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THE FREE ROOTS OF THE COMPLETE GRAPH

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ABSTRACT. There is a model-completion T_n of the theory of a (reflexive) n-coloured graph $\langle X, R_1, \ldots, R_n \rangle$ such that R_n is total, and $R_i \circ R_j \subseteq R_{i+j}$ for all i, j. For $n > 2$, the theory T_n is not simple, and does not have the strict order property. The theories T_n combine to yield a non-simple theory T_∞ without the strict order property, which does not eliminate hyperimaginaries.

1. INTRODUCTION

The first-order theory T_2 of the random graph R is the model completion of the theory of a graph. It is usually axiomatized by stating that R is a graph, and for any two finite disjoint sets A, B there is an element a such that $R(a, b)$ for all $b \in A$ and $\neg R(a, b)$ for all $b \in B$. An *n-th root* of a binary relation R is a binary relation S such that $S^n = R$, where S^n is the n-fold iterated composition $S \circ S \circ \ldots \circ S$. For example, xS^2y iff there is some z such that $xSz \wedge zSy$. If S is reflexive, we get $S \subseteq S^n$ and hence $S^n \subseteq S^{n+m}$.

The random graph is a square root of the complete graph. We shall investigate the n-th roots of the complete graph. There are, of course, different n-th roots. We are interested in studying the theory T_n of the free n-th root of the complete graph. We can also see T_n as the model completion of the theory of an n-coloured graph $R_1, \ldots, R_n$ such that R_n is complete and $R_iR_j \subseteq R_{i+j}$ for all i, j. (We then get $R_i = R_1^i$ for $i \leq n$.) Whereas T_2, the theory of the random graph, is an ω-categorical simple theory, for $n > 2$ the theory T_n of the free n-th root is ω-categorical without the strict order property, but it is not simple.

Another way to look at these graphs is the following: Define the *distance* $d(a, b)$ of two points a and b to be the minimal k such that aR_kb holds (and $d(a, a) = 0$). Then $R_iR_j \subseteq R_{i+j}$ is equivalent to the triangle inequality, and our n-coloured graphs are quantifier-freely bi-interpretable with metric spaces of diameter n and distances in $\{0, 1, \ldots, n\}$.

If we rename R_i from T_n as $S_{i/n}$, then in this language the theories T_n combine to a theory T_∞ without the strict order property, which eliminates quantifiers and is not simple. We will show that T_∞ does not eliminate hyperimaginaries. This

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seems to be the first example of a theory without the strict order property and without elimination of hyperimaginaries. All previously known examples involve infinitesimals with respect to some partial order, and therefore have the strict order property. It is still open whether there is a simple theory that does not eliminate hyperimaginaries [1].

All our relations are reflexive and symmetric.

2. The theory T_n

Definition 1. Let R be a binary relation. We say that two sets A and B are *R-connected*, denoted $A\,R\,B$, if $R(a,b)$ for all $a \in A$ and $b \in B$; they are *R-disjoint*, written $A\,\neg R\,B$, if $\neg R(a,b)$ for all $a \in A$ and $b \in B$.

Definition 2. T_n^- is the theory of all n-coloured graphs $R_1, \ldots, R_n$ such that R_n is total and $R_i R_j \subseteq R_{\min\{i+j,n\}}$ for all $0 < i, j < n$.

Remark 1. (1) T_n^- is a consistent universal theory; so it has existentially closed models.

(2) Occasionally, to simplify notation, we shall call equality R_0, and put R^0 to be equality, for any reflexive symmetric relation R. The axioms for T_n^- then also hold if $i = 0$ or $j = 0$.

Proposition 1. *An existentially closed model $\mathfrak{M}$ of T_n^- satisfies the axiom scheme:*

(†) *For all finite disjoint $A_1, \ldots, A_n$, if $A_i\,R_{i+j}\,A_j$ for all $i + j \leq n$ and $A_i\,\neg R_{j-i}\,A_{j+1}$ for all $i < j < n$, then there is some m such that $m\,R_i\,A_i$ and $m\,\neg R_i\,A_{i+1}$ for all $i < n$.*

In other words, there exists a point of distance i to all points in A_i, for $1 \leq i \leq n$, unless this violates some triangle inequality.

Proof. Let $A_1, \ldots, A_n$ be subsets of $\mathfrak{M}$ satisfying the hypothesis of (†). Let $\mathfrak{M}^*$ be the graph where we have added a point m^* and relations $m^* R_{\min\{i+j,n\}} m'$ whenever $m' \in \mathfrak{M}$ is R_j-related to a point in A_i, or $i + j \geq n$, for all possible $0 \leq i, j \leq n$. Then, in particular, $m^*\,R_i\,A_i$. Suppose $m, m' \in \mathfrak{M}$ and $mR_i m' R_j m^*$, with $0 < i, j$. If $i + j \geq n$ we get $mR_n m^*$; so we may assume $i + j < n$. Then there is some $k \leq j$ and $m'' \in A_k$ with $m' R_{j-k} m''$, whence $mR_{j-k+i} m''$ and $mR_{j+i} m^*$. On the other hand, if $m, m' \in \mathfrak{M}$ with $mR_i m^* R_j m'$, then either $i + j \geq n$ and $mR_n m'$, or $i + j < n$ and there are $k \leq i$ and $\ell \leq j$, and points $m_0 \in A_k$ and $m_1 \in A_\ell$ with $mR_{i-k} m_0$ and $m' R_{j-\ell} m_1$. Since $m_0 R_{k+\ell} m_1$, we get $mR_{i+j} m'$, and $\mathfrak{M}^* \models T_n^-$.

Suppose there is $m \in A_{j+1}$ with $m^* R_j m$, for some $j < n$. Then there is $i < j$ and $m' \in A_i$ with $mR_{j-i} m'$, contradicting $A_{j+1}\,\neg R_{j-i}\,A_i$. Since $\mathfrak{M}$ is existentially closed, we see that $\mathfrak{M} \models$ (†). □

Definition 3. T_n is the theory T_n^-, together with the axiom scheme (†).

Remark 2. Let $\mathfrak{M} \models T_n$. Then $R_i = R_1^i$ for all $i \leq n$.

Proof. T_n^- implies $R_1 R_i \subseteq R_{i+1}$ for all $i < n$. Suppose that there are $m, m' \in \mathfrak{M}$ with $mR_{i+1} m'$ and $m \neg R_i m'$. Put $A_1 = \{m\}$ and $A_i = \{m'\}$, and all other $A_j = \emptyset$. Then the hypotheses of (†) are satisfied. So there is $m'' \in \mathfrak{M}$ with $mR_1 m'' R_i m'$, whence $R_{i+1} \subseteq R_1 R_i$. It follows that $R_1 R_i = R_{i+1}$ for all $i < n$, whence $R_1^i = R_i$ for all $i \leq n$ by induction. □

Remark 3. Iterating Proposition 1 yields the consistency of any configuration, unless it violates some triangle inequality.

Proposition 2. *T_n is complete, ω-categorical, and eliminates quantifiers.*

Proof. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two models of T_n, and consider finite subsets $A \subseteq \mathfrak{M}$ and $B \subseteq \mathfrak{N}$ with a partial isomorphism $\sigma : A \to B$. Let $m \in \mathfrak{M} - A$. Put $A_i = \{a \in A : d(m,a) = i\}$ for $i \leq n$, and $B_i = \sigma(A_i)$. Then $A_1, \ldots, A_n$ satisfy the hypotheses of (†), as do $B_1, \ldots, B_n$. Since $\mathfrak{N} \models T_n$, we find $n \in \mathfrak{N} - B$ such that $\sigma \cup \{m \mapsto n\}$ is a partial isomorphism. It follows that the family of partial isomorphisms forms a back-and-forth system. Hence T_n is complete, ω-categorical, and eliminates quantifiers. □

Remark 4. It follows that T_n is the model-completion of T_n^-. Note that algebraic closure is trivial in T_n: $\mathrm{acl}(A) = A$ for any subset A.

Lemma 3. *Let A be a finite set in a model of T_n, and $p(\bar{x})$ a complete type over A. Suppose $(\bar{a}_i : i < \omega)$ is an infinite A-indiscernible sequence in p with $\bar{a}_0 \cap \bar{a}_1 = \emptyset$, and $R(\bar{x}, \bar{y})$ an A-definable reflexive and transitive relation satisfied by $\bar{a}_0\bar{a}_1$. Then $R(\bar{x}, \bar{y}) \wedge p(\bar{x}) \wedge p(\bar{y})$ is equivalent to $p(\bar{x}) \wedge p(\bar{y})$.*

Proof. Note that $p(\bar{x})$ is a single formula by ω-categoricity. Clearly we may assume that p implies that all of its variables are distinct, and that $R(\bar{x}, \bar{y})$ implies $p(\bar{x}) \wedge p(\bar{y})$. Write $\bar{a}_i = (a_i^0, a_i^1, \ldots)$, put $d_{i,j}(\bar{a}_0, \bar{a}_1) = d(a_0^i, a_1^j)$, and let $d_{i,j} = \max_{a \in A}\{|d(a_0^i, a) - d(a, a_0^j)|\}$; if $A = \emptyset$ put $d_{i,j} = 0$. By the triangle inequality $d_{i,j} \leq d_{i,j}(\bar{a}_0, \bar{a}_1)$ for any i, j. □

Claim 1. $d_{i,k} \geq |d_{i,j} - d_{j,k}|$.

Proof of Claim. We may assume $d_{i,j} \geq d_{j,k}$. Choose $a \in A$ with $d_{i,j} = |d(a_0^i, a) - d(a, a_0^j)|$. Then

$$\begin{aligned} 0 &\leq d_{i,j} - d_{j,k} \leq |d(a_0^i, a) - d(a, a_0^j)| - |d(a_0^j, a) - d(a, a_0^k)| \\ &\leq |d(a_0^i, a) - d(a, a_0^k)| \leq d_{i,k}. \end{aligned}$$

□

Note that $d_{i,i} = 0 < d(a_0^i, a_1^i)$ for any i. Choose j_0, j_1 such that $d(a_0^{j_0}, a_1^{j_1}) = d > d_{j_0,j_1}$ is maximal possible; if $d = 1$ take $j_0 = j_1$.

Claim 2. The complete type

$$\bigwedge_{i<\omega} p(\bar{x}_i) \wedge \bigwedge_{i<i'} \Big[d(x_i^{j_0}, x_{i'}^{j_1}) = d - 1 \wedge \bigwedge_{(j,j') \neq (j_0,j_1)} d(x_i^j, x_{i'}^{j'}) = d_{j,j'}(\bar{a}_0, \bar{a}_1)\Big]$$

is consistent.

Proof of Claim. Suppose not. By Remark 3 this means that some triangle inequality is violated. Obviously the triangle will have to involve $x_i^{j_0} x_{i'}^{j_1}$ as a non-maximal edge, for some i, i', and it cannot be degenerate. There are three cases:

(1) *The third point is some $a \in A$.*
This immediately contradicts our choice of $d > d_{j_0,j_1}$.

(2) *The third point is some $x_{i''}^j$, and the maximal side has length $\leq d$.*
Clearly impossible, because the other small side has length at least 1, since all points are distinct.

(3) *The third point is some* $x^j_{i''}$*, and the maximal side has length* $> d$.
If $x^{j_0}_i x^j_{i''}$ is the maximal side, then it has length $d_{j_0,j}$ by the maximal choice of d. Therefore,

$$d(x^{j_0}_i, x^{j_1}_{i'}) = d - 1 \geq d_{j_0,j_1} \geq d_{j_0,j} - d_{j,j_1} \geq d(x^{j_0}_i, x^j_{i''}) - d(x^j_{i''}, x^{j_1}_{i'}),$$

again a contradiction. □

Hence there is a realization $(\bar{a}'_i : i < \omega)$, which is an A-indiscernible sequence by quantifier elimination. We want to show that it is R-related. Put $q(\bar{x}\bar{y}) = \mathrm{tp}(\bar{a}_0\bar{a}_1)$. We claim that $q(\bar{a}'_0\bar{x}) \wedge q(\bar{x}\bar{a}'_1)$ is consistent. But if not, then again by Remark 3 there is some triangle inequality that is violated. Since $q(\bar{x}_0\bar{x}) \wedge q(\bar{x}, \bar{x}_1) \wedge q(\bar{x}_0, \bar{x}_1)$ is realized by $\bar{a}_0\bar{a}_1\bar{a}_2$ and thus is consistent, the triangle will have to involve ${a'}^{j_0}_0$ and ${a'}^{j_1}_1$, and we get a contradiction as in the proof of Claim 2.

By transitivity of R we get $R(\bar{a}'_i, \bar{a}'_j)$ for $i < j$. It follows iteratively that we find an A-indiscernible sequence $(\bar{a}''_i : i < \omega)$ in p with $R(\bar{a}''_0, \bar{a}''_1)$, and $d_{i,i}(\bar{a}''_0, \bar{a}''_1) = 1$ for all i except one, where the distance is zero. In other words, $|\bar{a}''_0 \cap \bar{a}''_1| = 1$, and $\bigcap_{i<\omega} \bar{a}_i$ is a single element a, say at position 0.

We now use induction on $|\bar{x}|$. If this is 0, then there is nothing to show. If $|\bar{x}| > 1$, define

$$R'(x_0, y_0) \equiv \exists\, x_1, y_1, \ldots, R(\bar{x}, \bar{y}).$$

Clearly R' is reflexive. Moreover, applying the inductive hypothesis to the Aa-indiscernible sequence $(\bar{a}''_i - a : i < \omega)$, we see that the relation $R(\bar{x}, \bar{y}) \wedge x_0 = y_0 = a$ is just $p(\bar{x}) \wedge p(\bar{y}) \wedge x_0 = y_0 = a$; note that this does not depend on the choice of a, whose type over A is determined by p. In other words, two realizations of p with the same first coordinate are R-related. Since R is transitive, R and $R' \wedge p(\bar{x}) \wedge p(\bar{y})$ are equivalent as formulas in $\bar{x}\bar{y}$. In particular, R' is transitive, and we may assume $R = R'$. We have reduced to the case $|\bar{x}| = 1$.

Let n' be minimal such that $p(x)$ implies $aR_{n'}x$ for some $a \in A$ and put $m = \min\{n, 2n'\}$. Then $p(x) \wedge p(y)$ implies $xR_m y$.

Claim 3. For any $0 < i < m$ there are $a, b, c \models p$ with $d(a, b) = d(a, c) = i$ and $d(b, c) = i + 1$. For any $0 < i \leq m$ there are $a, b, c \models p$ with $d(a, b) = d(a, c) = i$ and $d(b, c) = i - 1$.

Proof of Claim. We have to verify consistency with the triangle inequalities. Since $i > 0$, they are clearly satisfied for the triangle abc. For a triangle with only one point from $\{a, b, c\}$ the triangle inequalities follow from consistency of p. Finally, consider $a' \in A$, say $d(a', a) = d(a', b) = d(a', c) = d \geq \frac{m}{2}$. Hence $2d \geq i + 1$, and the triangle inequality is satisfied for triangles with two points from $\{a, b, c\}$.

The second assertion is proved similarly. □

Since there are $a \neq b \models p$ with $\models R(a, b)$, say with $d(a, b) = i \leq m$, we find $c \models p$ with $d(a, c) = i$ and $d(b, c) = i - 1$ (or $d(b, c) = i + 1$ if $i < m$). Now $p(a) \wedge p(b) \wedge d(a, b) = i$ is a complete type $\mathrm{tp}(ab/A)$ by quantifier eliminiation, which is satisfied by ca. Hence $R(c, a)$, and $R(c, b)$ by transitivity.

Thus either $R(x, y)$ implies $x = y$, which is impossible, or $R(x, y)$ is consistent with $d(x, y) = i$ for all $0 \leq i \leq m$. Since these are all the possibilities for $\mathrm{tp}(xy/A)$ consistent with $p(x) \wedge p(y)$, we see that $R(x, y)$ is equivalent to $p(x) \wedge p(y)$. □

Theorem 4. *T_n does not have the strict order property.*

Proof. Suppose $R(\bar{x}, \bar{y})$ defines a partial reflexive order with infinite chains. By Ramsey's Theorem and compactness there is an infinite indiscernible proper R-chain $(\bar{a}_i : i < \omega)$, where all a_i satisfy the same type $p(\bar{x})$. By indiscernibility the sequence is disjoint over $A = \bigcap_{i<\omega} \bar{a}_i$; if $\bar{a}_i = \bar{a}'_i A$, we consider the A-definable relation $R'(\bar{x}', \bar{y}') = R(\bar{x}'A, \bar{y}'A)$ on $q(\bar{x}') = \text{tp}(\bar{a}'_0/A)$. By Lemma 3 it is equivalent to $q(\bar{x}') \wedge q(\bar{y}')$ and hence symmetric, a contradiction. □

Theorem 5. *A definable equivalence relation on a complete type (over some parameters A) is definable (on that type) in the language of pure equality. In particular, a definable equivalence relation on a complete* 1*-type is either equality or complete.*

Proof. Suppose $E(\bar{x}, \bar{y})$ is an A-definable equivalence relation on $p(\bar{x}) \in S(A)$; clearly we may assume that A is finite, and inductively that the assertion is true for all definable equivalence relations on tuples of smaller length. If E has infinite classes, then by Ramsey's Theorem and compactness there is an infinite E-related A-indiscernible sequence $(\bar{a}_i : i < \omega)$ in p with $\bar{a}_0 \neq \bar{a}_1$. By indiscernibility it is disjoint over its common intersection $B = \bigcap_{i<\omega} \bar{a}_i$. We put $\bar{a}_i = \bar{a}'_i B$ and consider the induced equivalence relation $E'(\bar{x}', \bar{y}') = E(\bar{x}'B, \bar{y}'B)$ on $q(\bar{x}') = \text{tp}(\bar{a}'_0/AB)$. By Lemma 3 it is equivalent to $q(\bar{x}') \wedge q(\bar{y}')$. If $B = \emptyset$, we are done; otherwise, write $\bar{x} = \bar{x}'\bar{x}''$ and $\bar{y} = \bar{y}'\bar{y}''$ and consider

$$E''(\bar{x}'', \bar{y}'') \equiv \exists \bar{x}'\bar{y}' \ [p(\bar{x}) \wedge p(\bar{y}) \wedge E(\bar{x}, \bar{y})].$$

Then $E''(\bar{x}'', \bar{y}'') \wedge p(\bar{x}) \wedge p(\bar{y})$ is equivalent to $E(\bar{x}, \bar{y})$ by transitivity of E and the fact that $\text{tp}(\bar{x}'') = \text{tp}(\bar{y}'') = \text{tp}(B)$ is fixed. Thus E'' is a definable equivalence relation on $\text{tp}(B/A)$; since $|B| < |\bar{a}_i|$, the inductive hypothesis applies and E'' is definable on $\text{tp}(B/A)$ in the language of pure equality, as is E.

On the other hand, if all classes are finite, the size of any class is bounded, and $E(\bar{a}, \bar{a}')$ implies $\bar{a}' \in \text{acl}(\bar{a}) = \bar{a}$. In other words, E corresponds to a subgroup of $\text{Sym}(|\bar{x}|)$, and is definable on p by equality only. □

Theorem 6. *T_n is not simple for $n > 2$.*

Remark 5. Of course, T_1 is the complete graph and T_2 is the random graph, both of which are simple.

Proof. Suppose $n > 2$, and let $\mathfrak{M}$ be a monster model of T_n. By Remark 3 and quantifier elimination there are an indiscernible subset A of $\mathfrak{M}$ whose pairwise distances are 2, and an element $m \notin A$ with $m \, R_1 \, A$. □

Claim 4. $\text{tp}(m/A)$ forks over all proper $A' \subset A$.

Proof of Claim. We may assume $A = A' \cup \{a\}$. By Remark 3 we can construct a sequence $a = a_0, a_1, \dots$ with $d(a_i, a_j) = 3$ and $a_i \, R_2 \, A'$ for all $i < j < \omega$; by quantifier-elimination the sequence is A'-indiscernible. However, $R_1(x, a_i) \wedge R_1(x, a_j)$ is inconsistent for $i \neq j$. □

It follows that $\text{tp}(m/A)$ forks over all countable subsets, and T_n cannot be simple. □

3. The limit theory T_∞

If we rename, in the theory T_n, the relation R_i as $S_{i/n}$, then for $m, n < \omega$ both T_m and T_n are subtheories of T_{mn} in the language $\mathcal{L} = \{S_{p/q} : 0 < p \leq q < \omega\}$. We can thus put $T_\infty = \bigcup_{n<\omega} T_n$. It is the model companion of the theory of all graphs $\{S_{p/q} : 0 < p \le q < \omega\}$ such that $S_iS_j \subseteq S_{i+j}$ for all $i, j \in [0,1] \cap \mathbb{Q}$. It can be axiomatized just as the union of the axioms for the different T_n in the language $\{S_{p/q} : 0 < p \le q < \omega\}$. Elimination of quantifiers, failure of the strict order property, the characterisation of definable equivalence relations, and the lack of simplicity transfer immediately from T_n to T_∞.

Remark 6. If we want to consider T_∞ as a metric space, we have to interpret $S_{p/q}(a,b)$ as $d(a,b) \le p/q$. Note that the predicate $d(x,y) = p/q$ is not definable in the limit theory T_∞. A saturated model of T_∞ will be a metric space of diameter 1 and distances contained in a non-standard unit interval $[0,1]^*$. If $I \subseteq [0,1]^*$ realizes every 1-type over $[0,1] \cap \mathbb{Q}$ in the language $\{|x-y| \le p/q : 0 < p \le q < \omega\}$ precisely once (for instance, $I = [0,1] \cup ([0,1[\cap \mathbb{Q}) + \epsilon$, where ϵ is infinitesimal), we can take all distances of any model of T_∞ from I.

Since $S_iS_j \subseteq S_{i+j}$ for all $i + j \le 1$, the relation $\bigwedge_{0<p\le q<\omega} S_{p/q}(x,y)$ is a type-definable equivalence relation (in metric terms it means that x and y are infinitely close). It cannot be the intersection of definable equivalence relations. Since there is a unique 1-type over $\emptyset$, the only $\emptyset$-definable unary equivalence relations are equality and the complete graph.

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