PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 5, Pages 1473–1481
S 0002-9939(03)07210-1
Article electronically published on October 3, 2003

ISOMETRIES OF CERTAIN OPERATOR SPACES

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(Communicated by Joseph A. Ball)

Abstract. Let X and Y be Banach spaces, and $L(X,Y)$ be the spaces of bounded linear operators from X into Y. In this paper we give full characterization of isometric onto operators of $L(X,Y)$, for a certain class of Banach spaces, that includes ℓ^p, $1<p<\infty$. We also characterize the isometric onto operators of $L(c_0)$ and $K(\ell^1)$, the compact operators on ℓ^1. Furthermore, the multiplicative isometric onto operators of $L(\ell^1)$, when multiplication on $L(\ell^1)$ is taken to be the Schur product, are characterized.

0. Introduction

Let X and Y be Banach spaces, and $L(X,Y)$ the space of bounded linear operators from X into Y. An operator $T\in L(X,Y)$ is called an isometry if $\|Tx\|=\|x\|$ for all $x\in X$. Kadison [14] was the first to characterize the isometries of $L(\ell^2,\ell^2)$, while recently in [9], the isometries of $L(\ell^p,\ell^q), 1<q\le p<\infty$, were characterized. The isometries of the Schatten classes $C_p(H)$, $1\le p\ne 2\le\infty$, where characterized in [3]. Isometries of some operator ideals on Hilbert spaces were characterized in [24], while in [16], the isometries of $N(L^p)$, the nuclear operators on L^p, were characterized. Isometries of general Banach algebras are difficult to handle. In [12], it was proved that every isometry of a semisimple commutative Banach algebra that preserves the identity is multiplicative. In this paper, we give a full characterization of isometric onto operators of $L(X,Y)$, for a class of Banach spaces that includes ℓ^p, $1<p<\infty$. Isometric onto operators of $L(c_0)$, and $K(\ell^1)$, the compact operators on ℓ^1, are also fully characterized in this paper. Furthermore, multiplicative isometric onto operators of $L(\ell^1)$ when multiplication on $L(\ell^1)$ is taken to be the Schur product are characterized.

Throughout this paper, if X is a Banach space, X^* is the dual of X. If G is a subspace of X, then $G^\perp$ is the annihilator of G in X^*. G is called an M-ideal in X if $X^*=G^*\oplus G^\perp$, where the sum is a direct summand, in the sense, if $x^*=g^*+h$, then $\|x^*\|=\|g^*\|+\|h\|$. The concept of M-ideals was introduced in [1].

For any Banach spaces X and Y, we let $X\overset{\vee}{\otimes}Y$ and $X\overset{\wedge}{\otimes}Y$ denote the completed injective and projective tensor product spaces of X with Y, respectively. For $x\in X$ and $y\in Y$, the rank one operator $x\otimes y$ from X^* to Y, is called an **atom**. $K(X,Y)$ denotes the spaces of compact operators in $L(X,Y)$. For $x\in X$, we set $[x]$ to denote the span of $\{x\}$ in X.

Received by the editors June 17, 2002 and, in revised form, January 14, 2003.
2000 *Mathematics Subject Classification.* Primary 46B20; Secondary 46B04.
Key words and phrases. Isometries, operator spaces.

1. Isometries of $\mathbf{L(X,Y)}$

Let X, Y be Banach spaces. The pair (X,Y) is called an **ideal pair** if:
(i) X and Y are reflexive,
(ii) X and Y^* are strictly convex, [7],
(iii) X^* has the approximation property, [7],
(iv) $K(X,Y)$ is an M-ideal in $L(X,Y)$.

It is known in the literature that (ℓ^p, ℓ^q) is an ideal pair for $1 < p \leq q < \infty$; see [6], [23].

Theorem 1.1. *Let (X,Y) be a pair of Banach spaces that satisfies (i), (ii), and (iii), and let $J : X \overset{\wedge}{\otimes} Y^* \longrightarrow X \overset{\wedge}{\otimes} Y^*$ be an isometric onto operator. Then there exists $T_1 : X \longrightarrow X$ and $T_2 : Y^* \longrightarrow Y^*$ such that $J(T) = T_2 T T_1^*$ for every $T \in X \overset{\wedge}{\otimes} Y^*$.*

Let us postpone the proof of this theorem so that we can state and prove one of the main results of the paper.

Theorem 1.2. *Let (X,Y) be an ideal pair, and $J : L(X,Y) \longrightarrow L(X,Y)$, a bounded linear operator. The following are equivalent:*
(i) J is an isometric onto operator.
(ii) There are two isometric onto operators $U \in L(Y,Y)$ and $V \in L(X,X)$, such that $J(T) = UTV$ for all $T \in L(X,Y)$.

Proof. $(i) \longrightarrow (ii)$. Since the proof is a bit long, we divide the proof into steps. However, we have to point out some facts. Since (X,Y) is an ideal pair, it follows that
(a) $K(X,Y) = X^* \overset{\vee}{\otimes} Y$, [8, p. 247];
(b) $(K(X,Y))^* = X \overset{\wedge}{\otimes} Y^*$, [8, p. 247];
(c) $(L(X,Y))^* = X \overset{\wedge}{\otimes} Y^* \oplus K(X,Y)^\perp$, where the sum is a direct summand [1]. □

With these facts in mind, we can proceed to the first step.

Step I. If J is an isometric onto operator on $L(X,Y)$, and if there exists $x_0 \in X$ and $y_0^* \in Y^*$ such that $J^*(x_0 \otimes y_0^*) = z \otimes w^* \in X \overset{\wedge}{\otimes} Y^*$, then J^* preserves atoms (if J^* preserves one atom, then it preserves every atom in $X \overset{\wedge}{\otimes} Y^*$).

Proof. From fact (c), $(L(X,Y))^* = X \overset{\wedge}{\otimes} Y^* \oplus K(X,Y)^\perp$, where the sum is a direct summand [1]. So $ExtB_1(L(X,Y))^* = \{g : g \in extB_1(X \overset{\wedge}{\otimes} Y^*) \text{ or } g \in extB_1(K(X,Y)^\perp)\}$. Since J is an isometric onto operator, then J^* is an isometric onto operator. Consequently, J^* preserves extreme points. But $extB_1(X \overset{\wedge}{\otimes} Y^*)$ are just the atoms of norm 1 [22]. Hence, $J^*(x \otimes y^*)$ is either an atom or an element in $K(X,Y)^\perp$. Now, let $x \otimes y^* \in X \overset{\wedge}{\otimes} Y^*$ and $g = x_0 \otimes y_0^* + x \otimes y_0^*$. Then $J^*(g) = J^*((x_0 + x) \otimes y_0^*)$, which must be either an atom or an element in $K(X,Y)^\perp$. Since $J^*(x_0 \otimes y_0^*)$ is an atom, and $K(X,Y)^\perp$ has only trivial intersection with $X \overset{\wedge}{\otimes} Y^*$, we conclude that $J^*(g)$ and $J^*(x \otimes y_0^*)$ are both atoms. Similarly, $J^*(x_0 \otimes y^*)$ is an atom. Now, consider $x_0 \otimes y^* + x \otimes y^* = (x_0 + x) \otimes y^*$. Since J^* preserves extreme points and $J^*(x_0 \otimes y^*)$ is an atom, it follows that $J^*(x \otimes y^*)$ is an atom. □

Step II. If J is an isometric onto operator on $L(X,Y)$, then $K(X,Y)$ is an invariant subspace for J.

Proof. Let $T \in K(X,Y)$. We claim that $J(T) \in K(X,Y)$. To see this, let (x_n) be a sequence in X. With no loss of generality, assume that $J(T)x_n \neq 0$ for all n. Choose $y_n^* \in B_1(Y^*)$ such that $\langle J(T)x_n,\, y_n^*\rangle = \|J(T)x_n\|$. So $\|J(T)x_n\| = \langle T,\, J^*(x_n\otimes\, y_n^*)\rangle$.

Since J^* is an isometric onto operator, then as in step I, either $J^*(x_n\otimes\, y_n^*)$ is an atom or an element in $K(X,Y)^{\perp}$. But T being compact and $\|J(T)x_n\| \neq 0$ implies that $J^*(x_n\otimes\, y_n^*)$ is an atom. By step I, we get $J^*(X \overset{\wedge}{\otimes} Y^*) = X \overset{\wedge}{\otimes} Y^*$. By Theorem 1.1, it follows that $J^*(x\otimes y) = F_1(x)\otimes F_2(y^*)$ where F_1 is an isometric onto operator on X and F_2 on Y^*. Hence $\langle T,\, J^*(x_n\otimes\, y_n^*)\rangle = \langle T,\, F_1(x_n)\otimes F_2(y_n^*)\rangle$ $= \langle T(F_1(x_n)),\, F_2(y_n^*)\rangle$. Again, since T is compact, there exists (x_{n_k}) such that $T(F_1(x_{n_k}))$ is Cauchy. We claim that $(J(T)x_{n_k})$ is Cauchy (and so convergent). Indeed,

$$\begin{aligned}
&\left\|J(T)x_{n_k} - J(T)x_{n_j}\right\| \\
&\quad = \left|\langle J(T)(x_{n_k} - x_{n_j}), z^*_{n_{kj}}\rangle\right| \ \text{(for some } z^*_{n_{kj}} \in B_1(Y^*), \\
&\quad = \left|\langle T, J^*((x_{n_k} - x_{n_j}) \otimes z^*_{n_{kj}})\rangle\right| \\
&\quad = \left|\langle T, F_1(x_{n_k} - x_{n_j}) \otimes F_2(z^*_{n_{kj}})\rangle\right| \\
&\quad = \left|\langle TF_1(x_{n_k} - x_{n_j}), F_2(z^*_{n_{kj}})\rangle\right| \le \left\|TF_1(x_{n_k} - x_{n_j})\right\| \longrightarrow 0.
\end{aligned}$$

□

Consequently, $J(T)$ is compact.

Step III. If J is an isometric onto operator on $L(X,Y)$, then $J^*(X \overset{\wedge}{\otimes} Y^*) = X \overset{\wedge}{\otimes} Y^*$, and $J^*(K(X,Y)^{\perp}) = K(X,Y)^{\perp}$.

Proof. This follows from step I and the proof of step II. □

Step IV. $(i) \longrightarrow (ii)$.

Proof. Let J be an isometric onto operator on $L(X,Y)$. By step III, $J^*(X \overset{\wedge}{\otimes} Y^*) = X \overset{\wedge}{\otimes} Y^*$and $J(K(X,Y)^{\perp}) = K(X,Y)^{\perp}$. Hence, by Theorem 1.1, $J^*(x\otimes y^*) = F_1(x)\otimes F_2(y^*)$, where F_1 is an isometric onto operator on X and F_2 on Y^*. Thus $\langle J(T)x,\, y^*\rangle = \langle T, J^*(x\otimes y^*)\rangle = \langle T, F_1(x)\otimes F_2(y^*)\rangle = \langle F_2^*TF_1x,\, y^*\rangle$. Since this is true for all $y^* \in Y^*$, it follows that $J(T) = F_2^*TF_1$. Taking $U = F_2^*$ and $V = F_1$, the result follows.

The proof that $(ii) \longrightarrow (i)$ is immediate. This ends the proof of the theorem. □

Since the pair (ℓ^p, ℓ^r) is an ideal pair for $1 < p, r < \infty$.

Theorem 1.2 was proved for the pair (ℓ^p, ℓ^r) in [9]. We were unaware of reference [9]. The proof given in [9] uses smooth points of $L(\ell^p, \ell^r)$, while we use the concept of M-ideals and an idea of the first author in [16]. We thank the referee for calling our attention to that paper, and we thank Professor Grzaslewicz for supplying us with a copy of that paper later on.

Now, we go back to:

Proof of Theorem 1.1. As in Theorem 1.2, we split the proof into steps. □

Step I. If $x_1 \otimes y_1^* + x_2 \otimes y_2^* = f \otimes g$ in $X \overset{\wedge}{\otimes} Y^*$, then either x_1, x_2 are independent in X, or y_1^*, y_2^* are dependent in Y^*.

Proof. If either f or g is zero, there is nothing to prove. Assume $f \otimes g \neq 0 \otimes 0$, and x_1, x_2 are independent. Consider the operator $T : X^* \longrightarrow Y^*$ defined by $Tx^* = \langle f, x^* \rangle g = \langle x_1, x^* \rangle y_1^* + \langle x_2, x^* \rangle y_2^*$. Since x_1 and x_2 are independent, there exist x_1^* and x_2^* in X^* such that $\langle x_1, x_1^* \rangle = 1$, $\langle x_2, x_1^* \rangle = 1$, $\langle x_1, x_2^* \rangle = 0$, and $\langle x_2, x_2^* \rangle = 1$. Then $T(x_1^*) = y_1^* + y_2^*$ and $T(x_2^*) = y_2^*$. Since T is a rank one operator, $y_1^* + y_2^* = ay_2^*$. This happens only if y_1^*, y_2^* are dependent. □

One should remark here that the statement of step I appears in Jarosz [13]. The proof given here is different.

Step II. Let $J : X \overset{\wedge}{\otimes} Y^* \longrightarrow X \overset{\wedge}{\otimes} Y^*$ be an isometric onto operator, and assume that X is not isometrically isomorphic to Y^*. Then, for every $y^* \in Y^*$, there exists $g^* \in Y^*$ such that $J(X \overset{\wedge}{\otimes} [y^*]) = X \overset{\wedge}{\otimes} [g^*]$.

Proof. Any element in $X \overset{\wedge}{\otimes} [y^*]$ can be written in the form $z \otimes y^*$ for some $z \in X$. Since J is an isometry, J preserves extreme points, and consequently J preserves atoms, noting that extreme points of $X \overset{\wedge}{\otimes} Y^*$ are atoms of norm one [22]. Set $K = J(X \overset{\wedge}{\otimes} [y^*])$. Then any element in K is an atom.

Let $x_1 \otimes y^*$ and $x_2 \otimes y^*$ be any two elements in $X \overset{\wedge}{\otimes} [y^*]$, and let $J(x_1 \otimes y^*) = z_1 \otimes w_1^*$ and $J(x_2 \otimes y^*) = z_2 \otimes w_2^*$. Here, it is assumed that x_1 and x_2 are independent. But $z_1 \otimes w_1^* + z_2 \otimes w_2^* = J((x_1 + x_2) \otimes y^*)$ is an atom. Hence, by step I, there are two cases: (i) z_1 and z_2 are dependent and (ii) w_1^* and w_2^* are dependent.

Case (i). Assume w_1^* and w_2^* are dependent; so w_1^*, w_2^* are in $[g^*]$ for some $g^* \in Y^*$. We claim that any element in K has the form $z \otimes g^*$ with $z \in X$. To see this, let $x \otimes y^*$ be any element in $X \overset{\wedge}{\otimes} [y^*]$(with x independent of x_1 and x_2) and $J(x \otimes y^*) = p \otimes q^*$. Both $J(x \otimes y^* + x_1 \otimes y^*)$ and $J(x \otimes y^* + x_2 \otimes y^*)$ are atoms. If q^* and w_1^* are dependent, then we are done. If not, then by step I, we must have that p and z_1 are dependent, and p and z_2 are dependent. This contradicts the assumption that z_1 and z_2 are independent. Consequently, $K = W \overset{\wedge}{\otimes} [g^*]$. Furthermore, since J is an isometric onto operator, using step I, it follows easily that if $J(X \overset{\wedge}{\otimes} [y^*])$ has the form $X \overset{\wedge}{\otimes} [g^*]$, for one y^*, then it has the same form for all y^*, (g^* and W depend on y^*). We claim that W is the same for all y^*. Indeed, let $J(X \overset{\wedge}{\otimes} [y_1^*]) = W_1 \overset{\wedge}{\otimes} [w_1^*]$, and $J(X \overset{\wedge}{\otimes} [y_2^*]) = W_2 \overset{\wedge}{\otimes} [w_2^*]$, where w_1^* and w_2^* are independent (whenever y_1^* and y_2^* are independent). Take any element x in X, and let $J(x \otimes y_1^*) = \hat{x}_1 \otimes w_1^*$, and $J(x \otimes y_2^*) = \hat{x}_2 \otimes w_2^*$. Step I implies that $\hat{x}_1$ and $\hat{x}_2$ are dependent for all x in X, and so $W_1 = W_2$. Since J is onto, W must be X.

Case (ii). Assume w_1^* and w_2^* are independent. Hence z_1 and z_2 are dependent. But then, using the above argument again, one can show that there exists $\hat{x}$ and a closed subspace $E \subset Y^*$ such that $J(X \overset{\wedge}{\otimes} [y^*]) = [\hat{x}] \overset{\wedge}{\otimes} E$, where E is the same for all y^*. Since J is onto, then $E = Y^*$. But E is isometrically isomorphic to X, which implies that X is isometrically isomorphic to Y^*, contradicting our assumption. Hence, in both cases, $J(X \overset{\wedge}{\otimes} [y^*]) = X \overset{\wedge}{\otimes} [g^*]$.

We should **remark** that, if X is isometrically isomorphic to Y^*, then K can have the form $[h] \overset{\wedge}{\otimes} Y^*$. Furthermore, we **remark** that the same result of step II is true for $J([x] \overset{\wedge}{\otimes} Y^*)$. □

Step III. If $J : X \overset{\wedge}{\otimes} Y^* \longrightarrow X \overset{\wedge}{\otimes} Y^*$ is an isometric onto operator, then there are two isometric onto operators, $T_1 \in L(X, X)$ and $T_2 \in L(Y^*, Y^*)$ such that $J(x \otimes y) = T_1 x \otimes T_2 y$ for all $x \in X$ and $y \in Y^*$.

Proof. From step II, and the second part of the remark, we deduce that for $x \in X$ there exists $\hat{x} \in X$ such that $J(x \otimes y) = \hat{x} \otimes F(y)$, for all $y \in Y^*$. Define $T_1(x) = \hat{x}$. Then, one can easily see that T_1 is a well-defined isometric onto operator in $L(X, X)$. Similarly, one can define T_2 on Y^*. Furthermore, $J(x \otimes y^*) = T_1 x \otimes T_2 y^*$ for all $x \otimes y^* \in X \overset{\wedge}{\otimes} Y^*$. This ends the proof. □

Remark 1.3. Theorem 1.1 holds true for Banach spaces X and Y for which the closed convex hull of $ext(B_1(X))$ equals $B_1(X)$, or the closed convex hull of $ext(B_1(Y))$ equals $B_1(Y)$.

2. Isometries of operators on ℓ^1

In this section, we give full characterization of onto isometries of $L(c_0)$, $K(\ell^1)$ and multiplicative onto isometries in $L(\ell^1)$. First we recall some facts about bounded linear operators on sequence spaces. For a Banach space E and $p \in [1, \infty)$, we let $\ell^p\{E\} = \{(x_i) : x_i \in E$, and $\sup \sum |\langle x_i, x^* \rangle|^p < \infty$, $\|x^*\| = 1\}$. For $(x_i) \in \ell^p\{E\}$, one can define the norm $\|(x_i)\|_{\in(p)} = \sup(\sum |\langle x_i, x^* \rangle|^p)^{\frac{1}{p}}$, where the supremum is taken over the unit ball of E^*. It is known that ([21]) $(\ell^p\{E\}, \|\|_{\in(p)})$ is a Banach space. In [10, p. 86], Grothendieck proved that $L(\ell^q, E)$ is isometrically isomorphic to $\ell^p\{E\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \in (1, \infty)$; see also [6]. For $p = 1$, the result states that $\ell^1\{E\}$ is isometrically isomorphic to $L(c_0, E)$. For $x = (x_i) \in \ell^p\{E\}$, $p \in [1, \infty)$, the corresponding operator is defined by $T(a_i) = \sum a_i x_i$.

Another way to look at $L(\ell^1)$ is as a dual space. It is known that $L(E, F^*) = (E \overset{\wedge}{\otimes} F)^*$ [8, p. 230]. Thus, $L(\ell^1) = (\ell^1 \overset{\wedge}{\otimes} c_0)^*$. But ([8, p. 228]) $\ell^1 \overset{\wedge}{\otimes} E = \ell^1(E) = \{(x_i) : x_i \in E$, and $\sum \|x_i\| < \infty\}$, and $(\ell^1(E))^* = \ell^\infty(E^*) = \{(x_i^*) : x_i \in E^*$ and $\sup \|x_i^*\| < \infty\}$. Hence $L(\ell^1) = \ell^\infty(\ell^1)$, and to every $T \in L(\ell^1)$ there exists $(f_n) \in \ell^\infty(\ell^1)$ such that $\|T\| = \sup \|f_n\|$. Furthermore, $T((x_i)) = \sum x_i f_i$.

Let us start with $L(c_0)$.

Theorem 2.1. *Let $J : L(c_0) \longrightarrow L(c_0)$ be a bounded linear operator. The following are equivalent.*

(i) J is an onto isometry.

(ii) There are two isometric onto operators U, V, of c_0 such that $J(T) = UTV$ for all $T \in L(c_0)$.

Proof. $(i) \longrightarrow (ii)$. Let $J : L(c_0) \longrightarrow L(c_0)$ be an isometric onto operator. Then, $J^* : (L(c_0))^* \longrightarrow (L(c_0))^*$ is an isometric onto operator. It is known that [16], $(L(c_0))^* = (K(c_0))^* \oplus K(c_0)^\perp$, and $(K(c_0))^* = (\ell^1 \overset{\vee}{\otimes} c_0)^* = \ell^\infty \overset{\wedge}{\otimes} \ell^1$. If (δ_n) is the natural basis of ℓ^1, then for $T \in K(c_0)$ and for each n, there exists $y_n \in B_1(\ell^\infty)$ such that

$$\|(J(T))^* \delta_n\| = \langle J(T)^* \delta_n, y_n \rangle = \langle T, J^*(y_n \otimes \delta_n) \rangle. \tag{2.1}$$

With no loss of generality, assume that $\|(J(T)^*\delta_n\| \neq 0$. Since J^* is an isometric onto operator, it takes extreme points of $(K(c_0))^* \oplus K(c_0)^\perp$ to extreme points of $(K(c_0))^* \oplus K(c_0)^\perp$. But extreme points of $(K(c_0))^*$ are the atoms of the form $y \otimes \delta_n$, where y is an extreme point in $B_1(\ell^\infty)$. Since T is compact, and the extreme points in $B_1((K(c_0))^* \oplus K(c_0)^\perp)$ are either the extreme points of $B_1(K(c_0))^*$ or the extreme points of $B_1(K(c_0)^\perp)$, it follows that $J^*(y_n \otimes \delta_n)$ is an extreme point in $B_1(K(c_0))^*$, say $\hat{y}_n \otimes \delta_{\phi(n)}$. Thus, from (2.1) we get

$$\|(J(T))^*\delta_n\| = \left|\langle T, \hat{y}_n \otimes \delta_{\phi(n)}\rangle\right| = \left|\langle T^*\delta_{\phi(n)}, y_n\rangle\right| \leq \left\|T^*\delta_{\phi(n)}\right\| \longrightarrow 0.$$

Consequently, $J(T)$ is compact [20]. This implies that if $S \in K(c_0)^\perp$ and $A \in K(c_0)$, then $\langle A, J^*(S)\rangle = \langle J(A),\ S\rangle = 0$. Thus $J(K(c_0)^\perp) \subseteq K(c_0)^\perp$. Since J^* is onto, we get $J^*(\ell^\infty \overset{\wedge}{\otimes} \ell^1) = \ell^\infty \overset{\wedge}{\otimes} \ell^1$. By Remark 1.3, there exist two isometric onto operators $F_1 \in L(\ell^\infty)$ and $F_2 \in L(\ell^1)$ such that $J^* = F_1 \otimes F_2$ or $J^* = F_2 \otimes F_1$. Assume $J^* = F_1 \otimes F_2$. This gives

$$\langle J(T)x, y\rangle = \langle T, J^*(x \otimes y)\rangle = \langle TF_1x, F_2y\rangle = \langle F_2^*TF_1x, y\rangle.$$

This is (ii). As for $(ii) \longrightarrow (i)$, the proof is straightforward. □

As for isometries of $K(\ell^1)$, we have

Theorem 2.2. *Let $J : K(\ell^1) \longrightarrow K(\ell^1)$ be a bounded linear operator. The following are equivalent:*

(i) J is an isometric onto operator.

(ii) There are isometric onto operators S_r of ℓ^1, $r \in N$, and a permutation π of N such that, for $f = (f_n)$, $J(f) = (S_n f_{\pi(n)})$.

Proof. We need only show that $(i) \longrightarrow (ii)$. We recall that $T \in L(\ell^1)$ is compact if and only if $\lim_{n\longrightarrow\infty} \|T\delta_n\| = 0$ [21]. Now, $c_0(\ell^1) \subset \ell^\infty(\ell^1) = L(\ell^1)$, and if $(f_n) \in c_0(\ell^1)$ represents T in $L(\ell^1)$, then $T((x_i)) = \sum x_i f_i$, and $\|T\delta_n\| = \|f_n\|$, which is a null sequence. Hence, $c_0(\ell^1) \subset K(\ell^1)$. We claim $J(c_0(\ell^1)) \subset c_0(\ell^1)$. To prove this, let $A \in c_0(\ell^1)$. We claim $\|J(A)\delta_n\| \longrightarrow 0$. With no loss of generality, we can assume $\|J(A)\delta_n\| \neq 0$ for all n. For each $n \in N$, choose $y_n \in extB_1(\ell^\infty)$ such that $\langle J(A)\delta_n, y_n\rangle = \|J(A)\delta_n\| = \langle A, (J^*(\delta_n \otimes y_n))\rangle$. Now, $(K(\ell^1))^* = (\ell^\infty)^* \overset{\wedge}{\otimes} \ell^\infty$. But $(\ell^\infty)^* = M(\beta(N))$, the space of regular Borel measures on $\beta(N)$, the Stone-Cech compactification of N [17]. Since J^* is an isometric onto operator,

$$J^*(extB_1((M(\beta(N)) \overset{\wedge}{\otimes} \ell^\infty)) = extB_1((M(\beta(N)) \overset{\wedge}{\otimes} \ell^\infty)).$$

Consequently, since $extB_1((M(\beta(N)) \overset{\wedge}{\otimes} \ell^\infty)) = \{\delta_t \otimes y : t \in \beta(N),\ |y(n)| = 1$ for all $n\}$, $J^*(\delta_n \otimes y_n) = \delta_{t_n} \otimes z_n$, with $z_n \in extB_1(\ell^\infty)$. Furthermore, $t_n \in N$. Indeed, if $t_n \in \beta N \backslash N$, then there exists $k_i \in N$ such that $k_i \longrightarrow t_n$, and $\|J(A)\delta_n\| = |\langle J(A)\delta_n, y_n\rangle| = |\langle A, J^*(\delta_n \otimes y_n)\rangle| = |\langle A\delta_{t_n}, z_n\rangle| = \lim_{i\longrightarrow\infty} |\langle A\delta_{k_i}, z_n\rangle| \leq \lim_{i\longrightarrow\infty} \|A\delta_{k_i}\|$.

Since $A \in c_0(\ell^1)$, then $\lim_{i\longrightarrow\infty} \|A\delta_{k_i}\| = 0$. This implies that $\|J(A)\delta_n\| = 0$ for all n, which contradicts the assumption $\|J(A)\delta_n\| \neq 0$ for all n. Hence, $t_n \in N$. So, $\|J(A)\delta_n\| = |\langle A, \delta_{t_n} \otimes z_n\rangle| \leq \|A\delta_{t_n}\| \longrightarrow 0$, since $A \in c_0(\ell^1)$. Thus, $J(A) \in c_0(\ell^1)$, and since J is onto, we get $J : c_0(\ell^1) \longrightarrow c_0(\ell^1)$ is an isometric onto operator. Hence ([18]; see also [4, pp. 147-151]) there is a permutation π of the natural numbers N, and isometric onto operators S_n of ℓ^1 such that $J(f_1, f_{2,} f_3, ..., f_n, ...) = (S_n f_{\pi(n)})$, for all $f = (f_n) \in c_0(\ell^1)$.

Now, let $(f_n) \in K(\ell^1) \subset \ell^\infty(\ell^1)$. Then, for all $g^* \in \ell^1(c_0)$ we have $\langle J(f), g^*\rangle = \lim\langle J(H_n), g^*\rangle$, where $H_n = (f_1, f_2, f_3, ...f_n, 0, 0, 0, ...)$. Hence, using the result in [6], we get $J(f_1, f_{2,} f_3, ..., f_n, ...) = (S_n f_{\pi(n)})$. This ends the proof. □

Now, we turn to $L(\ell^1)$. Every element of $L(\ell^1)$ is an infinite matrix. For A, B in $L(\ell^1)$, we write $A * B$ for the Schur product of A and B, where $(A * B)_{ij} = A_{ij}B_{ij}$. It is not difficult to see that $A * B \in L(\ell^1)$, for A, B in $L(\ell^1)$. Schur multipliers were studied by many authors. We refer to [5] and [15] for the main results on Schur multipliers of $L(\ell^p, \ell^r)$ and $\ell^p \overset{\wedge}{\otimes} \ell^r$. An operator $J : L(\ell^1) \longrightarrow L(\ell^1)$ is called multiplicative if $J(A * B) = J(A) * J(B)$.

Theorem 2.3. *Let $J : L(\ell^1) \longrightarrow L(\ell^1)$ be a multiplicative bounded linear operator. The following are equivalent.*

(i) J is an isometric onto operator.

(ii) There are permutations π and φ_n of N and isometries S_n of ℓ^1 such that if $f = (x(n)) \in \ell^1$, then $J(0, 0, ..., 0, f, 0, ...) = (0, 0, ..., 0, S_n f, 0, 0, ...)$ where f appears in the nth-*coordinate, $S_n f$ appears in the* $\pi(n)$th-*coordinate and $S_n(f) = (x_{\varphi(n)}(i))$.*

Proof. $(i) \longrightarrow (ii)$. Suppose that $J : L(\ell^1) \longrightarrow L(\ell^1)$ is a multiplicative isometric onto operator. Any $A \in L(\ell^1)$ can be represented as $A = (f_1, f_2, ..., f_n, ...)$, with $f_n \in \ell^1$, and $\|A\| = \sup_n \|f_n\|$. Since each $f_n \in \ell^1$, then A has a matrix representation

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & \dots & a_{2n} & \dots \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & \dots \\ \vdots & & & \vdots & \end{pmatrix}$$

where the nth-row of A is f_n.

Let $\text{supp}(A) = \{(m, n) \in N \times N : a_{mn} \neq 0\}$. Then, $A * B = 0$ if and only if $\text{supp}(A) \cap \text{supp}(B) = \phi$. Since J is multiplicative, $J(A) * J(B) = 0$ if and only if $A * B = 0$.

Now, for $i, j \in N$, let $\delta_{i,j}$ be the matrix with entry 1 in the (i, j)th-coordinate and 0 elsewhere. We claim that J^{-1} (and hence J) maps $\delta_{ij}^{,s}$ into $\delta_{ij}^{,s}$. To prove this, fix $i, j \in N$, and let $A = (a_{ij}) \in L(\ell^1)$ with $J(A) = \delta_{ij}$. Write $A = A_1 + A_2$, where $A_1 = \delta_{11}$, and

$$A_2 = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & \dots & a_{2n} & \dots \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & \dots \\ \vdots & & & \vdots & \end{pmatrix}.$$

Notice that A_1 and A_2 have disjoint support, and so $J(A_1)$ and $J(A_2)$ have disjoint support. But, $J(A_1) + J(A_2) = J(A) = \delta_{ij}$. So, either $J(A_1) = \pm\delta_{ij}$ and $J(A_2) = 0$, or $J(A_2) = \pm\delta_{ij}$ and $J(A_1) = 0$. If $J(A_1) = \pm\delta_{ij}$ and $J(A_2) = 0$, then

$A_2 = 0$ and $A = A_1 = \pm\delta_{11}$. If $J(A_2) \neq 0$, write $A_2 = B_1 + B_2$, where

$$B_1 = \begin{pmatrix} 0 & a_{12} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & & 0 & & \\ 0 & & & 0 & \\ \vdots & & & \vdots & \ddots \end{pmatrix},$$

and $B_2 = A_2 - B_1$.

So, as in the argument above, either $B_2 = 0$ and $B_1 = \delta_{12}$, or else we continue the procedure. Since J is multiplicative, there must exist $(i_0, j_0) \in N \times N$ with $J(\delta_{i_0 j_0}) = \delta_{ij}$. Hence J^{-1} preserves the $\delta_{ij}^{;s}$, and so does J.

Since J is one-to-one and onto, there exists a permutation $\pi : N \longrightarrow N$, such that $J(\delta_{1n}) = \delta_{s_n \pi(n)}$. Furthermore, $\|J(\delta_{ir} + \delta_{jr})\| = \|\delta_{i_1 r_1} + \delta_{j_2 r_2}\| = 1$, for $r_1 \neq r_2$, and $\|\delta_{ir} + \delta_{jr}\| = 2$. This contradicts the fact that J is an isometry, unless $r_1 = r_2$. Hence, J maps a matrix with only one nonzero column into a matrix with only one nonzero column. Consequently, there are permutations φ_r of N, such that $J(\delta_{ir}) = \delta_{\varphi_r(i)\pi(r)}$, $i, r \in N$. Now, for $f = (x_1, x_2, x_3, ..., x_n, ...) \in \ell^1$ and a fixed $r \in N$, we have $J(\sum_{i=1}^{n} x_i \delta_{ir}) = \sum_{i=1}^{n} x_i \delta_{\varphi_r(i)\pi(r)}$, for all $n \in N$. Let $h_n = \sum_{i=1}^{n} x_i \delta_{ir} \in \ell^\infty(\ell^1)$. Then $h_n \longrightarrow (0, 0, 0, ..., f, 0, 0, 0, ...)$ where f appears in the rth-coordinate. Hence,

$$\begin{aligned} J(0, 0, 0, ..., f, 0, 0, 0, ...) &= \lim_n J(h_n) \\ &= \lim_n \sum_{i=1}^{n} x_i \delta_{\varphi_r(i)\pi(r)} = (0, 0, 0, ..., g, 0, 0, 0, ...), \end{aligned}$$

where $g = (x_{\varphi_r(1)}, x_{\varphi_r(2)}, ..., x_{\varphi_r(n)}, ...)$ appears in the $\pi(r)$th-coordinate. This proves $(i) \longrightarrow (ii)$.

$(ii) \longrightarrow (i)$. The proof is immediate and will be omitted. □

Remark 2.4. Theorem 2.2 holds true for isometries of $L(\ell^1, \ell^p)$.

Acknowledgment

The authors would like to thank the referee for his sound comments, and for calling the attention of the authors to the paper of Grzaslewicz [9], and pointing out some incomplete arguments.

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