\Omega |\nabla u_p|^2 = S_p$ and Lemma 2.1, we deduce that $\int_\Omega |\nabla u_p|^2 \to 0$ as p goes to infinity. By (2.4), we obtain the claim. ☐

Proof of Theorem 1.1. For any sequence $p_n \to +\infty$, let us set $z_n : \Omega_n = \frac{\Omega - x_{p_n}}{\varepsilon_n} \mapsto \mathbb{R}$,

$$z_n(x) = \frac{p_n}{u_{p_n}(x_{p_n})}\big(u_{p_n}(\varepsilon_n x + x_{p_n}) - u_{p_n}(x_{p_n})\big) \tag{2.5}$$

where $\varepsilon_n^2 = \frac{1}{p_n S_{p_n} u_{p_n}(x_{p_n})^{p_n-1}}$. From Lemma 2.1 and Lemma 2.2, we get $\varepsilon_n \to 0$ as $n \to \infty$ and "$\Omega_n \to \mathbb{R}^2$" as $n \to \infty$. Now let us write down the equation satisfied by z_n,

$$\begin{cases} -\Delta z_n = \left(1 + \frac{z_n}{p_n}\right)^{p_n} & \text{in } \Omega_n, \\ 0 < 1 + \frac{z_n}{p_n} \le 1 & \text{in } \Omega_n, \\ z_n = -p_n & \text{on } \partial\Omega_n. \end{cases} \tag{2.6}$$

We want to pass to the limit in (2.6). To do this we use some ideas in [2]. Let $B(0,R)$ be the ball centered at the origin with radius R, and let w_n be the solution of

$$\begin{cases} -\Delta w_n = \left(1 + \frac{z_n}{p_n}\right)^{p_n} & \text{in } B(0,R), \\ w_n = 0 & \text{on } \partial B(0,R). \end{cases} \tag{2.7}$$

By the maximum principle and the standard regularity theory, we have that $0 \le w_n \le C$ with C independent of n. For $x \in B(0,R)$ set $\psi_n(x) = z_n(x) - w_n(x)$. Hence ψ_n is a sequence of harmonic functions which are uniformly bounded above. Hence by Harnack's inequality [7] we have the alternative:
either
i) a subsequence of ψ_n is bounded in $L^\infty_{loc}(B(0,R))$,
or
ii) ψ_n converges uniformly to $-\infty$ on compact subsets of $(B(0,R)$.

Since $\psi_n(0) = z_n(0) - w_n(0) = -w_n(0) \ge -C$, case ii) cannot occur. Hence, up to a subsequence, which we denote again by ψ_n, we have that ψ_n is bounded in $L^\infty(B(0,R))$ for any $R > 0$ and the same holds for z_n. From (2.6), and the standard regularity theory, we derive that z_n is bounded in $C^2_{loc}(\mathbb{R}^2)$, and then it converges to a function $z \in C^2(\mathbb{R}^2)$. Passing to the limit in (2.6), we get that z satisfies

$$-\Delta z = e^z \quad \text{in } \mathbb{R}^2. \tag{2.8}$$

Let us prove that $\int_{\mathbb{R}^2} e^z < +\infty$. To do this we observe that, since $z_n \to z$ in $C^2_{loc}(\mathbb{R}^2)$, then

$$p_n\Big(\log\big(1+\frac{z_n}{p_n}\big)-\frac{z_n}{p_n}\Big)\to 0 \quad \text{pointwise in } \mathbb{R}^2. \tag{2.9}$$

Hence

$$z_n + p_n\Big(\log\big(1+\frac{z_n}{p_n}\big)-\frac{z_n}{p_n}\Big)\to z \quad \text{pointwise in } \mathbb{R}^2. \tag{2.10}$$

By Fatou's Lemma, we deduce

$$\begin{aligned}\int_{\mathbb{R}^2} e^z &\le \lim_{n\to\infty}\int_{\Omega_n} e^{z_n+p_n\Big(\log\big(1+\frac{z_n}{p_n}\big)-\frac{z_n}{p_n}\Big)} = \lim_{n\to\infty}\int_{\Omega_n}\Big(1+\frac{z_n}{p_n}\Big)^{p_n}\\ &= \lim_{n\to\infty}\frac{1}{\varepsilon_n^2 u_{p_n}^{p_n}(x_{p_n})}\int_\Omega u_{p_n}^{p_n} \le \lim_{n\to\infty}\frac{p_nS_{p_n}}{u_{p_n}(x_{p_n})}|\Omega|^{\frac{1}{p_n+1}}\le C\end{aligned} \tag{2.11}$$

since $u_{p_n}(x_{p_n})\ge C$ in Ω for n large (see [8], p. 755).

By a result of Chen and Li ([3]), the solutions of (2.8) satisfying $\int_{\mathbb{R}^2} e^z < +\infty$ are given by

$$z(x)=\log\frac{\mu}{\big(1+\frac{\mu}{8}|x-x_0|^2\big)^2}\quad \text{for } \mu>0 \text{ and } x_0\in\mathbb{R}^2. \tag{2.12}$$

Since $z(x)\le z(0)=0$ for any $x\in\mathbb{R}^2$, we derive that $\mu=1$ and $x_0=0$ in (2.12), and this gives the claim of Theorem 1.1. □

3. Proof of Theorem 1.2

The next estimate plays a role in the proof of Theorem 1.2. This estimate was proved in [9] but we stress that it follows easily by Theorem 1.1.

Lemma 3.1. *We have that*

$$\limsup_{n\to\infty}||u_{p_n}||_\infty \le \sqrt{e}. \tag{3.1}$$

Proof. It follows directly by Theorem 1.1.

Setting $u_{p_n}=u_n$ and $L=\limsup_{n\to\infty}||u_n||_\infty$, by using Fatou's Lemma, we obtain

$$\begin{aligned}1&=\int_\Omega u_n^{p+1} = u_n(x_n)^{p_n+1}\varepsilon_n^2\int_{\Omega_n}\big(1+\frac{z_n}{p_n}\big)^{p_n+1}\\ &=\frac{u_n(x_n)^2}{p_nS_{p_n}}\int_{\Omega_n}\big(1+\frac{z_n}{p_n}\big)^{p_n+1}\ge\frac{L^2}{8\pi e}\int_{\mathbb{R}^2}e^z.\end{aligned} \tag{3.2}$$

Recalling that $\int_{\mathbb{R}^2} e^z = 8\pi$, we deduce the claim. □

Let us consider the linearized operator associated to (1.3), i.e., $L_p : H^1_0(\Omega)\to H^{-1}(\Omega)$,

$$L_p = -\Delta - pS_p u_p^{p-1}(x)I, \quad x\in\Omega, \tag{3.3}$$

and let us denote by $\lambda_1(L_p)$, $\lambda_2(L_p)$ the first and the second eigenvalue of L_p. Now let us recall a property of $\lambda_2(L_p)$.

Lemma 3.2. *We have that*

$$\lambda_2(L_p)\ge 0. \tag{3.4}$$

Proof. The proof is standard since u_p is a minimizer of J on the manifold $\int_\Omega |u|^{p+1} = 1$. □

We consider, for $D \subset \Omega_p$, $L_{p,D} : H_0^1(D) \to H^{-1}(D)$,

$$L_{p,D} = -\Delta - \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{u_p^{p-1}(x_p)} I, \quad x \in D, \tag{3.5}$$

and let us denote by $\lambda_1(L_{p,D})$, $\lambda_2(L_{p,D})$ the first and the second eigenvalue of $L_{p,D}$.

Lemma 3.3. *We have that*

$$\lambda_2(L_{p,\Omega_p}) \geq 0. \tag{3.6}$$

Proof. Using the scaling $x \to \varepsilon_p x + x_p$ we get $\lambda_2(L_{p,\Omega_p}) = \varepsilon_p^2 \lambda_2(L_p)$ and (3.6) follows by Lemma 3.2. □

Lemma 3.4. *Let us denote by $B_1 = B(0,1)$. Let $p_n \to \infty$ such that $z_{p_n} \to z$ in $C^1_{loc}(\mathbb{R}^2)$. Then for large p_n, we have*

$$\lambda_1(L_{p_n,B_1}) < 0. \tag{3.7}$$

Proof.

$$w_p = x \cdot \nabla z_p + \frac{2}{p-1} z_p + \frac{2p}{p-1}. \tag{3.8}$$

By direct computation we get that w_p satisfies

$$-\Delta w_p = \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{u_p^{p-1}(x_p)} w_p. \tag{3.9}$$

Moreover, $w_{p_n}(0) \to 2$ and for $|x| = 1$, $w_{p_n}(x) \to -\frac{4}{9}$ as $p_n \to \infty$.

Hence, if we denote by $A_p = \{x \in B_1 : w_p > 0\}$ and

$$\tilde{w}_p = \begin{cases} w_p & \text{if } x \in A_p, \\ 0 & \text{if } x \in B_1 \setminus \bar{A}_p, \end{cases} \tag{3.10}$$

we derive that for p_n large, $\tilde{w}_{p_n} \in H_0^1(B_1)$. From (3.9) we get

$$\int_{B_1} |\nabla \tilde{w}_{p_n}|^2 - \int_{B_1} \frac{u_{p_n}^{p_n-1}(\varepsilon_{p_n} x + x_{p_n})}{u_{p_n}^{p_n-1}(x_{p_n})} \tilde{w}_{p_n}^2 = 0, \tag{3.11}$$

and this implies that $\lambda_1(L_{p_n,B_1}) < 0$. □

Lemma 3.5. *Let p_n be a sequence as in Lemma 3.4. Then for p_n large we have*

$$\lambda_1(L_{p_n,\Omega_{p_n} \setminus B_1}) > 0. \tag{3.12}$$

Proof. By contradiction let us suppose that $\lambda_1(L_{p_n,\Omega_{p_n} \setminus B_1}) \leq 0$. Then from Lemma 3.4, for large p_n, $\lambda_1(L_{p_n,B_1}) < 0$ and hence $\lambda_2(L_{p_n,\Omega_{p_n}}) < 0$. This gives a contradiction with Lemma 3.3. □

Remark 3.6. Lemma 3.4 implies that the operator $L_{p_n}, \Omega_{p_n \setminus B_1}$ satisfies the maximum principle in $\Omega_{p_n} \setminus B_1$.

Proof of Theorem 1.2. By Lemma 3.1 we know that (up to a subsequence) $\lim_{n\to\infty} ||u_{p_n}||_\infty \leq \sqrt{e}$. By contradiction let us suppose that there exists a subsequence of u_{p_n} (still denoted by u_{p_n}) such that

$$\lim_{n\to\infty} ||u_{p_n}||_\infty < \sqrt{e}. \tag{3.13}$$

Now, we will show that for large p_n, (3.13) implies the following estimate:

$$z_n(x) \leq C + \log \frac{1}{\left(1+\frac{|x|^2}{8}\right)^2} \quad \forall x \in \Omega_{p_n} \tag{3.14}$$

where C is a constant independent of n.

By Theorem 1.1, $z_n \to z$ in $C^\circ(\overline{B}_1)$ and hence (3.14) fails for $x \in B_1$. It is enough to prove 3.14 for $x \in \Omega_n \setminus B_1$. To prove this let us observe that the function z satisfies

$$-\Delta z = e^z \geq \left(1+\frac{z}{p}\right)^p \tag{3.15}$$

for any $p > 1$. Furthermore, let us consider $\psi_n = z_n - z$ in Ω_{p_n}. By computing ψ_n on $\partial(\Omega_{p_n} \setminus B_1)$ and by applying the maximum principle, if $x \in \partial\Omega_{p_n}$, we get

$$\begin{aligned} \psi_n(x) = z_n(x) - z(x) = -p_n + 2\log(1+\frac{|x|^2}{8}) &\leq -p_n + 2\log\frac{1}{\varepsilon_{p_n}^2} + C \\ \leq -p_n + 2\log u_{p_n}(x_{p_n})^{p_n-1} + C \leq C \end{aligned} \tag{3.16}$$

where we had used $u_{p_n}(x_{p_n}) < \sqrt{e}$.

Now if $x \in \partial B_1$, by Theorem 1.1 we can derive again that $\psi_n(x) \leq C$.

Finally, we write down the equation satisfied by ψ_n. Using the convexity of $F(s) = \left(1+\frac{s}{p}\right)^p$ for $p > 1$ we have

$$\begin{aligned} -\Delta\psi_n &= \left(1+\frac{z_n}{p_n}\right)^{p_n} - \left(1+\frac{z}{p_n}\right)^{p_n} \\ &\leq \left(1+\frac{z_n}{p_n}\right)^{p_n-1}\psi_n = \frac{u_{p_n}^{p_n-1}(\varepsilon_{p_n}x + x_{p_n})}{u_{p_n}^{p_n-1}(x_{p_n})}\psi_n. \end{aligned} \tag{3.17}$$

Since the maximum principle holds in $\Omega_{p_n} \setminus B_1$ for $L_{p_n,\Omega_{p_n}\setminus B_1}$, we now deduce that $\psi_n \leq C$ in $\Omega_{p_n} \setminus B_1$ and this gives (3.14).

From (3.14), a contradiction follows easily. Indeed, using Theorem 1.1 and Lebesgue's Theorem we derive

$$\begin{aligned} 1 &= \int_\Omega u_n^{p_n+1} = u_n(x_n)^{p_n+1}\varepsilon_n^2 \int_{\Omega_n} \left(1+\frac{z_n}{p_n}\right)^{p_n+1} \\ &= \frac{u_n^2(x_n)}{8\pi e + o(1)}\big(8\pi + o(1)\big), \end{aligned} \tag{3.18}$$

which proves that $\lim_{n\to\infty} ||u_n||_\infty = \sqrt{e}$, a contradiction with (3.13). □

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