PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 5, Pages 1453–1461
S 0002-9939(03)07190-9
Article electronically published on September 30, 2003

NUMERICAL RADIUS DISTANCE-PRESERVING MAPS ON $\mathcal{B}(H)$

ZHAOFANG BAI AND JINCHUAN HOU

(Communicated by Joseph A. Ball)

ABSTRACT. Let H be a complex Hilbert space, $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H, $\mathcal{H}(H)$ be the subset of all selfadjoint operators in $\mathcal{B}(H)$ and $\mathcal{V} = \mathcal{B}(H)$ or $\mathcal{H}(H)$. Denote by $w(A)$ the numerical radius of $A \in \mathcal{B}(H)$. We characterize surjective maps $\Phi : \mathcal{V} \to \mathcal{V}$ that satisfy $w(\Phi(A) - \Phi(B)) = w(A - B)$ for all $A, B \in \mathcal{V}$ without the linearity assumption.

1. INTRODUCTION

In the 1940s, L. K. Hua [4]–[11] initiated the study of geometry of matrices. Let $\mathbb{F}$ be a field and $M_n(\mathbb{F})$ be the space of all $n \times n$ matrices over $\mathbb{F}$. To $M_n(\mathbb{F})$ there is an associated group of motions consisting of the following:

$$T \to PTQ + R \text{ and } T \to PT^{tr}Q + R$$

for $T \in M_n(\mathbb{F})$, where P and Q are invertible $n \times n$ matrices, and R is an $n \times n$ matrix. The fundamental problem of geometry of matrices is to characterize the group of motions by as few geometry invariants as possible [17]. Hua discovered that, for some field $\mathbb{F}$, especially the real field $\mathbb{R}$ and the complex field $\mathbb{C}$, the invariant "adjacency" (T and S are adjacent if $\operatorname{rank}(T - S) = 1$) alone is sufficient to characterize the motions (to within automorphisms of the underlying field) on spaces of matrices, symmetric matrices, skew-symmetric matrices and Hermitian matrices, respectively. Motivated by the idea of geometry of matrices, a similar fundamental question may be raised for the infinite-dimensional case, especially for $\mathcal{B}(H)$, where H is an infinite-dimensional Hilbert space and $\mathcal{B}(H)$ is the von Neumann algebra of all bounded linear operators on H, and to develop a similar kind of geometry of operators. The purpose of this paper is to study this kind of problem. We find that the invariant "numerical radius distance" alone is sufficient to characterize the maps from $\mathcal{B}(H)$ onto itself. Furthermore, the numerical radius distance-preserving maps keep the algebraic structure and the geometric structure of $\mathcal{B}(H)$ nicely.

Let H be a nontrivial Hilbert space over the complex field $\mathbb{C}$ with inner products denoted by $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}_{\mathcal{K}}(H)$, $\mathcal{H}(H)$ and $\mathcal{H}_{\mathcal{K}}(H)$ denote the space of all operators in $\mathcal{B}(H)$ with the form $\alpha I + K$, where α is a scalar and K is compact, the real linear

Received by the editors September 27, 2002 and, in revised form, January 11, 2003.
2000 *Mathematics Subject Classification.* Primary 47H20, 47B49; Secondary 47A12.
Key words and phrases. Numerical radius, Isometry, C*-isomorphism.
This work was supported partially by NNSFC and PNSFS.

space of selfadjoint operators, and the real linear space of selfadjoint operators in $\mathcal{B}_{\mathcal{K}}(H)$, respectively. $A \in \mathcal{B}(H)$ is positive if $\langle Ax, x\rangle \geq 0$ for every $x \in H$. Note that A is positive if and only if A is selfadjoint with nonnegative spectrum. Let $\mathcal{V}$ be any one of these four spaces. Recall that [3], for every A in $\mathcal{B}(H)$, the numerical range and the numerical radius of A are defined, respectively, by

$$W(A) = \{\langle Ax, x\rangle \ \mid \ x \in H \text{ and } \|x\| = 1\},$$
$$w(A) = \sup\{|\lambda| \ \mid \ \lambda \in W(A)\}.$$

These concepts and their generalizations have been studied extensively because of their connections and applications to many different areas. In this paper, we show that a surjective map (no linearity assumed) acting on $\mathcal{V}$ preserves the numerical radius distance if and only if it is a composition of unitary equivalence, transpose, conjugate, multiplied by a scalar of modular 1 and added by an operator. We remark that the finite-dimensional case of our main result, that is, the case $\mathcal{V} = M_n(\mathbb{C})$, was recently obtained by C. K. Li and P. Šemrl in [14], where the numerical radius distance-preserving maps on upper triangular matrices were described. Based on the results in this paper and [14], we can characterize the numerical radius distance-preserving surjective maps between atomic nest algebras, which will be discussed in another paper. A characterization of a linear map from $\mathcal{V}$ onto itself preserving the closure of numerical range may be found in [13].

2. Results and proofs

The following is our main result. In the proof of Theorem 1, the argument in Claim 1 is borrowed from [14] by some modifications for the infinite-dimensional case.

Theorem 1. *Let $\mathcal{V} = \mathcal{B}(H)$ or $\mathcal{B}_{\mathcal{K}}(H)$. Let $\Phi : \mathcal{V} \to \mathcal{V}$ be a surjective map. Then Φ satisfies that $w(\Phi(A) - \Phi(B)) = w(A - B)$ for all $A, B \in \mathcal{V}$ if and only if there is a unitary $U \in \mathcal{B}(H)$, a complex unit μ and an operator $S \in \mathcal{V}$ such that one of the following is true:*

(I) $\Phi(A) = \mu UAU^* + S$ *for every* $A \in \mathcal{V}$;
(II) $\Phi(A) = \mu UA^{tr}U^* + S$ *for every* $A \in \mathcal{V}$;
(III) $\Phi(A) = \mu U(A^*)^{tr}U^* + S$ *for every* $A \in \mathcal{V}$;
(IV) $\Phi(A) = \mu UA^*U^* + S$ *for every* $A \in \mathcal{V}$.

Here A^{tr} is the transpose of A with respect to a fixed but arbitrary orthonormal basis of H.

Proof. It is obvious that every map of the form in (I)-(IV) preserves the numerical radius distance. So we only need to check the "only if" part.

Let $\Psi(A) = \Phi(A) - \Phi(0)$. Then $w(\Psi(A)) = w(A)$ for every $A \in \mathcal{B}(H)$ and $\Psi(0) = 0$. Since the numerical radius is a norm on $\mathcal{B}(H)$, it follows from the Mazur-Ulam theorem [15] that Ψ is real linear. Consequently, without loss of generality, we may assume that Φ itself is real linear and only need to prove that Φ is a C*-isomorphism or a conjugate C*-isomorphism multiplied by a scalar of modulus 1; that is, there exists a unitary operator U and a complex unit μ such that Φ has one of the following forms:

(i) $\Phi(A) = \mu UAU^*$ for every $A \in \mathcal{V}$;
(ii) $\Phi(A) = \mu UA^{tr}U^*$ for every $A \in \mathcal{V}$;
(iii) $\Phi(A) = \mu U(A^*)^{tr}U^*$ for every $A \in \mathcal{V}$;

(iv) $\Phi(A) = \mu U A^* U^*$ for every $A \in \mathcal{V}$.

We first assume that $\mathcal{V} = \mathcal{B}(H)$. It is clear that Φ is injective.

Claim 1. There is a unit $\mu \in \mathbb{C}$ such that $\Phi(\lambda I) = \mu\lambda I$ for every $\lambda \in \mathbb{C}$ or $\Phi(\lambda I) = \mu\overline{\lambda} I$ for every $\lambda \in \mathbb{C}$.

We first extend a result in [14] to the infinite-dimensional case.

Assertion. If $S, T \in \mathcal{B}(H)$ are real linearly independent and such that $w(sS + tT) \leq 1$ holds for any $s, t \in \mathbb{R}$ with $s^2 + t^2 \leq 1$, then conditions (a) and (b) are equivalent.

(a) There exists a complex unit μ such that $(S, T) = \mu(I, \pm iI)$.

(b) For any rank one operator $A \in \mathcal{B}(H)$, there are $s, t \in \mathbb{R}$ such that $s^2 + t^2 = 1$ and $w(sS + tT + A) = 1 + w(A)$.

It is obvious that (a) implies (b). Now assume (b) holds. For any $x \in H$ with $\|x\| = 1$ and $\theta \in [0, 2\pi)$, let $A_\theta = (\cos\theta + i\sin\theta)x \otimes x$. It follows from (b) that there are $s_\theta, t_\theta \in \mathbb{R}$ with $s_\theta^2 + t_\theta^2 = 1$ such that

$$w(s_\theta S + t_\theta T + A_\theta) = 1 + w(A_\theta) = 2.$$

So we have a sequence $\{x_n\}_{n=1}^\infty$ of unit vectors in H such that

$$\lim_{n\to\infty} |\langle (s_\theta S + t_\theta T)x_n, x_n\rangle + \langle A_\theta x_n, x_n\rangle| = 2;$$

hence

$$\lim_{n\to\infty} |\langle (s_\theta S + t_\theta T)x_n, x_n\rangle| = 1$$

and

$$\lim_{n\to\infty} |\langle A_\theta x_n, x_n\rangle| = 1.$$

Write $x_n = \alpha_n x + u_n$ for some $u_n \in \{x\}^\perp$. Then

$$|\langle A_\theta x_n, x_n\rangle| = |\alpha_n|^2 \, |\langle A_\theta x, x\rangle| = |\alpha_n|^2 \to 1.$$

Thus $x_n \to \alpha x$ for some unit α. Furthermore,

$$|\langle (s_\theta S + t_\theta T)x, x\rangle| + |\langle A_\theta x, x\rangle| = 2,$$
$$|\langle (s_\theta S + t_\theta T)x, x\rangle| = 1$$

and

$$|\langle (s_\theta S + t_\theta T)x, x\rangle + (\cos\theta + i\sin\theta)| = 2.$$

So

$$\langle (s_\theta S + t_\theta T)x, x\rangle = \cos\theta + i\sin\theta. \tag{2.1}$$

Suppose $\langle Sx, x\rangle = h_1 + ih_2$ and $\langle Tx, x\rangle = k_1 + ik_2$ with $h_1, h_2, k_1, k_2 \in \mathbb{R}$. Let $C = \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix} \in M_2(\mathbb{R})$. By (2.1), for any $\theta \in [0, 2\pi)$, we have

$$\begin{aligned}\langle (s_\theta S + t_\theta T)x, x\rangle &= s_\theta\langle Sx, x\rangle + t_\theta\langle Tx, x\rangle \\ &= s_\theta(h_1 + ih_2) + t_\theta(k_1 + ik_2) = (h_1 s_\theta + k_1 t_\theta) + i(h_2 s_\theta + k_2 t_\theta) \\ &= \cos\theta + i\sin\theta.\end{aligned}$$

Now let $u_\theta = (s_\theta, t_\theta)^{tr}$. Then u_θ is a unit vector in $\mathbb{R}^2$ and

$$Cu_\theta = \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix}\begin{pmatrix} s_\theta \\ t_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = (\cos\theta, \sin\theta)^{tr}.$$

Hence C maps the unit ball in $\mathbb{R}^2$ onto itself, and thus C is an isometry on $\mathbb{R}^2$ with the form

$$C = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \text{ or } C = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}.$$

It follows that $\langle Sx, x\rangle = h_1 + ih_2$ is a complex unit and

$$\langle Tx, x\rangle = \pm i\langle Sx, x\rangle = \langle \pm iSx, x\rangle.$$

Since x is arbitrary, we see that $T = \pm iS$. Also, $|\langle Sx, x\rangle| = 1$, together with the convexity of $W(S)$, implies that $S = \mu I$ for some complex unit μ. Hence $(S, T) = \mu(I, \pm iI)$.

It is easy to check that the real linearly independent pair (I, iI) satisfies $w(sI + tiI) \leq 1$ for any $s, t \in \mathbb{R}$ with $s^2 + t^2 \leq 1$ and the condition (b). Since Φ is real linear and preserves numerical radius, it follows that $(\Phi(I), \Phi(iI))$ meets condition (b) and hence, by the assertion, $(\Phi(I), \Phi(iI)) = \mu(I, \pm iI)$ for some complex unit μ. Now the assertion of Claim 1 is clearly true.

In the sequel, without loss of generality, let $\mu = 1$ and consider the case in which $\Phi(\lambda I) = \lambda I$ for every $\lambda \in \mathbb{C}$.

Claim 2. $\overline{W(\Phi(B))} = \overline{W(B)}$ for every $B \in \mathcal{B}(H)$ and, consequently, Φ preserves the positivity of operators in both directions (i.e., $\Phi(B)$ is positive if and only if B is positive).

Assume that there exists a $\nu \in \mathbb{C}$ such that $\nu \in W(\Phi(B))\backslash\overline{W(B)}$. Then there is a circle with sufficiently large radius and centered at a certain λ ($\lambda \in \mathbb{C}$) such that $\overline{W(B)}$ lies inside the circle, but ν lies outside the circle. Hence

$$w(B - \lambda I) < |\nu - \lambda| \leq w(\Phi(B) - \lambda I) = w(B - \lambda I),$$

which is a contradiction. So $W(\Phi(B)) \subseteq \overline{W(B)}$ and hence $\overline{W(\Phi(B))} \subseteq \overline{W(B)}$. Using the argument to Φ^{-1}, we see that $\overline{W(B)} \subseteq \overline{W(\Phi(B))}$. So for all B, $\overline{W(\Phi(B))} = \overline{W(B)}$.

Claim 3. $P \in \mathcal{B}(H)$ is a rank one projection if and only if $\Phi(P)$ is a rank one projection. Also, $P, Q \in \mathcal{B}(H)$ are two rank one projections orthogonal to each other if and only if $\Phi(P)$, $\Phi(Q)$ are projections of rank one orthogonal to each other.

Note that a positive operator $A \in \mathcal{B}(H)$ is a scalar multiple of a rank one projection if and only if, for any positive operators $B, C \in \mathcal{B}(H)$, $A = B + C$ implies that both B and C are scalar multiples of A [16, Proposition 3.1].

Choose any projection $P \in \mathcal{B}(H)$ of rank one. Then $\Phi(P)$ is positive. If $\Phi(P) = B + C$ with B, C positive, then $P = \Phi^{-1}(B) + \Phi^{-1}(C)$. Thus there is a real number $r \in [0, 1]$ such that $\Phi^{-1}(B) = rP$ and $\Phi^{-1}(C) = (1 - r)P$. From the real-linearity of Φ, we conclude that $B = r\Phi(P)$ and $C = (1 - r)\Phi(P)$. This implies that $\Phi(P)$ is a scalar multiple of a rank one projection, saying $\Phi(P) = \lambda Q$. Noting that $w(\lambda Q) = w(P) = 1$ and λQ is positive, we have $\lambda = 1$. So Φ maps rank one projections to rank one projections.

Suppose that $P, Q \in \mathcal{B}(H)$ are two rank one projections orthogonal to each other, i.e., $PQ = QP = 0$. Write $\Phi(P) = x \otimes x$ and

$$\Phi(Q) = [rx + (1 - r^2)^{\frac{1}{2}}y] \otimes [rx + (1 - r^2)^{\frac{1}{2}}y],$$

where x, y are orthonormal and $r \in [0, 1)$. Note that $w(P - Q) = 1$ and that $\Phi(P) - \Phi(Q)$ has a kernel of codimension two and two eigenvalues $-(1 - r^2)^{\frac{1}{2}}$ and

$(1-r^2)^{\frac{1}{2}}$, which forces that $w(\Phi(P)-\Phi(Q)) = (1-r^2)^{\frac{1}{2}}$ and consequently $r = 0$. Hence $\Phi(P)$ and $\Phi(Q)$ are orthogonal to each other.

Applying the above argument to Φ^{-1} we get the converse.

Claim 4. If $P \in \mathcal{B}(H)$ is a rank one projection and if $A \in \mathcal{B}(H)$ is such that either A or $-iA$ is positive, then $PA = AP = 0$ implies $\Phi(A)\Phi(P) = \Phi(P)\Phi(A) = 0$.

It is clear that for a sufficiently large real number M we have

$$w(A + MP) = M.$$

Thus $w(\Phi(A) + M\Phi(P)) = M$ since Φ is numerical radius preserving. Therefore, for any unit vector x in $\mathrm{rng}\Phi(P)$ (the range of $\Phi(P)$), we have

$$|\langle \Phi(A)x, x\rangle + M| \leq M. \tag{2.2}$$

If A is positive, (2.2) implies that $\langle \Phi(A)x, x\rangle + M \leq M$ and, consequently, $\langle \Phi(A)x, x\rangle = 0$. If $-iA$ is positive, (2.2) implies that $|\langle \Phi(A)x, x\rangle|^2 + M^2 \leq M^2$, and we also have $\langle \Phi(A)x, x\rangle = 0$. Now it is obvious that $\Phi(A)\Phi(P) = \Phi(P)\Phi(A) = 0$.

Claim 5. $\Phi(iP) = i\Phi(P)$ holds for every projection P of rank one.

Choose any projection P of rank one. By Claim 2, $-i\Phi(iP)$ is positive. If $\Phi(iP) = B + C$ with B, C such that $-iB, -iC$ are positive, then $iP = \Phi^{-1}(B) + \Phi^{-1}(C)$, $P = -i\Phi^{-1}(B) + (-i\Phi^{-1}(C))$. Since $-iB, -iC$ are positive, we have $-i\Phi^{-1}(B), -i\Phi^{-1}(C)$ are positive. Thus by [16, Proposition 3.1] there is a real number $r \in [0,1]$ such that $-i\Phi^{-1}(B) = rP$ and $-i\Phi^{-1}(C) = (1-r)P$. Consequently, $B = r\Phi(iP)$ and $C = (1-r)\Phi(iP)$. By [16, Proposition 3.1] again, $\Phi(iP) = \lambda Q$ for some $\lambda \in \mathbb{C}$ and rank-1 projection Q. Since $w(\Phi(iP)) = 1$ and $-i\Phi(iP)$ is positive, we must have $\lambda = i$.

In the following we verify that $Q = \Phi(P)$. Since $P(i(I-P)) = 0$, by Claim 4,

$$\Phi(P)\Phi(iI - iP) = i\Phi(P) - \Phi(P)\Phi(iP) = 0,$$

and hence $\Phi(P) = \Phi(P)Q$. Note that Q is a rank-1 projection; so $Q = \Phi(P)$.

Claim 6. $\Phi(iP) = i\Phi(P)$ for every projection P. Consequently, Φ is linear.

Let P be any projection in $\mathcal{B}(H)$. Then there exists an orthogonal set $\{P_\alpha\}_{\alpha\in\Lambda}$ of rank-1 projections such that $P = \sum_\alpha P_\alpha$ (converging in strong operator topology). Indeed, let $\{x_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis of the range of P, and $P_\alpha = x_\alpha \otimes x_\alpha$, which is a rank one projection. Then, by [12, Proposition 2.5.8, Remark 2.5.9 and 2.5.17], $\sum_\alpha P_\alpha$ converges strongly to $\bigvee_\alpha P_\alpha = P$ and $Px = \sum_\alpha P_\alpha x$ (converges in norm) for every $x \in H$, where $\bigvee_\alpha P_\alpha$ denotes the projection onto the closed linear subspace spanned by the union of the ranges of P_α. Note that $\sum_\alpha \Phi(P_\alpha)$ is a projection, since $\{\Phi(P_\alpha)\}_\alpha$ is an orthogonal set of rank-1 projections (see [12, Remark 2.5.17]). In the following, we will show that $\Phi(P) = \sum_\alpha \Phi(P_\alpha)$, and hence, Φ preserves projections in both directions.

By Claim 4, it follows from $(P - P_\alpha)P_\alpha = 0$ that $\Phi(P)\Phi(P_\alpha) = \Phi(P_\alpha)$ for every $\alpha \in \Lambda$. Thus $\Phi(P)\sum_\alpha \Phi(P_\alpha) = \Phi(P)\bigvee_\alpha \Phi(P_\alpha) = \bigvee_\alpha \Phi(P_\alpha) = \sum_\alpha \Phi(P_\alpha)$. Let nonzero vector $x \in \ker(\sum_\alpha \Phi(P_\alpha))$ be arbitrary and $Q = \frac{x}{\|x\|} \otimes \frac{x}{\|x\|}$; then $\Phi(P_\alpha)Q = 0$. By Claim 3, we have $P_\alpha \Phi^{-1}(Q) = 0$. As a consequence, $P\Phi^{-1}(Q) = 0$. Using the assertion of Claim 4 again, we get $\Phi(P)Q = 0$, which implies that $\Phi(P)x = 0$. Hence $\Phi(P) = \sum_\alpha \Phi(P_\alpha)$. Combining this with Claim 3, one sees easily that Φ preserves projections in both directions.

By Claim 4 and $P_\alpha(i(P-P_\alpha))=0$, we have $\Phi(P_\alpha)\Phi(iP-iP_\alpha)=\Phi(P_\alpha)\Phi(iP)-i\Phi(P_\alpha)=0$, which shows that

$$\Phi(iP)\Phi(P)=\Phi(iP)\sum_\alpha \Phi(P_\alpha)=i\sum_\alpha \Phi(P_\alpha)=i\Phi(P).$$

Let $x\in\ker(\Phi(P))$ be arbitrary and $Q=\frac{x}{\|x\|}\otimes\frac{x}{\|x\|}$; then $\Phi(P_\alpha)Q=0$. By Claim 3, we have $P_\alpha\Phi^{-1}(Q)=0$. As a consequence, $iP\Phi^{-1}(Q)=0$. Using the assertion of Claim 4 again, one gets $\Phi(iP)Q=0$, which implies that $\Phi(P)x=0$. Thus we must have $\Phi(iP)=i\Phi(P)$.

Now, since every operator in $\mathcal{B}(H)$ can be written as a linear combination of finitely many projections (see [2]), one concludes that Φ is linear.

Claim 7. There is a unitary $U\in\mathcal{B}(H)$ such that Φ is of the form

$$\Phi(A)=UAU^* \qquad \forall A\in\mathcal{B}(H)$$

or

$$\Phi(A)=UA^{tr}U^* \qquad \forall A\in\mathcal{B}(H),$$

where A^{tr} is the transpose of A with respect to an arbitrarily fixed orthonormal basis of H.

Apply [13, Theorem 2] directly.

Now let us treat the case in which $\Phi(\lambda I)=\overline{\lambda}I$ for every $\lambda\in\mathbb{C}$. In this case, take an orthonormal basis and define $\Psi:\mathcal{B}(H)\to\mathcal{B}(H)$ by $\Psi(A)=(\Phi(A)^*)^{tr}$ for every $A\in\mathcal{B}(H)$ with respect to this basis. Then Ψ is real linear, preserves numerical radius and $\Psi(\lambda I)=\lambda I$ for every $\lambda\in\mathbb{C}$. Thus, by what we have proved above, there is a unitary operator $V\in\mathcal{B}(H)$ such that Ψ is of the form

$$\Psi(A)=VAV^* \text{ or } \Psi(A)=VA^{tr}V^*.$$

Hence Φ is of the form

$$\Phi(A)=U(A^*)^{tr}U^* \qquad \forall A\in\mathcal{B}(H)$$

or

$$\Phi(A)=UA^*U^* \qquad \forall A\in\mathcal{B}(H),$$

for some unitary $U\in\mathcal{B}(H)$.

This completes the proof of the theorem for the case $\mathcal{V}=\mathcal{B}(H)$.

The proof of the case $\mathcal{V}=\mathcal{B}_\mathcal{K}(H)$ is similar. Indeed, the only difference is the proof of the linearity of Φ in Claim 6. Just as in the case of $\mathcal{V}=\mathcal{B}(H)$, we get $\Phi(\lambda P)=\lambda\Phi(P)$ for every projection $P\in\mathcal{B}_\mathcal{K}(H)$. It follows that $\Phi(\lambda A)=\lambda\Phi(A)$ for every finite rank operator $A\in\mathcal{B}_\mathcal{K}(H)$ and $\Phi(\lambda I)=\lambda\Phi(I)$. Note that Φ is continuous. So $\Phi(\lambda T)=\lambda\Phi(T)$ holds for every $T\in\mathcal{B}_\mathcal{K}(H)$. □

Theorem 1 can be restated as follows, which is sometimes convenient for applications. Recall that a conjugate unitary operator $U:H\to H$ is a conjugate linear bounded operator with $UU^*=U^*U=I$.

Theorem 1′. *Let $\mathcal{V}=\mathcal{B}(H)$ or $\mathcal{B}_\mathcal{K}(H)$. Let $\Phi:\mathcal{V}\to\mathcal{V}$ be a surjective map. Then Φ satisfies that $w(\Phi(A)-\Phi(B))=w(A-B)$ for all $A,B\in\mathcal{V}$ if and only if there is a unitary or conjugate unitary operator $U:H\to H$, a complex unit μ and an operator $S\in\mathcal{V}$ such that one of the following is true:*

(I) *$\Phi(A)=\mu UAU^*+S$, for every $A\in\mathcal{V}$;*
(II) *$\Phi(A)=\mu UA^*U^*+S$, for every $A\in\mathcal{V}$.*

From the above theorem, it follows that the group of surjective maps acting on $\mathcal{V} = \mathcal{B}(H)$ (or $\mathcal{B}_{\mathcal{K}}(H)$) which preserves the numerical radius distance is a group generated by the following five simple kinds of maps (motions):

(1) $A \mapsto UAU^*$, where $U \in \mathcal{B}(H)$ is a unitary operator;
(2) $A \mapsto UAU^*$, where $U : H \to H$ is a conjugate unitary operator;
(3) $A \mapsto A^*$;
(4) $A \mapsto \mu A$, where $\mu \in \mathbb{C}$ and $|\mu| = 1$;
(5) $A \mapsto A + S$, where $S \in \mathcal{V}$.

Note that the maps of the form (1)-(4) are linear or conjugate linear. So the nonlinear part essentially occurs only from (5). Furthermore, Theorem 1 implies that every numerical radius isometry (without the linearity assumption) from $\mathcal{B}(H)$ onto itself is also an operator norm isometry, but the converse proposition is not true because, for any unitary operators U and V, the map $\phi : A \mapsto UAV$ is norm preserving but is not numerical radius preserving if V is not U^* multiplied by a scalar of modulus 1.

In the rest of this paper we turn to the discussion of the maps on the real subspace of selfadjoint operators. The following result says that the group of surjective maps acting on $\mathcal{V} = \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$ that preserve the numerical radius distance is in fact a group generated by the maps of the form (1), (2), (4) with $\mu = 1$ or -1 and (5), as just mentioned above.

Theorem 2. *Let $\mathcal{V} = \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$, and let $\Phi : \mathcal{V} \to \mathcal{V}$ be a surjective map. Then Φ satisfies that $w(\Phi(A) - \Phi(B)) = w(A - B)$ for all $A, B \in \mathcal{V}$ if and only if there is a unitary $U \in \mathcal{B}(H)$, a real number $\mu \in \{1, -1\}$ and an operator $S \in \mathcal{V}$ such that one of the following is true:*

(I) *$\Phi(A) = \mu UAU^* + S$ for every $A \in \mathcal{V}$;*
(II) *$\Phi(A) = \mu UA^{tr}U^* + S$ for every $A \in \mathcal{V}$.*

Here A^{tr} is the transpose of A with respect to a fixed but arbitrary orthonormal basis of H.

Proof. Consider first the case $\mathcal{V} = \mathcal{H}(H)$. Assume that Φ preserves the numerical radius distance. Let $\Psi(A) = \Phi(A) - \Phi(0)$. Then $w(\Psi(A)) = w(A)$ for every $A \in \mathcal{H}(H)$. It follows from the Mazur-Ulam theorem [15] that Ψ is real linear. Consequently, we can assume that Φ itself is real linear, and we only need to prove that there is a unitary $U \in \mathcal{B}(H)$ and a scalar $\mu \in \{1, -1\}$ such that one of the following is true:

(i) $\Phi(A) = \mu UAU^*$ for every $A \in \mathcal{V}$;
(ii) $\Phi(A) = \mu UA^{tr}U^*$ for every $A \in \mathcal{V}$.

Claim 1. $\Phi(I) = \pm I$.

Following the proof of [1, Lemma 1], one easily checks that, for a selfadjoint operator $A \in \mathcal{H}(H)$ with $\|A\| = 1$, $A = \pm I$ if and only if for any selfadjoint operator $B \in \mathcal{V}$, either $w(A + B) = 1 + w(B)$ or $w(A - B) = 1 + w(B)$ holds. This together with the assumption that Φ preserves numerical radius implies that $\Phi(I) = \pm I$. Without loss of generality, we assume in the sequel that $\Phi(I) = I$.

Claim 2. $\overline{W(\Phi(B))} = \overline{W(B)}$ for every $B \in \mathcal{V}$.

We apply the same proof as in Claim 2 of Theorem 1, with μ and λ real. Now we finish the proof by using [13, Theorem 2].

The case $\mathcal{V} = \mathcal{H}_{\mathcal{K}}(H)$ is treated similarly. $\square$

Just like Theorem 1, Theorem 2 can be restated as

Theorem 2′. *Let $\mathcal{V} = \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$, and let $\Phi : \mathcal{V} \to \mathcal{V}$ be a surjective map. Then Φ satisfies that $w(\Phi(A)-\Phi(B)) = w(A-B)$ for every $A, B \in \mathcal{V}$ if and only if there is a unitary or conjugate unitary operator U on H, a real number $\mu \in \{1, -1\}$ and an operator $S \in \mathcal{V}$ such that $\Phi(A) = \mu UAU^* + S$ for all $A \in \mathcal{V}$.*

Corollary 3. *Let $\mathcal{V} = \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$, and let $\Phi : \mathcal{V} \to \mathcal{V}$ be a surjective map. Then Φ is an isometry if and only if there is a unitary $U \in \mathcal{B}(H)$, a scalar $\mu \in \{1, -1\}$ and an operator $S \in \mathcal{V}$ such that one of the following is true:*

(I) $\Phi(A) = \mu UAU^* + S$ *for every* $A \in \mathcal{V}$;

(II) $\Phi(A) = \mu UA^{tr}U^* + S$ *for every* $A \in \mathcal{V}$.

Here A^{tr} is the transpose of A with respect to a fixed but arbitrary orthonormal basis of H.

Proof. If Φ is an isometry, then $\|\Phi(A) - \Phi(B)\| = \|A - B\|$ holds for all $A, B \in \mathcal{V}$. However, $\|T\| = w(T)$ if T is selfadjoint. Now Theorem 2 is available. □

Remark. Let H and K be complex Hilbert spaces and $\Phi : \mathcal{V}(H) \to \mathcal{V}(K)$ be a map preserving the numerical radius distance. Then Theorem 1 is still true with $U : H \to K$ and $S \in \mathcal{V}(K)$, where $(\mathcal{V}(H), \mathcal{V}(K)) = (\mathcal{B}(H), \mathcal{B}(K))$ or $(\mathcal{B}_{\mathcal{K}}(H), \mathcal{B}_{\mathcal{K}}(K))$. A similar statement holds for Theorem 2.

Acknowledgment

The authors are grateful to the referees for valuable suggestions.

References

[1] J. T. Chan, Numerical radius preserving operators on $\mathcal{B}(H)$, Proc. Amer. Math. Soc. 123 (1995), 1437-1439. MR **95f:**47010

[2] P. A. Fillmore, Sums of operators with square zero, Acta. Sci. Math. (Szeged) 28 (1967), 285-288. MR **36:**4353

[3] P. R. Halmos, A Hilbert space problem book, 2nd edition, Springer-Verlag, New York, 1982. MR **84e:**47001

[4] L. K. Hua, Geometries of matrices I. Generalizations of von Staudt's theorem, Trans. Amer. Math. Soc. 57 (1945), 441-481. MR **7:**58b

[5] L. K. Hua, Geometries of matrices I_1. Arithmetical construction, Trans. Amer. Math. Soc. 57 (1945), 482-490. MR **7:**58c

[6] L. K. Hua, Geometries of symmetric matrices over the real field I, Dokl. Akad. Nauk. SSSR 53 (1946), 95-97. MR **8:**328b

[7] L. K. Hua, Geometries of symmetric matrices over the real field II, Dokl. Akad. Nauk. SSSR 53 (1946), 195-196. MR **8:**328c

[8] L. K. Hua, Geometries of matrices II. Study of involutions in the geometry of symmetric matrices, Trans. Amer. Math. Soc. 61 (1947), 193-228. MR **9:**171d

[9] L. K. Hua, Geometries of matrices III. Fundamental theorems in the geometry of symmetric matrices, Trans. Amer. Math. Soc. 61 (1947), 229-255. MR **9:**171e

[10] L. K. Hua, Geometry of symmetric matrices over any field with characteristic other than two, Ann. Math. 50 (1949), 8-31. MR **10:**424h

[11] L. K. Hua, A theorem on matrices over a field and its applications, Acta Math. Sinica 1 (1951), 109-163. MR **17:**123a

[12] R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras, Vol. I. Elementary Theory, Reprint of the 1983 original, Graduate Studies in Mathematics, Vol. 15, Amer. Math. Soc., Providence, RI, 1997. MR **98f:**46001a

[13] C. K. Li, L. Rodman, and P. Šemrl, Linear maps on selfadjoint operators preserving invertibility, positive definiteness, numerical range, Canad. Math. Bull. 46 (2003), 216-228.

[14] C. K. Li and P. Šemrl, Numerical radius isometries, Linear and Multilinear Algebra 50 (2002), 307-314. MR **2003i:**15025

[15] S. Mazur and S. Ulam, Sur les transformations isométriques d′espaces vectoriels normés, C. R. Acad. Sci. Paris 194 (1932), 946-948.

[16] M. Omladič, On operators preserving the numerical range, Linear Algebra Appl. 134 (1990), 31-51. MR **91i:**47006

[17] Z. Wan, Geometry of matrices, in memory of Professor L. K. Hua (1910–1985), World Scientific Publishing Co., River Edge, NJ, 1996. MR **98a:**51001

School of Science, Xi′an Jiaotong University, Xi′an, 710049, P. R. China; Department of Mathematics, Shanxi Teachers University, Linfen, 041004, P. R. China

Department of Mathematics, Shanxi Teachers University, Linfen, 041004, P. R. China; Department of Mathematics, Shanxi University, Taiyuan, 030000, P. R. China

E-mail address: jhou@dns.sxtu.edu.cn