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ESTIMATES FOR AN OSCILLATORY INTEGRAL OPERATOR RELATED TO RESTRICTION TO SPACE CURVES

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Abstract. We consider the oscillatory integral operator defined by

$$T_\lambda f(x) = \int_{\mathbb{R}} e^{i\lambda\phi(x,t)} a(x,t) f(t) dt$$

where $\lambda > 1$, $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$ and ϕ is a real-valued function in $C^\infty(\mathbb{R}^n \times \mathbb{R})$. This operator may be thought of as a variable-curve version of the adjoint of the Fourier restriction operator for space curves. Under a certain nondegeneracy condition on ϕ, we obtain $L^p - L^q$ estimates for T_λ with a suitable bound for the operator norm $\|T_\lambda\|_{L^p \to L^q}$. This generalizes a result of Hörmander for the plane to higher dimensions.

1. Introduction and statement of results

Let $n \geq 2$ and assume that ϕ is a real-valued function in $C^\infty(\mathbb{R}^n \times \mathbb{R})$, and let $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$. Let us consider the oscillatory integral operator T_λ defined by

$$T_\lambda f(x) = \int_{\mathbb{R}} e^{i\lambda\phi(x,t)} a(x,t) f(t) dt \tag{1.1}$$

for $\lambda > 1$. We are interested in the problem of obtaining $L^p - L^q$ estimates for T_λ with a suitable bound for the operator norm $\|T_\lambda\|_{L^p \to L^q}$, as $\lambda \to \infty$, with some suitable conditions imposed on the phase function ϕ. When $n = 2$, Hörmander [H] (see also [S]) showed that for $1/p + 3/q \leq 1$, $q > 4$, there is a constant C such that

$$\|T_\lambda f\|_{L^q(\mathbb{R}^2)} \leq C\lambda^{-2/q} \|f\|_{L^p(\mathbb{R})} \tag{1.2}$$

provided that the phase function ϕ satisfies the Carleson-Sjölin condition (see [CS]), namely,

$$\det\left(\partial_t(\nabla_x\phi), \partial_t^2(\nabla_x\phi)\right) \neq 0$$

on the support of the cutoff function a. Here ∇_x is the gradient with respect to x. By rescaling and duality, (1.2) implies the sharp restriction theorem for plane curves with nonvanishing curvature (see *Remark* 1.2). We may think of T_λ as a variable-curve version of the adjoint of the Fourier restriction operator for nondegenerate curves in $\mathbb{R}^n$. The purpose of this note is to obtain an extension of the estimate (1.2) to higher dimensions.

If we set $\phi(x,t) = x \cdot \gamma(t)$ and $\lambda = 1$ in (1.1), then T_λ is precisely the adjoint of the Fourier restriction operator for the curve $t \to \gamma(t)$. The problem of obtaining

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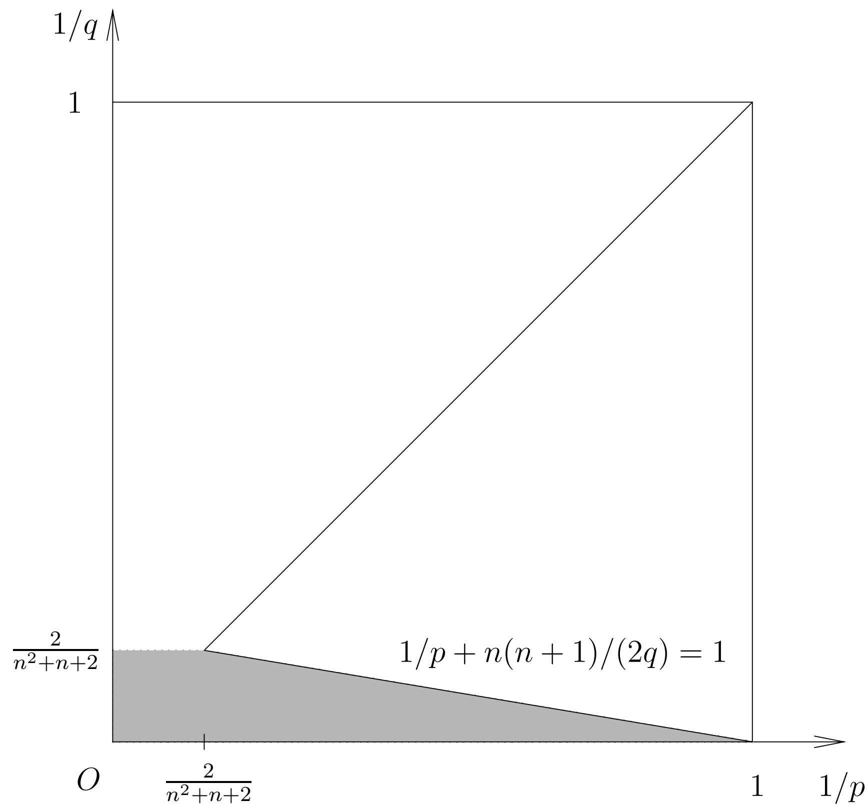

FIGURE 1. The known region of boundedness of T_λ with $\|T_\lambda\|_{L^p \to L^q} \le C\lambda^{-n/q}$

the $L^p - L^q$ estimates for the restriction of the Fourier transform to nondegenerate curves in $\mathbb{R}^n$ has been studied by several authors including Prestini [P1], [P2], Christ [C], and Drury [D1] (see also [F], [DM2], [D2], [BO]). In particular, Drury [D1] showed that the adjoint of the Fourier restriction operator for the curve $(t, t^2, \dots, t^n)$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R}^n)$ for any pair (p, q) satisfying $1/p + n(n+1)/(2q) \le 1$, $q > (n^2 + n + 2)/2$. By rescaling this result, which corresponds to the case $\phi(x, t) = x \cdot (t, t^2, \dots, t^n)$ and $\lambda = 1$ in (1.1), we get $\|T_\lambda\|_{L^p \to L^q} = O(\lambda^{-n/q})$ provided that $1/p + n(n + 1)/(2q) \le 1$, $q > (n^2 + n + 2)/2$. (See Figure 1.) So it seems to be reasonable to expect that a similar estimate holds for T_λ when ϕ is a more general phase function satisfying suitable conditions. In view of the adjoint of the restriction operator for space curves, a natural nondegeneracy condition for the phase function might be that, for each fixed x, the curve $t \to \nabla_x\phi(x, t)$ has nonvanishing torsion, that is,

$$\det\big(\partial_t(\nabla_x\phi), \partial_t^2(\nabla_x\phi), \dots, \partial_t^n(\nabla_x\phi)\big) \ne 0 \qquad (1.3)$$

on the support of the cutoff function a.

Our result may be stated as follows.

Theorem 1.1. *Let T_λ be given by (1.1). Suppose that ϕ satisfies (1.3) on the support of a. If $1/p + n(n + 1)/(2q) \le 1$, $q > (n^2 + n + 2)/2$, then there is a* *constant C, independent of f and $\lambda > 1$, such that*

$$\|T_\lambda f\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{-n/q}\|f\|_{L^p(\mathbb{R})}. \tag{1.4}$$

Remark 1.2. The condition $1/p + n(n+1)/(2q) \leq 1$ is necessary for (1.4) to hold. To see this, just replace f by $e^{-i\lambda\phi(x_0,t)}f$, and replace x by $x_0 + x/\lambda$ in (1.1). Then (1.4) implies that

$$\left\| \int_{\mathbb{R}} e^{i\lambda(\phi(x_0+x/\lambda,t)-\phi(x_0,t))} a(x_0 + x/\lambda, t) f(t) dt \right\|_q \leq C\|f\|_p.$$

Letting $\lambda \to \infty$, we get

$$\left\| \int_{\mathbb{R}} e^{ix\cdot\nabla_x\phi(x_0,t)} a(x_0, t) f(t) dt \right\|_q \leq C\|f\|_p \tag{1.5}$$

by Fatou's lemma and the Dominated Convergence Theorem. By adapting the proof of the necessary conditions for the $L^p - L^q$ estimates for the restriction to nondegenerate curves (see [P2]), we may see that (1.5) cannot hold unless $1/p + n(n+1)/(2q) \leq 1$. More precisely, let us assume that $a(x_0, t_0) > 0$ and apply Taylor's theorem to each component of $\nabla_x\phi(x_0, t)$ to get

$$\begin{aligned}\nabla_x\phi(x_0, t) =& \nabla_x\phi(x_0, t_0) + \partial_t(\nabla_x\phi)(x_0, t_0)(t - t_0) + \partial_t^2(\nabla_x\phi)(x_0, t_0)(t - t_0)^2/2! \\ &+ \cdots + \partial_t^n(\nabla_x\phi)(x_0, t_0)(t - t_0)^n/n! + E(t)\end{aligned}$$

where each component of $E(t)$ is $O(|t - t_0|^{n+1})$ near t_0. So

$$|x \cdot (\nabla_x\phi(x_0, t) - \nabla_x\phi(x_0, t_0))| < 1$$

if $|t - t_0| < c\delta$ for $c > 0$ sufficiently small and if $|x \cdot u_j| < \delta^{-j}$, $1 \leq j \leq n$, where $u_j = \partial_t^j(\nabla_x\phi)(x_0, t_0)/j!$. Thus, if we take f to be the characteristic function of the interval $(t_0, t_0 + c\delta)$ for δ small, then the integral on the left-hand side of (1.5) is bounded below in absolute value by a multiple of δ on the parallelepiped given by $|x \cdot u_j| < \delta^{-j}$, $1 \leq j \leq n$. Since (1.3) holds, this parallelepiped is nondegenerate and it has volume comparable to $\delta^{-n(n+1)/2}$. So (1.5) implies that

$$\delta^{1-n(n+1)/2q} \leq C\delta^{1/p}$$

for every small $\delta > 0$. Therefore, we must have $1/p + n(n+1)/(2q) \leq 1$ if (1.4) holds.

It seems to be a lot harder to prove that the condition $q > (n^2 + n + 2)/2$ is necessary for (1.4) and (1.5), although there appears to be some evidence that this is the case. In the special case of $\phi(x,t) = x \cdot (t, t^2, \ldots, t^n)$, the condition $q > (n^2 + n + 2)/2$ is known to be necessary for (1.5) for $n = 2$, 3 (see [M]). When $n \geq 4$, for the same choice of ϕ, Mockenhaupt [M] showed the weaker necessary condition that if (1.5) holds for some p, then $q \geq (n^2 + n + 2)/2$.

The estimate (1.4) remains valid under small smooth perturbations of the phase function ϕ and the amplitude a. This stability (or uniformity) plays an important role in the proof of our result. The proof is essentially an adaptation of an argument of Drury [D1], but we also use some methods of Hörmander and Bourgain. It is an inductive argument based on the fact that a known $L^p - L^q$ estimate for T_λ for a pair $(1/p, 1/q)$ on the critical line, i.e., satisfying $1/p + n(n+1)/(2q) = 1$, can be combined with a related L^2 estimate to obtain estimates for a larger range of pairs of exponents on the critical line. In this way we can obtain all estimates for T_λ on the half-open segment $1/p + n(n+1)/(2q) = 1$, $0 \le 1/q < 2/(n^2+n+2)$ in the $(1/p, 1/q)$-plane.

Remark 1.3. A geometric formulation of the Carleson-Sjölin condition is contained in a paper by Mockenhaupt, Seeger and Sogge [MSS]. The hypothesis (1.3) is a natural generalization of that condition. A related curvature condition appeared also in a paper by Greenleaf and Seeger [GS]. Our assumption that the curve $t \to \phi'_x(x,t) = \nabla_x \phi(x,t)$ has nonvanishing torsion for every (x,t) is invariant under the changes of variables $x = x(u)$, $t = t(s)$ when these functions are diffeomorphisms. Indeed these transformations induce *linear* changes of variables in the fibers of the canonical relation $\{(x, \phi'_x, t, -\phi'_t)\}$, and the torsion condition is clearly invariant under linear changes of variables. (The latter assertion may also be checked directly.)

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2. Preliminaries

Before beginning the proof we state three well-known lemmas which will be used later. A function F on $\mathbb{R}^n$ is called *symmetric* if $F(x_1, \dots, x_n) = F(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any permutation σ on n letters. Let $g_1, \dots, g_n$ be the *elementary symmetric polynomials* in n variables, that is,

$$\begin{aligned} g_1(x_1, \dots, x_n) &= x_1 + \cdots + x_n, \\ g_2(x_1, \dots, x_n) &= x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_2x_n + \cdots + x_{n-1}x_n, \\ &\vdots \\ g_n(x_1, \dots, x_n) &= x_1x_2 \cdots x_n. \end{aligned}$$

The following result may be found in Glaeser [G]. (See also [GG].)

Lemma 2.1. *If F is a smooth symmetric function on $\mathbb{R}^n$, then there is a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ satisfying $F(x) = H(g_1(x), \dots, g_n(x))$.*

The next lemma is just a restatement of Lemma 1 in [DM1].

Lemma 2.2. *Let $v(h) = v(h_1, \dots, h_{n-1}) = |h_1 \cdots h_{n-1}| \prod_{1 \le i < j \le n-1} |h_i - h_j|$. Then for $\lambda > 0$,*

$$|\{h \in \mathbb{R}^{n-1} : v(h) \le \lambda\}| \le C\lambda^{2/n}.$$

Proof. We just note that, by homogeneity, it suffices to show that

$$|\{h \in \mathbb{R}^{n-1} : v(h) \le 1\}| \le C.$$

For the proof of this fact the reader is referred to [DM1]. □

Let $L^{p,q}$ denote the Lorentz space with the (quasi-)norm $\|\cdot\|_{p,q}$. (See e.g. [BL].) The following result on interpolation is implicit in [B]. A general version may be found in [CSWW].

Lemma 2.3. *Let ε_0, $\varepsilon_1 > 0$. Suppose that T_j is a sequence of linear operators such that $\|T_j f\|_{q_0,\infty} \le M_0 2^{-\varepsilon_0 j} \|f\|_{p_0,1}$ and $\|T_j f\|_{q_1,\infty} \le M_1 2^{\varepsilon_1 j} \|f\|_{p_1,1}$, where*

$1 \le p_0, p_1 \le \infty$ *and* $1 < q_0, q_1 \le \infty$. *Then* $T = \sum_{j=-\infty}^{\infty} T_j$ *is bounded from* $L^{p,1}$ *to* $L^{q,\infty}$ *with*

$$\|Tf\|_{q,\infty} \le C M_0^{1-\theta} M_1^{\theta} \|f\|_{p,1}$$

where $\theta = \varepsilon_0/(\varepsilon_0 + \varepsilon_1)$, $1/q = (1-\theta)/q_0 + \theta/q_1$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$.

3. Proof of Theorem 1.1

We will deduce Theorem 1.1 from the following result.

Proposition 3.1. *Let* $n \ge 3$. *Suppose that for some* $r, s \in [1, \infty]$ *satisfying* $1/r + n(n+1)/(2s) = 1$, $s > (n^2+n+2)/2$, *the estimate*

$$\|T_\lambda f\|_{L^{s,\infty}(\mathbb{R}^n)} \le C\lambda^{-n/s} \|f\|_{L^{r,1}(\mathbb{R})} \tag{3.1}$$

holds, and suppose also that this estimate holds with the same constant C *under small smooth perturbations of* ϕ *and* a. *If* ϕ *satisfies* (*1.3*), *then*

$$\|T_\lambda f\|_{L^{q,\infty}(\mathbb{R}^n)} \le C\lambda^{-n/q} \|f\|_{L^{p,1}(\mathbb{R})} \tag{3.2}$$

holds for $1/p + n(n+1)/(2q) = 1, 1/q = (n-2)/[n(n+2)s] + 2/[n(n+2)]$. *Here the new constant* C *is again stable under small smooth perturbations of* ϕ *and* a .

Proof of Theorem 1.1. It is easy to deduce Theorem 1.1 from Proposition 3.1 by using induction. We begin induction by applying Proposition 3.1 to the trivial estimate $\|T_\lambda f\|_\infty \le C\|f\|_1$. Obviously this constant C is stable under small smooth perturbations of ϕ and a. Applying Proposition 3.1 repeatedly yields a sequence of points $(1/p_i, 1/q_i)$ for which (3.2) holds, which are recursively given by the relations $(1/p_0, 1/q_0) = (1, 0)$ and

$$\frac{1}{p_i} + \frac{n(n+1)}{2q_i} = 1, \quad \frac{1}{q_{i+1}} = \frac{(n-2)}{n(n+2)q_i} + \frac{2}{n(n+2)}, \quad i \ge 0. \tag{3.3}$$

Since the desired strong-type estimates can be obtained by real interpolation from these restricted weak-type estimates, we only need to observe that

$$q_i \to (n^2+n+2)/2$$

as $i \to \infty$. But this is easy to see by the monotonicity of q_i. (Note that, given any point (p_*, q_*) on the critical line, the strong-type estimate at this point can be obtained after a finite number of applications of Proposition 3.1.) Thus we have established Theorem 1.1, assuming Proposition 3.1. □

Proof of Proposition 3.1. The support of a may be assumed to be as small as we like, since T_λ can be written as a finite sum of operators with small support by using a partition of unity. Let us consider an n-linear operator L_λ defined by

$$\begin{aligned} L_\lambda(f_1, f_2, \ldots, f_n)(x) &= \prod_{i=1}^{n} T_\lambda f_i(x) \\ &= \int_{\mathbb{R}^n} e^{i\lambda(\phi(x,t_1)+\cdots+\phi(x,t_n))} \prod_{i=1}^{n} [a(x,t_i) f_i(t_i)]\, dt_1 \ldots dt_n. \end{aligned}$$

The singularities of this oscillatory integral appear on the set $\{t \in \mathbb{R}^n : \prod_{i<j} |t_i - t_j| = 0\}$, where $\det \frac{\partial^2}{\partial x_i \partial t_j}(\sum_\ell \phi(x, t_\ell))$ vanishes. So we decompose L_λ dyadically

away from this set. Thus, for any integer k, put

$$S_k = \{t \in \mathbb{R}^n : 2^{-k-1} \le \prod_{1\le i<j\le n} |t_i - t_j| < 2^{-k}\}$$

and $\chi_k = \chi_{S_k}$. Now let

$$(3.4)\qquad L^k_\lambda(f_1, f_2, \dots, f_n)(x) = \int_{\mathbb{R}^n} e^{i\lambda(\phi(x,t_1)+\cdots+\phi(x,t_n))}\chi_k(t_1,\dots,t_n)\prod_{i=1}^n [a(x,t_i)f_i(t_i)]\,dt_1\dots dt_n.$$

An L^2 estimate for L^k_λ: We write $s = (s_1, s_2, \dots, s_n)$ and $t = (t_1, \dots, t_n)$. Let us make the change of variables given by

$$s = G(t) = (g_1(t_1,\dots,t_n), g_2(t_1,\dots,t_n), \dots, g_n(t_1,\dots,t_n))$$

where the g_j are the elementary symmetric polynomials mentioned in Section 2. Since the mapping $t \to s$ is defined by polynomials of degree at most n, Bézout's theorem implies that the multiplicity of the mapping $t \to s$ is at most $n!$ except on a set of measure zero. This means that we can decompose $\mathbb{R}^n$ (except a set of measure zero) into a finite number of sets $\Omega_1, \dots, \Omega_M$ so that on each Ω_i the map $t \to s$ is one-to-one. So it follows that

$$(3.5)\qquad L^k_\lambda(f_1, f_2, \dots, f_n)(x) = \sum_{j=1}^M \int_{G(\Omega_j)} e^{i\lambda\Phi(x,s)}\chi_k(t(s))A(x,s)\prod_{i=1}^n f_i(t_i(s))\left|\frac{\partial t}{\partial s}\right| ds$$

where

$$\Phi(x,s) = \phi(x,t_1(s)) + \cdots + \phi(x,t_n(s)) \quad\text{and}\quad A(x,s) = \prod_{i=1}^n a(x,t_i(s)).$$

The condition (1.3) implies that if the support of a is sufficiently small, then there is a constant $c > 0$ such that

$$(3.6)\qquad \left|\det\left(\frac{\partial^2\Phi(x,s)}{\partial x_i \partial s_j}\right)\right| \ge c$$

for all $(x, s) \in \operatorname{supp} A$.

We may see this as follows. A simple computation shows that

$$(3.7)\qquad \left|\frac{\partial s}{\partial t}\right| = \left|\frac{\partial(s_1\cdots s_n)}{\partial(t_1\cdots t_n)}\right| = \prod_{1\le i<j\le n}|t_i - t_j|.$$

Now set

$$\gamma_x(t) = \nabla_x\phi(x,t).$$

By a generalization of the mean value theorem (see [PS], part V, Chapter 1, problem 95), there exist $u_1, \dots, u_n \in (\min_i t_i(s), \max_i t_i(s))$ such that

$$\det(\gamma'_x(t_1(s)), \cdots, \gamma'_x(t_n(s))) = \frac{1}{c_n}\det(\gamma'_x(u_1), \dots, \gamma^{(n)}_x(u_n))\prod_{i<j}|t_i(s) - t_j(s)|$$

where $c_n = 2!3!\cdots(n-1)!$. So (3.6) follows from (1.3), since

$$\left|\det\left(\frac{\partial^2\Phi(x,s)}{\partial x_i \partial s_j}\right)\right| = |\det(\gamma_x'(t_1(s)),\ldots,\gamma_x'(t_n(s)))| \left|\frac{\partial t}{\partial s}\right|$$
$$= \frac{1}{2}|\det(\gamma_x'(u_1),\ldots,\gamma_x^{(n)}(u_n))|.$$

Lemma 2.1 implies that Φ and A are smooth functions with respect to s. This fact and (3.6) allow us to apply a well-known oscillatory integral estimate of Hörmander (see [S], p. 377) to (3.5). Thus we obtain

$$\|L_\lambda^k(f_1,f_2,\ldots,f_n)\|_2^2 \le C\lambda^{-n}\int \chi_k(t(s))\prod_{i=1}^n |f_i(t_i(s))|^2 \left|\frac{\partial t}{\partial s}\right|^2 ds.$$

By reversing the change of variables made above, we get

$$\|L_\lambda^k(f_1,f_2,\ldots,f_n)\|_2^2 \le C\lambda^{-n}\int \chi_k(t)\prod_{i=1}^n |f_i(t_i)|^2 \prod_{1\le i<j\le n}|t_i-t_j|^{-1}dt.$$

Since

$$\sup_{t_1}|\{(t_2,\ldots,t_n) : \prod_{1\le i<j\le n}|t_i-t_j| \sim 2^{-k}\}| = |\{h\in\mathbb{R}^{n-1} :\ v(h)\sim 2^{-k}\}|,$$

it follows that

$$\|L_\lambda^k(f_1,f_2,\ldots,f_n)\|_2 \le C\lambda^{-n/2}2^{k/2}|\{h\in\mathbb{R}^{n-1} :\ v(h)\le 2^{-k}\}|^{1/2}\|f_1\|_2\prod_{i=2}^n\|f_i\|_\infty.$$

Hence, by Lemma 2.2, we have

$$\|L_\lambda^k(f_1,f_2,\ldots,f_n)\|_2 \le C\lambda^{-n/2}2^{(n-2)k/2n}\|f_1\|_2\prod_{i=2}^n\|f_i\|_\infty. \tag{3.8}$$

An estimate following from the hypothesis (3.1): For $h=(h_1,\ldots,h_{n-1})\in\mathbb{R}^{n-1}$, define ϕ_h by

$$\phi_h(x,t) = \phi(x,t)+\phi(x,t+h_1)+\cdots+\phi(x,t+h_{n-1}).$$

We make the change of variables $t_1\to\tau, t_2\to\tau+h_1,\ldots,t_n\to\tau+h_{n-1}$ in (3.4) to get

$$L_\lambda^k(f_1,f_2,\ldots,f_n)(x) = \int_{\{h\in\mathbb{R}^{n-1}:\ v(h)\sim 2^{-k}\}}\int e^{i\lambda\phi_h(x,\tau)}a_h(x,\tau)f_h(\tau)d\tau dh \tag{3.9}$$

where

$$a_h(x,\tau) = a(x,\tau)\prod_{i=2}^n a(x,\tau+h_{i-1}) \quad\text{and}\quad f_h(\tau) = f_1(\tau)\prod_{i=2}^n f_i(\tau+h_{i-1}).$$

For h and x fixed let

$$\gamma_x^h(t) = \nabla_x\phi_h(x,t).$$

From the smoothness of ϕ, it follows immediately by continuity that if $|h|$ is sufficiently small, then $\frac{1}{n}\left(\frac{d}{d\tau}\right)^k\gamma_x^h(\tau)$ is close to $\left(\frac{d}{d\tau}\right)^k\gamma_x(\tau)$. Therefore, if the support of a is sufficiently small, then the linear independence of

$$\frac{d}{d\tau}\gamma_x^h(\tau),\ldots,\left(\frac{d}{d\tau}\right)^n\gamma_x^h(\tau)$$

is a consequence of the linear independence of $\gamma_x'(\tau), \gamma_x''(\tau), \ldots, \gamma_x^{(n)}(\tau)$, which is the condition (1.3). Thus the phase function ϕ_h satisfies the assumption (1.3) uniformly in small h. Therefore, after a use of Minkowski's inequality we may apply the estimate in (3.1) to the inner integral in (3.9) with a bound uniform in small h. This yields

$$\|L_\lambda^k(f_1, f_2, \ldots, f_n)\|_{s,\infty} \le C\lambda^{-n/s}|\{h \in \mathbb{R}^{n-1} : v(h) \le 2^{-k}\}| \cdot \|f_1\|_{r,1} \prod_{i=2}^{n} \|f_i\|_\infty.$$

Using Lemma 2.2 again, we obtain

$$\|L_\lambda^k(f_1, f_2, \ldots, f_n)\|_{s,\infty} \le C\lambda^{-n/s} 2^{-2k/n} \|f_1\|_{r,1} \prod_{i=2}^{n} \|f_i\|_\infty. \tag{3.10}$$

Summation of the estimates for L_λ^k: We have $L_\lambda = \sum_{k=-\infty}^{\infty} L_\lambda^k$. Let us fix $f_2, \ldots, f_n \in L^\infty$ and apply Lemma 2.3 to (3.8) and (3.10). This gives

$$\|L_\lambda(f_1, f_2, \ldots, f_n)\|_{b,\infty} \le C\lambda^{-n/b} \|f_1\|_{a,1} \prod_{i=2}^{n} \|f_i\|_\infty \tag{3.11}$$

where

$$\frac{1}{a} = \frac{n-2}{n+2}\left(\frac{1}{r}\right) + \frac{2}{n+2}, \quad \frac{1}{b} = \frac{n-2}{n+2}\left(\frac{1}{s}\right) + \frac{2}{n+2}.$$

Interpolating the n estimates obtained by permuting the functions in (3.11), and then setting each $f_j = f$, we get

$$\|T_\lambda f\|_{q,\infty} \le C\lambda^{-n/q} \|f\|_{p,1}$$

where $p = na$ and $q = nb$. The constant C is stable under small smooth perturbations of ϕ and a, since the same holds for the constants C in (3.8) and (3.10). From the fact that $1/r + n(n+1)/(2s) = 1$, it follows that

$$\frac{1}{p} + \frac{n(n+1)}{2q} = 1.$$

Note also that q and s are related by

$$\frac{1}{q} = \frac{(n-2)}{n(n+2)s} + \frac{2}{n(n+2)}.$$

This completes the proof of Proposition 3.1. □

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