LOCAL AUTOMORPHISMS AND DERIVATIONS ON $M_n$

SANG OG KIM AND JU SEON KIM

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Abstract. The aim of this note is to give a short proof that 2-local derivations on $M_n$, the $n \times n$ matrix algebra over the complex numbers are derivations and to give a shorter proof that 2-local $*$-automorphisms on $M_n$ are $*$-automorphisms.

A mapping $\phi$ of an algebra $\mathcal{A}$ into itself is called a local automorphism (respectively, local derivation) if for every $A \in \mathcal{A}$ there exists an automorphism (respectively, local derivation) $\phi_A$ of $\mathcal{A}$, depending on $A$, such that $\phi(A) = \phi_A(A)$. These notions were introduced by Kadison [Kad] and Larson and Sourour [LaSo]. In fact, their definitions were stronger. They have assumed that these mappings are also linear. Larson and Sourour proved that every local derivation on $B(\mathcal{X})$, the algebra of all bounded linear operators on a Banach space $\mathcal{X}$, is a derivation, and provided that $\mathcal{X}$ is infinite dimensional, every surjective linear local automorphism of $B(\mathcal{X})$ is an automorphism. In [BrSe], they proved that the surjectivity assumption in the last result can be dropped if $\mathcal{X}$ is a separable Hilbert space.

It is easy to see that if we drop the assumption of linearity of the local maps, then the corresponding statements are no longer true. However, in [KoSl], they obtained the following result: If $\mathcal{A}$ is a unital Banach algebra and if $\phi : \mathcal{A} \rightarrow \mathbb{C}$ is a map (no linearity is assumed) having the property that $\phi(I) = 1$ and for every $A, B \in \mathcal{A}$, there exists a multiplicative linear functional $\phi_{A,B}$ on $\mathcal{A}$ such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$, then $\phi$ is linear and multiplicative.

Motivated by the above considerations, Šemrl [Sem] introduced the following definition.

Definition. Let $\mathcal{A}$ be an algebra. A mapping $\phi : \mathcal{A} \rightarrow \mathcal{A}$ is called a 2-local automorphism (respectively, 2-local derivation) if for every $A, B \in \mathcal{A}$ there is an automorphism (respectively, derivation) $\phi_{A,B} : \mathcal{A} \rightarrow \mathcal{A}$, depending on $A$ and $B$, such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$.

Also, they showed the following result.

Theorem 1 ([Sem]). Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Then every 2-local automorphism $\phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ (no linearity, surjectivity or continuity of $\phi$ is
assumed) is an automorphism and every 2-local derivation \( \theta : B(H) \to B(H) \) (no linearity or continuity of \( \theta \) is assumed) is a derivation.

In [Sem, Remark] they say that they get the same results in the case that \( H \) is finite dimensional by a long proof involving tedious computations. We found very recently that Molnár [Mol2] gave a short proof that every 2-local automorphism on \( M_n(C) \) is an automorphism. It is the aim of this note to make a short proof that 2-local derivations on \( M_n(C) \) are derivations and to make the proof shorter in the case of 2-local *-automorphisms (its definition is self-explanatory, i.e., the \( \phi_{A,A} \) is a *-automorphism for every \( A \) and \( B \)). Note that the proof of the following Theorem 2 is similar to that in [Mol1].

**Theorem 2.** Let \( M_n \) be the \( n \times n \) matrix algebra over \( C \) and \( \phi : M_n \to M_n \) a 2-local *-automorphism. Then \( \phi \) is a *-automorphism.

**Proof.** By the well-known result that every *-automorphism of \( A \) is of the form \( A \mapsto UAU^* \) for some unitary \( U \in M_n \), for every \( A, B \in M_n \) there is a unitary \( U \) in \( M_n \) such that

\[
\phi(A) = UAU^* \quad \text{and} \quad \phi(B) = UBU^*.
\]

Then if we let \( \text{tr} \) be the usual trace functional, we have

\[
\text{tr}(\phi(A)\phi(B)^*) = \text{tr}(AB^*).
\]

Then for any \( C \in M_n \),

\[
\text{tr}[(\phi(A + B) - \phi(A) - \phi(B))\phi(C)^*] = \text{tr}[(A + B - A - B)C^*] = 0
\]

by the linearity of \( \text{tr} \), and then we obtain that

\[
\text{tr}[(\phi(A + B) - \phi(A) - \phi(B))(\phi(A + B) - \phi(A) - \phi(B))^*] = 0.
\]

Consequently, it follows that \( \phi \) is additive. Let \( A \) be an element of \( M_n \) and let \( \lambda \) be any scalar. If we use the 2-locality of \( \phi \) to the elements \( A \) and \( \lambda A \), we have

\[
\phi(\lambda A) = \phi_{A,\lambda A}(\lambda A) = \lambda \phi_{A,\lambda A}(A) = \lambda \phi(A).
\]

Then \( \phi \) is homogeneous, and hence \( \phi \) is a linear map. Since the set of eigenvalues of \( \phi(A) \) according to multiplicity is the same as that of \( A \in M_n \) and \( \phi(A^*) = \phi_{A,A^*}(A^*) = \phi_{A,A^*}(A)^* = \phi(A)^* \), there exists by [MaMo] Theorem 4] a unitary \( U \in M_n \) such that \( \phi \) is either of the form

\[
\phi(A) = UAU^* \quad (A \in M_n)
\]

or of the form

\[
\phi(A) = UA^TU^* \quad (A \in M_n).
\]

Suppose, on the contrary, that \( \phi(A) = UA^TU^* \) for all \( A \in M_n \). Take two matrices \( A \) and \( B \) in \( M_n \) such that \( AB \neq 0 \) and \( BA = 0 \). Then

\[
0 \neq \phi_{A,B}(AB) = \phi(A)\phi(B) = UA^TU^*UB^TU^* = U(BA)^TU^* = 0.
\]

This shows that \( \phi(A) = UAU^*(A \in M_n) \), completing the proof. \( \square \)

Now we consider 2-local derivations on \( M_n \). We have to mention that the idea for the proof of the following Theorem 3 comes from that of [Sem].

**Theorem 3.** Let \( M_n \) be the \( n \times n \) matrix algebra over \( C \) and \( \phi : M_n \to M_n \) a 2-local derivation. Then \( \phi \) is a derivation.
Proof. Let \(\{e_1, e_2, \ldots, e_n\}\) be the standard basis for \(\mathbb{C}^n\). We define two matrices \(A, N \in M_n\) by

\[
A = \begin{pmatrix}
\frac{1}{2} & 0 & \ldots & 0 \\
0 & \frac{1}{2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{2}
\end{pmatrix}, \quad N = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]

It is easy to see that \(T \in M_n\) commutes with \(A\) if and only if it is diagonal, and if \(U\) commutes with \(N\), then \(U\) is of the form \(Ue_k = \mu_k e_1 + \mu_{k-1} e_2 + \cdots + \mu_1 e_k\) \((k = 1, 2, \ldots, n)\) for some \(\{\mu_1, \mu_2, \ldots, \mu_n \mid \mu_k \in \mathbb{C}\}\). That is, \(U\) is of the form

\[
U = \begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 & \ldots & \mu_n \\
0 & \mu_1 & \mu_2 & \ldots & \mu_{n-1} \\
0 & 0 & \mu_1 & \ldots & \mu_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \mu_1 & \mu_2 \\
0 & 0 & \ldots & 0 & \mu_1
\end{pmatrix}.
\]

Replacing \(\phi\) by \(\phi - \phi_{A,N}\), if necessary, we can assume that \(\phi(A) = \phi(N) = 0\). Every derivation on \(M_n\) is inner. It follows that for every \(T \in M_n\) there exist diagonal \(D\) and \(U\) of the above form, depending on \(T\), such that \(\phi(T) = TD - DT = TU - UT\).

Let \(\{E_{ij}\}_{i,j=1,\ldots,n}\) be the system of matrix units of \(M_n\). Then for any fixed \(i\) and \(j\), we have \(\phi(E_{ij}) = E_{ij}D - DE_{ij} = E_{ij}U - UE_{ij}\) for some \(D = \text{diag}(\lambda_1, \ldots, \lambda_n)\) and \(U\) of the above form. Since \(E_{ij}D - DE_{ij} = (\lambda_j - \lambda_i)E_{ij}\) and \(E_{ij}U - UE_{ij}\) has 0 as \((i,j)\)-entry, it follows that \(\phi(E_{ij}) = 0\). Noting that \(E_{ij}\) is the rank one operator \(e_i \otimes e_j\), we then have for any \(T \in M_n\),

\[
E_{ij}\phi(T)E_{ij} = \phi_{E_{ij},T}(E_{ij}TE_{ij}) \\
= \langle Te_i, e_j \rangle \phi(E_{ij}) \\
= \langle Te_i, e_j \rangle \phi(E_{ij}) \\
= 0.
\]

From this equation it follows that \(\langle \phi(T)e_i, e_j \rangle E_{ij} = 0\) and hence \(\phi(T) = 0\), completing the proof.

\begin{thebibliography}{9}
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Department of Mathematics, Hallym University, Chuncheon 200-702, Korea
E-mail address: sokim@hallym.ac.kr

Department of Mathematics Education, Seoul National University, Seoul, 151-742, Korea