LOCATION OF FOCAL POINTS

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(Communicated by Carmen C. Chicone)

Abstract. The effect of various factorizations of a disconjugate linear operator on focal points is studied.

1.

The differential equation

\[ L_n y + p(x)y = 0, \]

where \( L_n \) is an \( n \)th-order disconjugate linear differential operator and \( p(x) \) is continuous and sign definite, will be discussed. The operator \( L_n \) is assumed to be factored according to a Polya factorization

\[ \begin{align*}
L_0 y &= \rho_0 y, \\
L_i y &= \rho_i (L_{i-1} y)', \quad i = 1, \ldots, n,
\end{align*} \]

with \( \rho_i > 0 \) and \( \rho_i \in C^{n-i}_a \) for \( i = 0, 1, \ldots, n \). Boundary value problems of the following form are considered:

\[ \begin{align*}
L_i y(a) &= 0 \quad \text{for } i = 0, \ldots, k - 1, \\
L_j y(b) &= 0 \quad \text{for } j = m, m + 1, \ldots, n - k - 1 + m.
\end{align*} \]

Let \( \eta_{k,n-k}(a) \), called the first \( (k, n-k) \) conjugate point of \( a \), be the least value of \( b \) such that with \( m = 0 \) the boundary value problem (1.3) has a nontrivial solution, and let \( \xi_{k,n-k}(a) \), called the first \( (k, n-k) \) focal point of \( a \), be such a point for \( m = k \).

It is well known that if \( \eta_{k,n-k}(a) \) exists, then so does \( \xi_{k,n-k}(a) \) and \( \xi_{k,n-k}(a) < \eta_{k,n-k}(a) \) \([2]\).

The point \( \eta_{k,n-k}(a) \) is independent of the Polya factorization of \( L_n \). The point \( \xi_{k,n-k}(a) \), on the other hand, depends on the factorization of \( L_n \). This fact is illustrated in the book by Elias \([2]\) in examples on pages 119 and 124.

The purpose of this note is to prove the following.

**Theorem 1.1.** Suppose \( \eta_{k,n-k}(a) = b \leq \infty \). Then for any \( c \), with \( a < c < b \), there is a Polya factorization of \( L_n \) on \( [a, b] \) so that \( \xi_{k,n-k}(a) = c \) for that factorization.

*Received by the editors October 7, 2002 and, in revised form, January 9, 2003.
2000 Mathematics Subject Classification. Primary 34C10.
Key words and phrases. Conjugate point, focal point, disconjugate differential operator.*
There is a fundamental system \( \{u_0, u_1, ..., u_{n-1}\} \) of solutions for
\[
L_n y = 0
\]
so that on \((a, b)\) their Wronskian determinants \(W(u_{i_0}, u_{i_1}, ..., u_{i_k}) > 0\), for \(0 \leq i_0 < i_1 < \cdots < i_k < n - 1\) and \(k = 0, 1, ..., n - 1\). Such a fundamental system is called Descartes \([1]\). A Descartes fundamental system \(\{u_0, u_1, ..., u_{n-1}\}\) determines a factorization of \(L_n\) on \((a, b)\) \([3]\). In fact,
\[
\frac{1}{\rho_0} = u_0, \quad \frac{1}{\rho_1} = \left(\frac{u_1}{u_0}\right), \quad \frac{1}{\rho_n} = \frac{W(u_0, ..., u_{n-2})}{W(u_0, ..., u_{n-1})}
\]
and
\[
\frac{1}{\rho_i} = \left(\frac{W(u_0, ..., u_{i-2}, u_i)}{W(u_0, ..., u_{i-1})}\right)' \quad \text{for } i = 2, ..., n - 1.
\]
Furthermore,
\[
L_i y = \frac{W(u_0, ..., u_{i-1}, y)}{W(u_0, ..., u_i)} \quad \text{for } i = 1, 2, ..., n - 1,
\]
\[
L_n y = \frac{W(u_0, ..., u_{n-1}, y)}{W(u_0, ..., u_{n-1})}.
\]
If \(u_0(a) > 0\) and \(W(u_0, ..., u_i)(a) > 0\) for \(i = 1, ..., n - 1\), the factorization is valid on \([a, b]\). If the Descartes fundamental system is chosen, as is always possible, so that \(u_i\) has a zero of order \(i\) at \(a\) and a zero of order \(n - 1 - i\) at \(b\), then
\[
\int^b_{a} \rho_i^{-1} dx = \infty \quad \text{for } i = 1, ..., n - 1.
\]
and we have the unique canonical factorization of \(L_n\) on \([a, b]\) due to Trench \([4]\).

We now proceed with the proof of our theorem.

**Proof.** Suppose \(\eta_{k,n-k}(a) = \infty \). Let \(\{u_0, u_1, ..., u_{n-1}\}\) be the Descartes fundamental system of solutions for \((2.1)\) such that \(u_i\) has a zero of order \(i\) at \(a\) and a zero of order \(n - 1 - i\) at \(b\). Let \(a < c < b\) and \(z\) be a solution of \((1.1)\) so that
\[
L_i z(a) = 0 \quad \text{for } i = 1, ..., k - 1
\]
and
\[
L_i z(c) = 0 \quad \text{for } i = k + 1, ..., n - 1.
\]
Since \((1.1)\) is \((k, n-k)\) disfocal (i.e., there is no solution to boundary value problem \((1.3)\) with \(m = k\)) on \([a, b]\) \([2]\) with respect to the factorization of \(L_n\) determined by \(\{u_0, u_1, ..., u_{n-1}\}\), then \(L_k z(c) \neq 0\). Thus, assume \(L_k z(c) = 1\). It now follows \([2]\) that \(L_k z > 0\) and \(L_{k+1} z < 0\) on \([a, c]\). It follows from \((2.4)\) that
\[
\left(\frac{W(u_0, u_1, ..., u_{k-2}, u_k, z)}{W(u_0, u_1, ..., u_{k-1}, u_k)}\right)' = \frac{W(u_0, u_1, ..., u_{k-2}, u_k W(u_0, u_1, ..., u_k, z)}{[W(u_0, u_1, ..., u_k)]^2} < 0
\]
on \((a, c)\). Since \(W(u_0, u_1, ..., u_{k-2}, u_k, z)(a) = 0\), it follows that
\[
W(u_0, u_1, ..., u_{k-2}, u_k, z) < 0
\]
on \((a,c]\). Therefore, there is a constant \(\alpha > 0\) so that
\[
0 = W(u_0, \ldots, u_{k-1}, z)(c) + \alpha W(u_0, u_1, \ldots, u_{k-1}, u_k, z)(c)
\]
\[
= W(u_0, u_1, \ldots, u_{k-1}, u_k + \alpha u_k, z)(c).
\]
Let
\[
y_i = u_i \quad \text{for} \quad i \neq k - 1 \quad \text{and} \quad y_{k-1} = u_{k-1} + \alpha u_k.
\]
If \(j < k - 1\) or \(j > k - 1\), then
\[
W(y_i, y_{i+1}, \ldots, y_j) = W(u_i, u_{i+1}, \ldots, u_j) > 0,
\]
while
\[
W(y_i, y_{i+1}, \ldots, y_{k-1}) = W(u_i, u_{i+1}, \ldots, u_{k-1})
\]
\[
+ \alpha W(u_i, u_{i+1}, \ldots, u_{k-2}, u_k) > 0.
\]
Thus, \(\{y_0, y_1, \ldots, y_{n-1}\}\) is a Descartes fundamental system on \((a,b)\). (See [1].)

Let \(L_n\) be factored as in (2.2) through (2.5), but where the factorization is determined by the Descartes system \(\{y_0, y_1, \ldots, y_{n-1}\}\). Letting the \(i\)th quasiderivative of \(z\) according to that factorization be denoted by \(J_i(z)\), it follows from (2.4) and (2.8) that \(L_i z = J_i z\) for \(i \neq k\). Consequently,
\[
J_i z(a) = 0 \quad \text{for} \quad i = 0, 1, \ldots, k - 1,
\]
\[
J_i z(c) = 0 \quad \text{for} \quad i = k + 1, \ldots, n - 1.
\]
Also by (2.4) and (2.7),
\[
J_k z(c) = \frac{W(y_0, y_1, \ldots, y_{k-1}, z)(c)}{W(y_0, y_1, \ldots, y_{k-1}, y_k)(c)}
\]
\[
= \frac{W(u_0, u_1, \ldots, u_{k-1}, z)(c) + \alpha W(u_0, u_1, \ldots, u_{k-1}, u_k, z)(c)}{W(y_0, y_1, \ldots, y_{k-1}, y_k)(c)} = 0.
\]
Thus, for the factorization determined by \(\{y_0, y_1, \ldots, y_{n-1}\}\), \(c = \xi_k, n - k(a)\).

References


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