ON BOHMAN'S CONJECTURE RELATED TO
A SUM PACKING PROBLEM OF ERDŐS

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ABSTRACT. Motivated by a sum packing problem of Erdős, Bohman discussed
an extremal geometric problem which seems to have an independent interest.
Let H be a hyperplane in $\mathbb{R}^n$ such that $H \cap \{0, \pm 1\}^n = \{0^n\}$. The problem is
to determine
$$f(n) \triangleq \max_H |H \cap \{0, \pm 1, \pm 2\}^n|.$$
Bohman (1996) conjectured that
$$f(n) = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n.$$
We show that for some constants c_1, c_2 we have $c_1(2,538)^n < f(n) < c_2(2,723)^n$—disproving the conjecture. We also consider a more general question of the estimation of $|H \cap \{0, \pm 1, \dots, \pm m\}|$, when $H \cap \{0, \pm 1, \dots, \pm k\} = \{0^n\}$, $m > k > 1$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Let H be a hyperplane in $\mathbb{R}^n$ so that $H \cap \{0, \pm 1\}^n = \{0^n\}$. Let
$$f(n) = \max_H |H \cap \{0, \pm 1, \pm 2\}^n|.$$

The problem (of determination of $f(n)$) was raised by Bohman [1] in connection
with a subset sum problem of Erdős [2].

A set S of positive integers $b_1 < b_2 < \cdots < b_n$ has distinct subset sums if all
sums of subsets are distinct. Erdős [2] asked for the value of

$$g(n) \triangleq \min\{a_n : S \text{ has distinct subset sums}, |S| = n\}.$$

A long-standing conjecture of Erdős claims that $g(n) \geq c2^n$ for some constant c.

In [1] Bohman explained the relationship between functions $f(n)$ and $g(n)$, and
noticed that studying the function $f(n)$ might be helpful for further investigation
of the problem of Erdős.

Suppose a hyperplane H defined by the equation

(1.1) $$\sum_{i=0}^{n-1} a_i x_i = 0; \ a_0, \dots, a_{n-1} \in \mathbb{N}$$

satisfies $H \cap \{0, \pm 1\}^n = \{0^n\}$. This clearly means that $\{a_0, \ldots, a_{n-1}\}$ has distinct subset sums. A simple example of such a set with $a_{n-1} \leq 2^{n-1}$ is $\{1, 2, 2^2, \ldots, 2^{n-1}\}$. For more complicated examples see [1], [3].

For $f(n)$, Bohman [1] conjectured that

$$f(n) = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n,$$

showing that this number can be achieved, taking $a_i = 2^i$ $(i = 0, \ldots, n-1)$ in (1.1).

Let us consider now the hyperplanes defined by

$$(1.2) \qquad \sum_{i=0}^{n-1} 2^i \lambda_i x_i = 0,$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are odd integers.

One can easily see that the set $\{\lambda_0, 2\lambda, \ldots, 2^{n-1}\lambda_{n-1}\}$ has distinct subset sums.

Let $f^*(n)$ denote the maximum possible number of solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of equation (1.2) over all choices of odd integers $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$.

Theorem 1. *For some constants c', c''*

$$c'(2,538)^n < f^*(n) < c''(2,547)^n.$$

Clearly this means that $f(n) > c_1(2,538)^n$ and the conjecture of Bohman fails.

Our next goal is to give an upper bound for $f(n)$. A simple upper bound is

$$(1.3) \qquad f(n) \leq 3^n.$$

Indeed, let X be the set of solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of equation (1.1). Then observe that for any $u^n, v^n \in \{0, 1\}^n$, $u^n \neq v^n$, we have $(X + u^n) \cap (X + v^n) = \varnothing$. This implies that $|X + \{0, 1\}^n| = |X||\{0, 1\}^n| = |X|2^n$. On the other hand, $\{X + \{0, 1\}^n\} \subset \{0, \pm 1, \pm 2, 3\}^n$. Hence $|X|2^n \leq 6^n$ and thus (1.3). The next result improves bound (1.3).

Theorem 2. *For some constant c,*

$$f(n) < c(2,723)^n.$$

Conjecture 1. *For some constant c,*

$$f(n) \sim c\beta^n,$$

where β is the largest real root of the equation $z^8 - 8z^6 + 10z^4 + 1 = 0$ $(\beta = 2,5386\ldots)$. The construction attaining this number is given in section 2.

We also consider a more general problem. Let $Q \subset \mathbb{Z}$ be finite and $F = \{0, \pm 1, \ldots, \pm k\}$. Then

$$f(n, Q, F) \triangleq \max\{|H \cap Q^n| : H \text{ is a hyperplane and } H \cap F^n = \{0^n\}\}.$$

In some cases we succeed in giving the exact answer.

Theorem 3. (i) *Let $Q = \{0, \pm 1, \ldots, \pm m\}$, $F = \{0, \pm 1, \cdots \pm k\}$ and $k + 1 | 2m + 1$. Then*

$$f(n, Q, F) = \left(\frac{2m + 1}{k + 1}\right)^{n-1}.$$

(ii) *Let $Q = \{0, \pm 1, \ldots, \pm(m-1), m\}$, $F = \{0, \pm 1, \ldots, \pm k\}$ and $k + 1 | 2m$. Then*

$$f(n, Q, F) = \left(\frac{2m}{k + 1}\right)^{n-1}.$$

An interesting case is

$$Q = \{0, \pm 1, \ldots, \pm(k+1)\}, \quad F = \{0, \pm 1, \ldots, \pm k\}, \quad k \geq 1.$$

Note that for $k = 1$ we have Bohman's problem. It can be shown that

$$(1 + \sqrt{2})^n \leq f(n, Q, F) \leq 3^n.$$

The upper bound is derived exactly as above for $k = 1$. For the lower bound consider the equation

$$(1.4) \qquad x_0 + (k+1)x_1 + \cdots + (k+1)^{n-1}x_{n-1} = 0.$$

Let $X \subset Q^n$ denote the set of solutions of (1.4). Clearly, $X \cap F^n = \{0^n\}$. On the other hand, one can show that $|X| = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n$ (as for $k = 1$). We believe that Bohman's conjecture is true for $k \geq 2$, that is,

Conjecture 2. For $Q = \{0, \pm 1, \ldots, \pm(k+1)\}$, $F = \{0, \pm 1, \ldots, \pm k\}$ and $k \geq 2$ (or a weaker condition for $k > k_0$) one has

$$f(n, Q, F) = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n.$$

2. Proof of Theorem 1

We start with an auxiliary statement. Let $f_\lambda^*(n)$ denote the maximum number of solutions $x^n = (x_0, \ldots, x_{n-1}) \in \{0, \pm 1, \pm 2\}^n$ of the equation

$$(2.1) \qquad \sum_{i=0}^{n-1} 2^i \lambda_i x_i = \lambda$$

over all choices of odd integers $\lambda_0, \ldots, \lambda_{n-1}$ and a given integer λ. Remember that $f_0^*(n) = f^*(n)$.

Lemma 1.

$$f^*(n) \geq \frac{1}{25} f_\lambda^*(n).$$

Proof. Suppose we have an optimal equation (2.1). That is, for the solutions of (2.1), $X \subset \{0, \pm 1, \pm 2\}^n$, we get $|X| = f_\lambda^*(n)$.

For an integer μ consider the equation

$$(2\mu + 1)y + 2z + 4\lambda_0 x_0 + \cdots + 2^{n+1}\lambda_{n-1}x_{n-1} = 0.$$

Then taking $y = -2$, $z = 1$, we come to equation $\sum_{i=0}^{n-1} 2^i \lambda_i x_i = \mu$, which implies that $f^*(n+2) \geq \max_\mu f_\mu(n)$. On the other hand, clearly

$$\max_\mu f_\mu(n) \geq \frac{1}{25} f_\lambda^*(n+2).$$

$\square$

Consider the equation

$$(2.2) \qquad x_0 + 2x_1 + \cdots + 2^{n-1}x_{n-1} = \lambda.$$

Let $X(\lambda)$ be the set of all solutions (from $\{0, \pm 1, \pm 2\}^n$) of (2.2). With the help of this lemma we can get a lower bound using an average argument. There are 5^n vectors $(x_0, \ldots, x_{n-1}) \in \{0, \pm 1, \pm 2\}^n$. On the other hand, there are $4(2^n - 1) + 1$

possible values for λ for which equation (2.2) has solutions. Hence there exists a λ such that

$$|X(\lambda)| \geq \frac{5^n}{4(2^n - 1) + 1}.$$

This together with Lemma 1 implies that $f(n) \geq c(2,5)^n$ for some constant c, which actually disproves the conjecture of Bohman. However, we can improve this bound constructively.

Lower bound. As above let $X(\lambda) = H \cap \{0, \pm 1, \pm 2\}^n$, where H is the hyperplane defined by (2.2).

Also, let $h_\lambda(n)$ denote the number of solutions of (2.2), that is, $h_\lambda(n) = |X(\lambda)|$. Suppose that $\lambda = 2s$, where s is an integer. Then observe that

$$(2.3) \qquad h_{2s}(n) = h_{s-1}(n-1) + h_s(n-1) + h_{s+1}(n-1).$$

Correspondingly, if $\lambda = 2s + 1$, then

$$(2.4) \qquad h_{2s+1}(n) = h_s(n-1) + h_{s+1}(n-1).$$

For a positive integer n, define

$$(2.5) \qquad S_n = \begin{cases} 2^{n-1} + 2^{n-3} + \cdots + 2^3 + 2, & \text{if } 2 \mid n; \\ 2^{n-1} + 2^{n-3} + \cdots + 2^2 + 1, & \text{if } 2 \nmid n. \end{cases}$$

Claim. For $2 \mid n$ and some constant c,

$$(2.6) \qquad h_{S_n}(n) > c(2,538)^n.$$

Proof. In view of (2.3) we have

$$(2.7) \qquad h_{S_n}(n) = h_{S_{n-1}-1}(n-1) + h_{S_{n-1}}(n-1) + h_{S_{n-1}+1}(n-1).$$

Correspondingly,

$$\begin{aligned}
h_{S_{n-1}-1}(n-1) &= h_{S_{n-2}-1}(n-2) + h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2), \\
(2.8) \qquad h_{S_{n-1}}(n-1) &= h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2), \\
h_{S_{n-1}+1}(n-1) &= h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2) + h_{S_{n-2}+2}(n-2).
\end{aligned}$$

It is easy to see that $h_{S_n}(n)$ can be represented by linear combinations of the functions $h_{S_{n-i}-1}(n-i), h_{S_{n-i}}(n-i), h_{S_{n-i}+1}(n-i), h_{S_{n-i}+2}(n-i)$.

In view of (2.7) and (2.8) we can write

$$
\begin{aligned}
h_{s_n}(n) &= h_{S_{n-1}+1}(n-1) + h_{S_{n-1}}(n-1) + h_{S_{n-1}}(n-1) \\
&= h_{S_{n-2}-1}(n-2) + 3h_{S_{n-2}}(n-2) + 3h_{S_{n-2}+1}(n-2) \\
&\quad + h_{S_{n-2}+2}(n-2) \\
&= 4h_{S_{n-3}-1}(n-3) + 8h_{S_{n-3}}(n-3) + 7h_{S_{n-3}+1}(n-3) \\
&\quad + h_{S_{n-3}+2}(n-3) \\
&= 4h_{S_{n-4}-1}(n-4) + 19h_{S_{n-4}}(n-4) + 20h_{S_{n-4}+1}(n-4) \\
&\quad + 8h_{S_{n-4}+2}(n-4) \\
&\quad \ldots \\
&= a_i h_{S_{n-i}-1}(n-i) + b_i h_{S_{n-i}}(n-i) + c_i h_{S_{n-i}+1}(n-i) \\
&\quad + d_i h_{S_{n-i}+2}(n-i) \\
&\quad \ldots \\
&= a_{n-1} h_{S_1-1}(1) + b_{n-1} h_{S_1}(1) + c_{n-1} h_{S_1+1}(1) + d_{n-1} h_{S_1+2}(1) \\
&= a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}.
\end{aligned}
$$
(2.9)

From (2.7), (2.8) and (2.9) we obtain the following recurrences for the coefficients a_i, b_i, c_i, d_i in (2.9):

(2.10)
$$
\begin{aligned}
a_{2i} &= a_{2i-1}, \\
b_{2i} &= a_{2i-1} + b_{2i-1} + c_{2i-1}, \\
c_{2i} &= a_{2i-1} + b_{2i-1} + c_{2i-1} + d_{2i-1}, \\
d_{2i} &= c_{2i-1} + d_{2i-1};
\end{aligned}
$$

(2.11)
$$
\begin{aligned}
a_{2i+1} &= a_{2i} + b_{2i}, \\
b_{2i+1} &= a_{2i} + b_{2i} + c_{2i} + d_{2i}, \\
c_{2i+1} &= b_{2i} + c_{2i} + d_{2i}, \\
d_{2i+1} &= d_{2i} \ (i = 1, 2, \ldots).
\end{aligned}
$$

Here are the first ten values of a_i, b_i, c_i, d_i.

a_i:	1	1	4	4	23	23	144	144	921	921,
b_i:	1	3	8	19	51	121	328	777	2113	5003,
c_i:	1	3	7	20	47	129	305	832	1969	5363,
d_i:	0	1	1	8	8	55	55	360	360	2329.

From (2.10) and (2.11) we obtain by elementary algebraic transformations the following recurrences:

$$
t_{i+8} = 8t_{i+6} - 10t_{i+4} - t_i \text{ for } t_i \in \{a_i, b_i, c_i, d_i\}, \ i = 1, 2, \ldots .
$$

In particular, we have

(2.12)
$$
c_{2i+8} = 8c_{2i+6} - 10c_{2i+4} - c_{2i}
$$

with initial values $c_2 = 3$, $c_4 = 20$, $c_6 = 129$, $c_8 = 832$.

The characteristic equation of (2.12),

(2.13)
$$
z^8 - 8z^6 + 10z^4 + 1 = 0,
$$

has a largest real root $\beta = 2,5386\ldots$.

Thus c_{2i} can be estimated from below by $c_{2i} \geq c\beta^{2i} > c(2.538)^{2i}$, for some constant c definable from the initial values of c_{2i}.

Furthermore, in view of (2.9) and (2.10) for $n = 2k$ we have

$$h_{S_n}(n) = a_{2k-1} + b_{2k-1} + c_{2k-1} + d_{2k-1} = c_n,$$

which implies that $h_{S_n}(n) > c(2,538)^n$.

Thus we have proved that $f^*_{S_n}(n) > (2,538)^n$. This with Lemma 1 completes the proof of the lower bound.

Upper bound. Consider the equation

$$(2.14) \qquad \lambda_0 x_0 + 2\lambda_1 x_1 + \cdots + 2^{n-1}\lambda_{n-1} x_{n-1} = \lambda.$$

We distinguish the three cases:

(α) $\lambda \equiv 2 \pmod 4$: Then denote by $h_\alpha(n)$ the maximum possible number of solutions (from $\{0, \pm 1, \pm 2\}^n$ of equation (2.14)).
(β) $\lambda \equiv 0 \pmod 4$: The corresponding notation for this case is $h_\beta(n)$.
(γ) $\lambda \equiv 1$ or $3 \pmod 4$: The corresponding notation for this case is $h_\gamma(n)$.

Then one can easily observe that the following recurrence relations hold:

$$
\begin{aligned}
&h_\alpha(n) \leq h_\alpha(n-1) + h_\beta(n-1) + h_\gamma(n-1),\\
(2.15) \qquad &h_\beta(n) \leq \max\{h_\alpha(n-1), h_\beta(n-1)\} + 2h_\gamma(n-1),\\
&h_\gamma(n) \leq \max\{h_\alpha(n-1), h_\beta(n-1)\} + h_\gamma(n-1).
\end{aligned}
$$

We also have that $h_\alpha(1) = h_\beta(1) = h_\gamma(1) = 1$.

Now introduce the functions $g_\alpha(n)$, $g_\beta(n)$, and $g_\gamma(n)$, so that $g_\alpha(1) = g_\beta(1) = g_\gamma(1) = 1$, and

$$
\begin{aligned}
g_\alpha(n) &= g_\alpha(n-1) + g_\beta(n-1) + g_\gamma(n-1),\\
g_\beta(n) &= \max\{g_\alpha(n-1), g_\beta(n-1)\} + 2g_\gamma(n-1),\\
g_\gamma(n) &= \max\{g_\alpha(n-1), g_\beta(n-1)\} + g_\gamma(n-1).
\end{aligned}
$$

Clearly, we have that $g_\alpha(n) \geq h_\alpha(n), g_\beta(n) \geq h_\beta(n), g_\gamma(n) \geq h_\gamma(n)$.

Observe also that for $n \geq 3$ we have $g_\alpha(n) > g_\beta(n) > g_\gamma(n)$.

Hence, finally, we come to the recurrences

$$
\begin{aligned}
&g_\alpha(n) = g_\alpha(n-1) + g_\beta(n-1) + g_\gamma(n-1),\\
(2.16) \qquad &g_\beta(n) = g_\alpha(n-1) + 2g_\gamma(n-1),\\
&g_\gamma(n) = g_\alpha(n-1) + g_\gamma(n-1).
\end{aligned}
$$

From (2.16) we obtain the recurrence

$$(2.17) \qquad g_\alpha(n) = 2g_\alpha(n-1) + g_\alpha(n-2) + g_\alpha(n-3)$$

with initial values $g_\alpha(1) = 1, g_\alpha(2) = 3, g_\alpha(3) = 8$.

Now to estimate the function $f^*(n)$ it remains to solve recurrence (2.17), since $f^*(n) \leq g_\alpha(n)$. The latter gives the estimation

$$g_\alpha(n) \leq c''(2,547)^n$$

for some constant c'' definable from the initial values. This completes the proof of Theorem 1. $\qquad\square$

3. Proof of Theorem 2

Suppose that $\{a_1, \ldots, a_n\} \subset \mathbb{N}$ has distinct subset sums. Let X denote the set of all solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of the equation $\sum\limits_{i=1}^{n} a_i x_i = \lambda$.

Consider two mappings φ_0 and φ_1 from $\{0, \pm 1, \pm 2\}$ to $\{0, \pm 1\}$ $\varphi_0(-2) = \varphi_1(-2) = -1$, $\varphi_0(2) = \varphi_1(2) = 1$, $\varphi_0(\pm 1) = \varphi_1(\pm 1) = 0$, and $\varphi_0(0) = -1$, $\varphi_1(0) = 1$.

Next for $x^n \in X$ define

$$\varphi(x^n) = \big\{ (\varphi_{\varepsilon_1}(x_1), \ldots, \varphi_{\varepsilon_n}(x_n)) : \varepsilon_i \in \{0, 1\}, i = 1, \ldots, n \big\}.$$

Claim 1. For $x^n, y^n \in X$, $x^n \neq y^n$,

$$\varphi(x^n) \cap \varphi(y^n) = \varnothing.$$

Proof. Suppose the opposite. Then it is not hard to verify that $x^n - y^n \in \{0, \pm 2\}^n \setminus \{0^n\}$, a contradiction. $\square$

Let us define

$$\alpha(x^n) = \text{ the number of zero coordinates in } x^n.$$

Claim 2. For any $x^n \in X$,

$$|\varphi(x^n)| = 2^{\alpha(x^n)}.$$

Proof. This immediately follows from the definition of $\varphi(x^n)$. $\square$

Combining Claims 1 and 2, we conclude that

(3.1) $$\sum_{x^n \in X} 2^{\alpha(x^n)} \leq 3^n.$$

Now consider the mapping $\Psi : X \to \{0, \pm 1\}^n$, defined by $\Psi(x^n) = \big(\Psi_0(x_1), \ldots, \Psi_0(x_n)\big)$, where

$$\Psi_0(x_i) = \begin{cases} -1, & \text{if } x_i = -2, -1, \\ 1, & \text{if } x_i = 2, 1, \\ 0, & \text{if } x_i = 0; i = 1, \ldots, n. \end{cases}$$

Claim 3. For $x^n, z^n \in X$, $x^n \neq z^n$ holds for $\Psi(x^n) \neq \Psi(z^n)$.

Proof. Assuming the opposite, we will get $x^n - z^n \in \{0, \pm 1\}^n \setminus \{0^n\}$, a contradiction. $\square$

Note (and this is important for us) that Ψ leaves the zero coordinates fixed. This with (3.1) implies that

$$\sum_{y^n \in \Psi(X)} 2^{\alpha(y^n)} \leq 3^n.$$

Since $|X| = |\Psi(X)|$, we can bound $|X|$ by the maximum cardinality of a set $Y \subset \{0, \pm 1\}^n$ satisfying

(3.2) $$\sum_{y^n \in Y} 2^{\alpha(y^n)} \leq 3^n.$$

Define

$$Y_i = \big\{ y^n \in Y : \alpha(y^n) = i \big\}, i = 0, 1, \ldots, n.$$

Note that $|Y_i| \leq 2^{n-i} \binom{n}{i}$.

Now (3.2) can be rewritten in the form

$$(3.3) \qquad \sum_{i=0}^{n} |Y_i| 2^i \leq 3^n.$$

Observe that to maximize $|Y| = \sum_{i=0}^{n} |Y_i|$ we have to take

$$|Y_i| = \begin{cases} \binom{n}{i} 2^{n-i}, & \text{if } i \leq \ell(n); \\ 0, & \text{if } i > \ell(n) \end{cases}$$

where $\ell(n)$ is the maximal index for which we get

$$\sum_{i=0}^{\ell(n)} 2^{n-i} \binom{n}{i} 2^i \leq 3^n.$$

This gives (using a standard technique) that $\ell(n) \geq \lfloor 0,1402\,n \rfloor$. Correspondingly, we get an estimation for $|Y|$ and consequently for $|X|$,

$$|X| \leq |Y| < c \frac{3^n}{2^{0,14n}} < c(2,723)^n,$$

for some constant c. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Proof of Theorem 3

Let $Q = \{0, \pm 1, \ldots, \pm m\}$, $F = \{0, \pm 1, \ldots, \pm k\}$ with $\alpha = (2m+1)/(k+1)$.

(a) First we will show that $f(n, Q, F) \leq \alpha^{n-1}$. Let H be defined by

$$(4.1) \qquad \sum_{i=1}^{n} a_i x_i = 0.$$

Also, let $H \cap F^n = \{0^n\}$ and $H \cap Q^n = X$ with $|X| = f(n, Q, F)$.

Define $Q_j = \{a \in Q : a \equiv j \pmod{\alpha}\}$, $j = 0, 1, \ldots, \alpha - 1$.

Then consider the mapping $\varphi : X \to \mathbb{Z}_\alpha^n$, defined by the transformation of coordinates, $\varphi(x_1, \ldots, x_n) = \big(\varphi_0(x_1), \ldots, \varphi_0(x_n)\big)$, where $\varphi_0(x_i) = j$ $(i = 1, \ldots, n)$ if $x_i \in Q_j$; $j \in \{0, \ldots, \alpha - 1\}$. Observe that φ is an injection. Hence $|X| = |\varphi(X)|$.

Note that now

$$\dim\big(\operatorname{span}\varphi(X)\big) \leq \dim\big(\operatorname{span}(X)\big) = n - 1.$$

This implies that

$$(4.2) \qquad\qquad |X| = |\varphi(X)| \leq \alpha^{n-1}.$$

(b) Next, we will show that bound (4.2) can be achieved by taking the hyperplane H defined by

$$(4.3) \qquad x_0 + (k+1)x_1 + \cdots + (k+1)^{n-1}x_{n-1} = 0.$$

In fact, $H \cap F^n = \{0^n\}$. Moreover, we claim that for any $-m \leq \lambda \leq m$, the equation

$$(4.4) \qquad\qquad \sum_{i=0}^{n-1} x_i (k+1)^i = \lambda$$

has exactly α^{n-1} solutions $x^n \in Q^n$. This can be shown using induction on n.

The case $n = 1$ is trivial.

Induction step from $n - 1$ to n: Clearly, $x_0 \in \big\{a : -m \leq a \leq m, a \equiv \lambda$ mod $(k+1)\big\}$. Thus x_0 can take α many values $x_0 \in [-m, m]$. For each x_0 we come to the equation

$$x_1 + (k+1)x_2 + \cdots + (k+1)^{n-2}x_{n-1} = \frac{\lambda - x_0}{k+1}$$

with $\left|\frac{\lambda - x_0}{k+1}\right| \leq \frac{2m}{k+1} \leq m$. Hence we get the result by the induction hypothesis. This completes the proof of Theorem 3 in the case (i). The case (ii) can be proved similarly. $\qquad\square$

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