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LOCATION OF FOCAL POINTS

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Abstract. The effect of various factorizations of a disconjugate linear operator on focal points is studied.

1.

The differential equation

$$L_n y + p(x)y = 0, \tag{1.1}$$

where L_n is an nth-order disconjugate linear differential operator and $p(x)$ is continuous and sign definite, will be discussed. The operator L_n is assumed to be factored according to a Pólya factorization

$$\begin{aligned} &L_0 y = \rho_0 y, \\ &L_i y = \rho_i (L_{i-1} y)', \quad i = 1, \ldots, n, \end{aligned} \tag{1.2}$$

with $\rho_i > 0$ and $\rho_i \in C^{n-i}$ for $i = 0, 1, \ldots, n$. Boundary value problems of the following form are considered:

$$\begin{aligned} &L_i y(a) = 0 \quad \text{for } i = 0, \ldots, k-1, \\ &L_j y(b) = 0 \quad \text{for } j = m, m+1, \ldots, n-k-1+m. \end{aligned} \tag{1.3}$$

Let $\eta_{k,n-k}(a)$, called the first $(k, n-k)$ conjugate point of a, be the least value of b such that with $m = 0$ the boundary value problem (1.3) has a nontrivial solution, and let $\xi_{k,n-k}(a)$, called the first $(k, n-k)$ focal point of a, be such a point for $m = k$.

It is well known that if $\eta_{k,n-k}(a)$ exists, then so does $\xi_{k,n-k}(a)$ and $\xi_{k,n-k}(a) < \eta_{k,n-k}(a)$ [2].

The point $\eta_{k,n-k}(a)$ is independent of the Pólya factorization of L_n. The point $\xi_{k,n-k}(a)$, on the other hand, depends on the factorization of L_n. This fact is illustrated in the book by Elias [2] in examples on pages 119 and 124.

The purpose of this note is to prove the following .

Theorem 1.1. *Suppose $\eta_{k,n-k}(a) = b \leq \infty$. Then for any c, with $a < c < b$, there is a Pólya factorization of L_n on $[a, b)$ so that $\xi_{k,n-k}(a) = c$ for that factorization.*

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2.

There is a fundamental system $\{u_0, u_1, ..., u_{n-1}\}$ of solutions for

$$L_n y = 0 \tag{2.1}$$

so that on (a,b) their Wronskian determinants $W(u_{i_0}, u_{i_1}, ..., u_{i_k}) > 0$, for $0 \le i_0 < i_1 < \cdots < i_k \le n-1$ and $k = 0, 1, \ldots, n-1$. Such a fundamental system is called Descartes [1]. A Descartes fundamental system $\{u_0, u_1, ..., u_{n-1}\}$ determines a factorization of L_n on (a,b) [3]. In fact,

$$\frac{1}{\rho_0} = u_0, \quad \frac{1}{\rho_1} = \left(\frac{u_1}{u_0}\right)', \quad \frac{1}{\rho_n} = \frac{W(u_0, \ldots, u_{n-2})}{W(u_0, \ldots, u_{n-1})} \tag{2.2}$$

and

$$\frac{1}{\rho_i} = \left(\frac{W(u_0, \ldots, u_{i-2}, u_i)}{W(u_0, \ldots, u_{i-1})}\right)' \text{ for } i = 2, \ldots, n-1. \tag{2.3}$$

Furthermore,

$$L_i y = \frac{W(u_0, \ldots, u_{i-1}, y)}{W(u_0, \ldots, u_i)} \quad \text{for } i = 1, 2, \ldots, n-1, \tag{2.4}$$

$$L_n y = \frac{W(u_0, \ldots, u_{n-1}, y)}{W(u_0, \ldots, u_{n-1})}. \tag{2.5}$$

If $u_0(a) > 0$ and $W(u_0, \ldots, u_i)(a) > 0$ for $i = 1, \ldots, n-1$, the factorization is valid on $[a,b)$. If the Descartes fundamental system is chosen, as is always possible, so that u_i has a zero of order i at a and a zero of order $n-1-i$ at b, then

$$\int^b \rho_i^{-1}\, dx = \infty \quad \text{for } i = 1, \ldots, n-1. \tag{2.6}$$

and we have the unique canonical factorization of L_n on $[a,b)$ due to Trench [4].

We now proceed with the proof of our theorem.

Proof. Suppose $\eta_{k,n-k}(a) = b \le \infty$. Let $\{u_0, u_1, ..., u_{n-1}\}$ be the Descartes fundamental system of solutions for (2.1) such that u_i has a zero of order i at a and a zero of order $n-1-i$ at b. Let $a < c < b$ and z be a solution of (1.1) so that

$$L_i z(a) = 0 \quad \text{for } i = 1, \ldots, k-1$$

and

$$L_i z(c) = 0 \quad \text{for } i = k+1, \ldots, n-1.$$

Since (1.1) is $(k, n-k)$ disfocal (i.e., there is no solution to boundary value problem (1.3) with $m = k$) on $[a,b)$ [2] with respect to the factorization of L_n determined by $\{u_0, u_1, ..., u_{n-1}\}$, then $L_k z(c) \ne 0$. Thus, assume $L_k z(c) = 1$. It now follows [2] that $L_k z > 0$ and $L_{k+1} z < 0$ on $[a,c)$. It follows from (2.4) that $W(u_0, \ldots, u_{k-1}, z) > 0$ and $W(u_0, \ldots, u_k, z) < 0$ on $[a,c)$. Consequently,

$$\left(\frac{W(u_0, u_1, \ldots, u_{k-2}, u_k, z)}{W(u_0, u_1, \ldots, u_{k-1}, u_k)}\right)' = \frac{W(u_0, u_1, \ldots, u_{k-2}, u_k) W(u_0, u_1, \ldots, u_k, z)}{[W(u_0, u_1, \ldots, u_k)]^2} < 0$$

on (a,c). Since $W(u_0, u_1, \ldots, u_{k-2}, u_k, z)(a) = 0$, it follows that

$$W(u_0, u_1, \ldots, u_{k-2}, u_k, z) < 0$$

on $(a, c]$. Therefore, there is a constant $\alpha > 0$ so that

$$\begin{aligned} 0 &= W(u_0, \dots, u_{k-1}, z)(c) + \alpha W(u_0, u_1, \dots, u_{k-2}, u_k, z)(c) \\ &= W(u_0, u_1, \dots, u_{k-2}, u_{k-1} + \alpha u_k, z)(c). \end{aligned} \tag{2.7}$$

Let

$$y_i = u_i \quad \text{for } i \neq k-1 \quad \text{and} \quad y_{k-1} = u_{k-1} + \alpha u_k. \tag{2.8}$$

If $j < k-1$ or $j > k-1$, then

$$W(y_i, y_{i+1} \dots, y_j) = W(u_i, u_{i+1} \dots, u_j) > 0, \tag{2.9}$$

while

$$\begin{aligned} W(y_i, y_{i+1}, \dots, y_{k-1}) = W(u_i, u_{i+1}, \dots, u_{k-1}) \\ + \alpha W(u_i, u_{i+1}, \dots, u_{k-2}, u_k) > 0. \end{aligned} \tag{2.10}$$

Thus, $\{y_0, y_1, ..., y_{n-1}\}$ is a Descartes fundamental system on (a, b). (See [1].)

Let L_n be factored as in (2.2) through (2.5), but where the factorization is determined by the Descartes system $\{y_0, y_1, ..., y_{n-1}\}$. Letting the ith quasiderivative of z according to that factorization be denoted by $J_i(z)$, it follows from (2.4) and (2.8) that $L_i z = J_i z$ for $i \neq k$. Consequently,

$$\begin{aligned} J_i z(a) &= 0 \quad \text{for } i = 0, 1, \dots, k-1, \\ J_i z(c) &= 0 \quad \text{for } i = k+1, \dots, n-1. \end{aligned}$$

Also by (2.4) and (2.7),

$$\begin{aligned} J_k z(c) &= \frac{W(y_0, y_1, \dots, y_{k-1}, z)(c)}{W(y_0, y_1, \dots, y_{k-1}, y_k)(c)} \\ &= \frac{W(u_0, u_1, \dots, u_{k-1}, z)(c) + \alpha W(u_0, u_1, \dots, u_{k-1}, u_k, z)(c)}{W(y_0, y_1, \dots, y_{k-1}, y_k)(c)} = 0. \end{aligned}$$

Thus, for the factorization determined by $\{y_0, y_1, ..., y_{n-1}\}$, $c = \xi_{k,n-k}(a)$. □

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