PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 5, Pages 1403–1410
S 0002-9939(03)07219-8
Article electronically published on August 28, 2003

SECOND COHOMOLOGY GROUP OF GROUP ALGEBRAS WITH COEFFICIENTS IN ITERATED DUALS

A. POURABBAS

(Communicated by Joseph A. Ball)

ABSTRACT. In this paper we show that the first cohomology group $\mathcal{H}^1(\ell^1(G),(\ell^1(S))^{(n)})$ is zero for every odd $n\in\mathbb{N}$ and for every G-set S. In the case when G is a discrete group, this is a generalization of the following result of Dales et al.: for any locally compact group G, $L^1(G)$ is $(2n+1)$-weakly amenable.

Next we show that the second cohomology group $\mathcal{H}^2(\ell^1(G),(\ell^1(S))^{(n)})$ is a Banach space. Finally, for every locally compact group G we show that $\mathcal{H}^2(L^1(G),(L^1(G))^{(n)})$ is a Banach space for every odd $n\in\mathbb{N}$.

1. Introduction

In this paper we shall be concerned with the structure of the second cohomology group of a group algebra $L^1(G)$ with coefficients in the nth dual space $(L^1(G))^{(n)}$. We begin by recalling some terminology.

Suppose that $\mathcal{A}$ is a Banach algebra and that $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule, so that $\mathcal{X}$ is an $\mathcal{A}$-bimodule, $\mathcal{X}$ is a Banach space for a norm $\|\cdot\|$, and $\|a\cdot x\|\leq\|a\|\,\|x\|$ and $\|x\cdot a\|\leq\|a\|\,\|x\|$ for $a\in\mathcal{A}$ and $x\in\mathcal{X}$. For example, $\mathcal{A}$ itself is a Banach $\mathcal{A}$-bimodule. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then the $\mathcal{X}'$ is also a Banach $\mathcal{A}$-bimodule for the products $a\cdot\lambda$ and $\lambda\cdot a$ specified by

$$a\cdot\lambda(x)=\lambda(x\cdot a),\quad \lambda\cdot a(x)=\lambda(a\cdot x)\qquad (a\in\mathcal{A},\,x\in\mathcal{X},\lambda\in\mathcal{X}').$$

Similarly, the higher duals $\mathcal{X}^{(n)}$ are Banach $\mathcal{A}$-bimodules. The canonical embedding of $\mathcal{X}$ in $\mathcal{X}^{(2)}=\mathcal{X}''$ is denoted by $\widehat{\ }$. Since $\widehat{a\cdot x}=a\cdot\widehat{x}$, then $\mathcal{X}^{(n)}$ is a submodule of $\mathcal{X}^{(n+2)}$ for every $n\in\mathbb{Z}^+$. We also set $\mathcal{X}^{(0)}=\mathcal{X}$. If S is a topological space and G is a (discrete) group, then we say that S is a G-set if the product gx is defined for all g in G and x in S in such a way that $g(hx)=(gh)x\,(g,h\in G,x\in S)$ and $x\mapsto gx$ is a homeomorphism of S onto S for every g in S. The cohomology complex is

$$0\longrightarrow\mathcal{X}\xrightarrow{\delta^0}\mathcal{C}^1(\mathcal{A},\mathcal{X})\xrightarrow{\delta^1}\mathcal{C}^2(\mathcal{A},\mathcal{X})\xrightarrow{\delta^2}\cdots,$$

where for $n\in\mathbb{Z}^+$, $\mathcal{C}^n(\mathcal{A},\mathcal{X})$ is the set of all bounded n-linear maps from $\mathcal{A}$ to $\mathcal{X}$. The map $\delta^0:\mathcal{X}\longrightarrow\mathcal{C}^1(\mathcal{A},\mathcal{X})$ is given by $\delta^0(x)(a)=a\cdot x-x\cdot a$ and for $n\in\mathbb{Z}^+$,

Received by the editors January 14, 2002 and, in revised form, December 31, 2002.
2000 *Mathematics Subject Classification.* Primary 43A20; Secondary 46M20.
This research was supported by a grant from Amir Kabir University. The author would like thank the Institute for their kind support.

the map $\delta^n : \mathcal{C}^n(\mathcal{A}, \mathcal{X}) \longrightarrow \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{X})$ is given by

$$\begin{aligned} \delta^n T(a_1, \ldots, a_{n+1}) &= a_1 \cdot T(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i T(a_1, \ldots, a_i a_{i+1}, \ldots a_{n+1}) \\ &\quad + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}, \end{aligned}$$

where $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{X})$ and $a_1, \ldots, a_{n+1} \in \mathcal{A}$. The space $\ker \delta^n$ of bounded n-cocycle is denoted by $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ and the space $\operatorname{Im} \delta^{n-1}$ of bounded n-coboundary is denoted by $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$. We recall that $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is a subspace of $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ and that the nth cohomology group $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is defined by the quotient

$$\mathcal{H}^n(\mathcal{A}, \mathcal{X}) = \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{X})}{\mathcal{B}^n(\mathcal{A}, \mathcal{X})},$$

which is called the nth continuous or Banach cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$. The space $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ is a Banach space, but in general $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is not closed; we regard $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ as a complete seminormed space with respect to the quotient seminorm. This seminorm is a norm if and only if $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is a closed subspace of $\mathcal{C}^n(\mathcal{A}, \mathcal{X})$, which means that $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space. It is unknown whether or not $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space for every n.

The Banach algebra $\mathcal{A}$ is amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}') = 0$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$. This definition of amenability was introduced by Johnson in (1972) [9]. The Banach algebra $\mathcal{A}$ is weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}') = 0$. This definition generalizes the one that was introduced by Bade, Curtis and Dales in [1], where it was noted that a commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = 0$ for every symmetric Banach $\mathcal{A}$-bimodule $\mathcal{X}$. The Banach algebra $\mathcal{A}$ is permanently weakly amenable if $\mathcal{A}$ is n-weakly amenable for every $n \in \mathbb{N}$, that is, $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = 0$. This notion was introduced by Dales, Ghahramani and Grønbæk in [2].

It was shown in [9] that the group algebra $L^1(G)$ is amenable if and only if G is an amenable group, and in [11] that $L^1(G)$ is weakly amenable for every locally compact group G; see also [3] for a shorter proof. Also it was shown in [2] that $L^1(G)$ is $(2n+1)$-weakly amenable for every locally compact group G. Johnson [9] proved that for the free group on two generators $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C}) \neq 0$ which by [15, Theorem 8.3.1] implies that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^1(\mathbb{F}_2)) \neq 0$ and $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^\infty(\mathbb{F}_2)) \neq 0$.

In [8] Ivanov and in [12] Matsumoto and Morita showed that $\mathcal{H}^2(\ell^1(G), \mathbb{C})$ is a Banach space for every discrete group G with trivial action on $\mathbb{C}$. A. Pourabbas and M. C. White [13] showed that the second cohomology group of $L^1(G)$ with coefficients in $L^\infty(G)$ is a Banach space for every locally compact group G. In this paper first we show that $\mathcal{H}^1(\ell^1(G), (\ell^1(S))^{(2n+1)}) = 0$ for every discrete group G and every G-set S. Next we show that the second cohomology group of $L^1(G)$ with coefficients in $(L^1(G))^{(2n+1)}$ is a Banach space for every locally compact group G and every $n \in \mathbb{N}$.

The following proposition is standard. We include the proof as a base for later results.

Proposition 1.1. *Let X and Y be Banach spaces and let $\Phi : X \to Y$ be a bounded linear map. If there exists a constant M such that for every $y \in \operatorname{Im} \Phi$ there exists an element $x \in X$ such that $\|x\| \leq M \|y\|$ and $y = \Phi(x)$, then $\operatorname{Im} \Phi$ is closed.*

Proof. Fix $y \in \overline{\operatorname{Im}\Phi}$ and $\epsilon > 0$. There exists an element $y_1 \in \operatorname{Im}\Phi$ such that

$$\|y - y_1\| \leq \frac{1}{2}\epsilon.$$

By considering $y - y_1$, there exits an element $y_2 \in \operatorname{Im}\Phi$ such that

$$\|y - y_1 - y_2\| \leq \frac{1}{2^2}\epsilon.$$

By induction, this process defines a sequence $\{y_n\} \subseteq \operatorname{Im}\Phi$ such that

$$\|y - y_1 - \cdots - y_n\| \leq \frac{1}{2^n}\epsilon. \tag{1.1}$$

Note that

$$\begin{aligned} \|y_{n+1}\| &\leq \|y - y_1 - \cdots - y_n\| + \frac{1}{2^{n+1}}\epsilon \\ &\leq \frac{1}{2^{n+1}}\epsilon + \frac{1}{2^n}\epsilon \leq \frac{2}{2^n}\epsilon. \end{aligned} \tag{1.2}$$

By the hypothesis, for every $y_n \in \operatorname{Im}\Phi$, there exists an $x_n \in X$ such that

$$\Phi(x_n) = y_n \qquad \text{and} \qquad \|x_n\| \leq M\,\|y_n\|\,. \tag{1.3}$$

For every $n \in \mathbb{N}$ by (1.3) and (1.2) we have

$$\|x_{n+1}\| \leq M\,\|y_{n+1}\| \leq \frac{M}{2^{n-1}}\epsilon. \tag{1.4}$$

If we set $s_n = x_1 + \cdots + x_n$, then (1.4) shows that $\{s_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an $x \in X$ such that $s_n \to \sum_{n=1}^{\infty} x_n = x$. Since Φ is continuous $\Phi(s_n) \to \Phi(x)$. By (1.1),

$$\Phi(s_n) = y_1 + \cdots + y_n \to y.$$

Hence $\Phi(x) = y$ so that $y \in \operatorname{Im}\Phi$. Thus the range of Φ is closed. □

2. Second cohomology group of discrete group algebras

In this section we will show that the first cohomology group of $\ell^1(G)$ with coefficients in $(\ell^1(S))^{(2n+1)}$ for an arbitrary G-set S is trivial. Next we will show that $\mathcal{H}^2(\ell^1(G), (\ell^1(S))^{(2n+1)})$ is a Banach space for every $n \in \mathbb{N}$.

Remark 2.1. Set $\mathcal{X} = (\ell^1(S))^{(2n)}$. We note that $(\ell^1(S))' = \ell^\infty(S)$ is a commutative unital C^*-algebra. Because the second dual of a commutative unital C^*-algebra is a commutative von Neumann algebra, then $\mathcal{X}' = (\ell^1(S))^{(2n+1)}$ is the underlying space of a commutative von Neumann algebra, and hence it is an L^∞-space. The space $\mathcal{X}'_{\mathbb{R}}$ of real-valued functions in $\mathcal{X}'$ forms a complete lattice in the sense that every nonempty subset of $\mathcal{X}'_{\mathbb{R}}$ that is bounded above has a supremum.

Theorem 2.2. *Let G be a discrete group. Then $\mathcal{H}^1(\ell^1(G), (\ell^1(S))^{(2n+1)}) = 0$ for every G-set S and for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{X} = (\ell^1(S))^{(2n)}$ and $D \in \mathcal{Z}^1(\ell^1(G), \mathcal{X}')$. We show that there exists a function ϕ in $\mathcal{X}'$ that satisfies

$$D(\delta_g)(x) = \phi \cdot \delta_g(x) - \delta_g \cdot \phi(x),$$

where δ_g is the point mass at g. Let $\operatorname{Re} D$ denote the real part of D and let $S = \{\operatorname{Re} D(\delta_h) \cdot \delta_{h^{-1}} : h \in G\}$. Then S is a subset of $\mathcal{X}'_{\mathbb{R}}$ and is bounded above by $\|D\|$ in $\mathcal{X}'_{\mathbb{R}}$. Since $\mathcal{X}'_{\mathbb{R}}$ is a complete lattice, $\phi_r = \sup(S)$ exists in $\mathcal{X}'_{\mathbb{R}}$.

For every $g \in G$ and $x \in \mathcal{X}$ we have

$$\begin{aligned}\delta_g \cdot \phi_r(x) &= \sup_{h\in G}\{\delta_g \cdot \operatorname{Re} D(\delta_h)(\delta_{h^{-1}} \cdot x)\}\\ &= \sup_{h\in G}\{\operatorname{Re} D(\delta_g * \delta_h)(\delta_{h^{-1}} \cdot x) - \operatorname{Re} D(\delta_g)\cdot \delta_h(\delta_{h^{-1}} \cdot x)\}\\ &= \sup_{gh\in G}\left\{\operatorname{Re} D(\delta_{gh})(\delta_{(gh)^{-1}} \cdot (\delta_g \cdot x)) - \operatorname{Re} D(\delta_g)(x)\right\}\\ &= \phi_r(\delta_g \cdot x) - \operatorname{Re} D(\delta_g)(x).\end{aligned}$$

Therefore, $\operatorname{Re} D(\delta_g)(x) = \phi_r \cdot \delta_g(x) - \delta_g \cdot \phi_r(x)$. A similar result holds for the imaginary part of $D(\delta_g)$, and so we see that there exists $\phi \in \mathcal{X}'$ such that

$$D(\delta_g)(x) = \phi \cdot \delta_g(x) - \delta_g \cdot \phi(x).$$

□

Theorem 2.3. *Let G be a discrete group. Then $\mathcal{H}^2(\ell^1(G), (\ell^1(S))^{(2n+1)})$ is a Banach space for every G-set S and for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{X} = (\ell^1(S))^{(2n)}$ and $\psi \in \mathcal{C}^1(\ell^1(G), \mathcal{X}')$. Then for every $g, h \in G$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$ we have

$$|\delta\psi(\delta_g, \delta_h)(x)| = \left|\psi(\delta_g)\cdot\delta_h(x) - \psi(\delta_{(gh)}(x) + \delta_g \cdot \psi(\delta_h)(x)\right| \leq \|\delta\psi\|. \tag{2.1}$$

Since $\left\{\operatorname{Re}\psi(\delta_g)\cdot\delta_g^{-1} : g \in G\right\}$ is bounded above by $\|\psi\|$ in $\mathcal{X}'_{\mathbb{R}}$, then

$$f_r = \sup_{g\in G}\left\{\operatorname{Re}\psi(\delta_g)\cdot\delta_g^{-1}\right\}$$

exists in $\mathcal{X}'_{\mathbb{R}}$. For every $h \in G$ by (2.1) we have

$$\begin{aligned}f_r \cdot \delta_h(x) &= \sup_{g\in G}\left\{\operatorname{Re}\psi(\delta_g)\cdot\delta_{g^{-1}h}(x)\right\}\\ &= \sup_{k\in G}\{\operatorname{Re}\psi(\delta_{hk})\cdot\delta_{k^{-1}}(x)\}\\ &\leq \sup_{k\in G}\{\operatorname{Re}\psi(\delta_h)(x) + \operatorname{Re}\delta_h\cdot\psi(\delta_k)\cdot\delta_{k^{-1}}(x) + \|\delta\psi\|\}\\ &= \operatorname{Re}\psi(\delta_h)(x) + \delta_h \cdot f_r(x) + \|\delta\psi\|,\end{aligned} \tag{2.2}$$

where $g^{-1}h = k^{-1}$. On the other hand,

$$\begin{aligned}f_r \cdot \delta_h(x) &= \sup_{g\in G}\left\{\operatorname{Re}\psi(\delta_g)\cdot\delta_{g^{-1}h}(x)\right\}\\ &\geq \operatorname{Re}\psi(\delta_h)(x) + \delta_h \cdot f_r(x) - \|\delta\psi\|.\end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we have

$$|\delta_h \cdot f_r(x) - f_r \cdot \delta_h(x) + \operatorname{Re}\psi(\delta_h)(x)| \leq \|\delta\psi\|.$$

Similarly, by considering imaginary parts we have

$$|\delta_h \cdot f_i(x) - f_i \cdot \delta_h(x) + \operatorname{Im}\psi(\delta_h)(x)| \leq \|\delta\psi\|.$$

By putting $f = f_r + if_i$ we obtain

$$|\delta_h \cdot f(x) - f \cdot \delta_h(x) + \psi(\delta_h)(x)| \leq 2\|\delta\psi\|.$$

Now let us define

$$\bar{\psi}(\delta_h)(x) = (\delta f)(\delta_h)(x) + \psi(\delta_h)(x).$$

It is clear that $\bar{\psi} \in \mathcal{C}^1(\ell^1(G), \mathcal{X}')$, $\delta\bar{\psi} = \delta\psi$ and $\left|\bar{\psi}(\delta_h)(x)\right| \leq 2\|\delta\psi\|\,\|x\|$ for every $h \in G$ and $x \in \mathcal{X}$. Thus $\left\|\bar{\psi}\right\| \leq 2\|\delta\psi\|$ and this completes the proof. □

3. Second cohomology group of locally compact group algebras

Now we state the final result of this paper. We show $\mathcal{H}^2(L^1(G),(L^1(G))^{(2n+1)})$ is a Banach space for every locally compact group G. Note that $L^\infty(G)=(L^1(G))'$ is also an $M(G)$-module, where for every $f\in L^\infty(G), a\in L^1(G)$ and $\mu\in M(G)$ the module actions are defined by

$$(f\mu)(a)=f(\mu * a) \quad \text{and} \quad (\mu f)(a)=f(a*\mu).$$

Thus as noted in the Introduction the higher duals $(L^1(G))^{(n)}$ are Banach $M(G)$-bimodules.

Proposition 3.1. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity, and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Let $\psi\in\mathcal{C}^1(\mathcal{A},\mathcal{X}')$ such that $|\psi(a)(b\cdot x\cdot c)|\le \|\delta\psi\|$ for every $x\in\mathcal{X}$ with $\|x\|\le 1$ and $a,b,c\in\mathcal{A}$ with $\|a\|\le 1$, $\|b\|\le 1$ and $\|c\|\le 1$. Then there exists $\widehat{\psi}\in\mathcal{X}'$ such that*

$$\left|\psi(a)(x)-\delta\widehat{\psi}(a)(x)\right|<5\left\|\delta\psi\right\|.$$

Proof. For every $x\in\mathcal{X}$ with $\|x\|\le 1$ and $a,b\in\mathcal{A}$ with $\|a\|\le 1$ and $\|b\|\le 1$, we have

$$|\delta\psi(a,b)(x)|=|a\cdot\psi(b)(x)-\psi(ab)(x)+\psi(a)\cdot b(x)|\le\left\|\delta\psi\right\|. \tag{3.1}$$

Choose a bounded approximate identity $\{e_\alpha\}$ in $\mathcal{A}$ such that the iterated weak*-limit $\widehat{\psi}=\lim_\alpha\lim_\beta(e_\alpha\cdot\psi(e_\beta)-\psi(e_\beta)\cdot e_\alpha)$ exists. Then for every $x\in\mathcal{X}$ and $a\in\mathcal{A}$, by (3.1) we have

$$\begin{aligned}
\psi(a)(x) &= \lim_\alpha\lim_\beta\left[\psi(e_\alpha a e_\beta)(x)\right]\\
&\le \lim_\alpha\lim_\beta\left[\psi(e_\alpha)\cdot ae_\beta(x)+e_\alpha\cdot\psi(ae_\beta)(x)\right]+\left\|\delta\psi\right\|\\
&\le \lim_\alpha\lim_\beta\left[\psi(e_\alpha)\cdot ae_\beta(x)+e_\alpha a\cdot\psi(e_\beta)(x)+e_\alpha\cdot\psi(a)\cdot e_\beta(x)\right]+2\left\|\delta\psi\right\|\\
&\le \lim_\beta\left[\psi(e_\beta)\cdot a+a\cdot\psi(e_\beta)\right](x)+3\left\|\delta\psi\right\|\\
&\le \lim_\alpha\lim_\beta a\cdot(e_\alpha\cdot\psi(e_\beta)-\psi(e_\beta)\cdot e_\alpha)(x)\\
&\quad -\lim_\alpha\lim_\beta(e_\alpha\cdot\psi(e_\beta)-\psi(e_\beta)\cdot e_\alpha)\cdot a(x)+5\left\|\delta\psi\right\|\\
&= a\cdot\widehat{\psi}(x)-\widehat{\psi}\cdot a(x)+5\left\|\delta\psi\right\|,
\end{aligned}$$

where we have used several times the fact that $|\psi(a)(b\cdot x\cdot c)|\le\|\delta\psi\|$. Then $\psi(a)(x)-\delta\widehat{\psi}(a)(x)\le 5\left\|\delta\psi\right\|$. On the other hand,

$$\begin{aligned}
\psi(a)(x) &= \lim_\alpha\lim_\beta\left[\psi(e_\alpha a e_\beta)(x)\right]\\
&\ge \lim_\alpha\lim_\beta\left[\psi(e_\alpha)\cdot ae_\beta(x)+e_\alpha\cdot\psi(ae_\beta)(x)\right]-\left\|\delta\psi\right\|\\
&\ge \lim_\alpha\lim_\beta\left[\psi(e_\alpha)\cdot ae_\beta(x)+e_\alpha a\cdot\psi(e_\beta)(x)\right]+3\left\|\delta\psi\right\|\\
&\ge a\cdot\widehat{\psi}(x)-\widehat{\psi}\cdot a(x)-5\left\|\delta\psi\right\|.
\end{aligned}$$

Then $\psi(a)(x)-\delta\widehat{\psi}(a)(x)\ge -5\left\|\delta\psi\right\|$. Thus

$$\left|\psi(a)(x)-\delta\widehat{\psi}(a)(x)\right|\le 5\left\|\delta\psi\right\|.$$

∎

For the rest of this section we set $\mathcal{X} = (L^1(G))^{(2n)}$. Observe that, as in Remark 2.1, since the dual space $\mathcal{X}'$ is an L^∞-space, then the space $\mathcal{X}'_{\mathbb{R}}$ of real-valued functions in $\mathcal{X}'$ forms a complete lattice.

Proposition 3.2. *Let $\psi \in \mathcal{C}^1(L^1(G), \mathcal{X}')$. Then $\tilde{\psi} \in \mathcal{C}^1(M(G), \mathcal{X}')$ with*

(i) *$\tilde{\psi}|_{L^1(G)} = \psi$ and $\delta\tilde{\psi}|_{L^1(G)\times L^1(G)} = \delta\psi$.*

(ii) *Let μ and ν be in $M(G)$ with $\|\mu\|, \|\nu\| \leq 1$, and let x be in $\mathcal{X}$ with $\|x\| \leq 1$. If $\{\mu_\alpha\}$ is a net in $M(G)$ with $\|\mu_\alpha\| \leq 1$ such that* so-lim$\mu_\alpha = \mu$*, then there exists $\widehat{\psi} \in \mathcal{X}'$ such that*

$$\left|(\overline{\lim}_\alpha \operatorname{Re}\tilde{\psi}(\mu_\alpha)(x) + i\,\overline{\lim}_\alpha \operatorname{Im}\tilde{\psi}(\mu_\alpha)(x)) - \tilde{\psi}(\mu)(x) - \delta\widehat{\psi}(\mu)(x)\right| \leq 15\left\|\delta\tilde{\psi}\right\|.$$

Proof. (i) We follow the proof of [9, Lemma 1.10] for this particular case. Let $\mu \in M(G)$ and let $\{e_\alpha\}$ be a bounded approximate identity of norm one for $L^1(G)$. Defining

$$\psi_\alpha(\mu) = \psi(\mu * e_\alpha),$$

we see that $\{\psi_\alpha\}$ is a bounded net in $\mathcal{C}^1(M(G), \mathcal{X}')$ and so has a subnet $\{\psi_\beta\}$ convergent to a limit $\tilde{\psi}$ in the weak*-topology induced by identifying $\mathcal{C}^1(M(G), \mathcal{X}')$ with $\mathcal{C}_1(M(G), \mathcal{X})'$. Thus

$$\lim_\beta \psi(\mu * e_\beta)(x) = \tilde{\psi}(\mu)(x)$$

for all $\mu \in M(G)$, $x \in \mathcal{X}$. Since for all $a \in L^1(G)$, $\psi(a * e_\beta) \to \psi(a)$ in norm, then $\tilde{\psi}|_{L^1(G)} = \psi$. Also, $\delta\tilde{\psi}|_{L^1(G)\times L^1(G)} = \delta\psi$.

To prove (ii) let us consider $\mu, \nu \in M(G)$ with $\|\mu\|, \|\nu\| \leq 1$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$. Then

$$\left|\delta\tilde{\psi}(\mu,\nu)(x)\right| = \left|\mu\cdot\tilde{\psi}(\nu)(x) - \tilde{\psi}(\mu*\nu)(x) + \tilde{\psi}(\mu)\cdot\nu(x)\right| \leq \left\|\delta\tilde{\psi}\right\|. \tag{3.2}$$

For $a, b \in L^1(G)$ with $\|a\|_1 \leq 1, \|b\|_1 \leq 1$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$ by (3.2) we have

$$\begin{aligned} -\operatorname{Re}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b) &= -\operatorname{Re}\tilde{\psi}(\mu_\alpha)\cdot a(x\cdot b) \\ &\leq \operatorname{Re}\mu_\alpha\cdot\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu_\alpha * a)(x\cdot b) + \left\|\delta\tilde{\psi}\right\| \end{aligned}$$

and so

$$\begin{aligned} -\overline{\lim}\operatorname{Re}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b) &\leq \underline{\lim}\left\{\operatorname{Re}\mu_\alpha\cdot\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu_\alpha * a)(x\cdot b) + \left\|\delta\tilde{\psi}\right\|\right\} \\ &= \operatorname{Re}\mu\cdot\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu * a)(x\cdot b) + \left\|\delta\tilde{\psi}\right\|. \end{aligned}$$

On the other hand,

$$-\overline{\lim}\operatorname{Re}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b) \geq \operatorname{Re}\mu\cdot\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu * a)(x\cdot b) - \left\|\delta\tilde{\psi}\right\|.$$

Hence

$$\left|\mu\cdot\operatorname{Re}\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu * a)(x\cdot b) + \overline{\lim}\operatorname{Re}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b)\right| \leq \left\|\delta\tilde{\psi}\right\|.$$

Similarly for imaginary parts we have

$$\left|\mu\cdot\operatorname{Im}\psi(a)(x\cdot b) - \operatorname{Im}\psi(\mu * a)(x\cdot b) + \overline{\lim}\operatorname{Im}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b)\right| \leq \left\|\delta\tilde{\psi}\right\|.$$

Therefore,

$$\begin{aligned}\Big|\mu\cdot\psi(a)(x\cdot b)-\psi(\mu*a)(x\cdot b)\\ +\left(\overline{\lim}\,\mathrm{Re}\,\tilde\psi(\mu_\alpha)-i\,\overline{\lim}\,\mathrm{Im}\,\tilde\psi(\mu_\alpha)\right)(a\cdot x\cdot b)\Big|\le 2\left\|\delta\tilde\psi\right\|.\end{aligned}\tag{3.3}$$

But from (3.2) we also have

$$\left|\mu\cdot\psi(a)(x\cdot b)-\psi(\mu*a)(x\cdot b)+\tilde\psi(\mu)(a\cdot x\cdot b)\right|\le\left\|\delta\tilde\psi\right\|.\tag{3.4}$$

Hence (3.3) and (3.4) imply that

$$\left|\left(\overline{\lim}\,\mathrm{Re}\,\tilde\psi(\mu_\alpha)-i\,\overline{\lim}\,\mathrm{Im}\,\tilde\psi(\mu_\alpha)\right)(a\cdot x\cdot b)-\tilde\psi(\mu)(a\cdot x\cdot b)\right|\le 3\left\|\delta\tilde\psi\right\|.$$

By Proposition 3.1 there exists $\widehat\psi\in\mathcal{X}'$ such that

$$\left|\left(\overline{\lim}\,\mathrm{Re}\,\tilde\psi(\mu_\alpha)-i\,\overline{\lim}\,\mathrm{Im}\,\tilde\psi(\mu_\alpha)\right)(x)-\tilde\psi(\mu)(x)-\delta\widehat\psi(\mu)(x)\right|\le 15\left\|\delta\tilde\psi\right\|.$$

□

Theorem 3.3. *Let G be a locally compact group. Then $\mathcal{H}^2(L^1(G),\mathcal{X}')$ is a Banach space, where $\mathcal{X}=(L^1(G))^{(2n)}$.*

Proof. Let $\phi\in\mathcal{C}^1(L^1(G),\mathcal{X}')$ and consider $\tilde\phi\in\mathcal{C}^1(M(G),\mathcal{X}')$ as in Proposition 3.2. Set

$$S=\left\{\mathrm{Re}\,\delta_{g^{-1}}\tilde\phi(\delta_g):g\in G\right\}.$$

Since S is bounded above by $\left\|\tilde\phi\right\|$ in $\mathcal{X}'_{\mathbb{R}}$, then $\psi_r=\sup_{g\in G}S$ exists in $\mathcal{X}'_{\mathbb{R}}$. For every $h\in G$ and $x\in\mathcal{X}$ with $\|x\|\le 1$ by (3.2) we have

$$\begin{aligned}\delta_h\cdot\psi_r(x)&=\sup_{k\in G}\left\{\mathrm{Re}(\delta_h*\delta_{k^{-1}})\cdot\tilde\phi(\delta_k)(x)\right\}=\sup_{g\in G}\left\{\mathrm{Re}\,\delta_{g^{-1}}\cdot\tilde\phi(\delta_g*\delta_h)(x)\right\}\\&\le\sup_{g\in G}\left\{\mathrm{Re}(\delta_{g^{-1}}*\delta_g)\cdot\tilde\phi(\delta_h)(x)+\mathrm{Re}\,\delta_{g^{-1}}\cdot\tilde\phi(\delta_g)\cdot\delta_h(x)+\left\|\delta\tilde\phi\right\|\right\}\\&=\mathrm{Re}\,\tilde\phi(\delta_h)(x)+\psi_r\cdot\delta_h(x)+\left\|\delta\tilde\phi\right\|,\end{aligned}$$

where $hk^{-1}=g^{-1}$. On the other hand,

$$\delta_h\cdot\psi_r(x)\ge\mathrm{Re}\,\tilde\phi(\delta_h)(x)+\psi_r\cdot\delta_h(x)-\left\|\delta\tilde\phi\right\|.$$

Therefore,

$$\left|\delta_h\cdot\psi_r(x)-\psi_r\cdot\delta_h(x)-\mathrm{Re}\,\tilde\phi(\delta_h)(x)\right|\le\left\|\delta\tilde\phi\right\|.\tag{3.5}$$

Similarly, by considering imaginary parts we obtain ψ_i such that

$$\left|\delta_h\cdot\psi_i(x)-\psi_i\cdot\delta_h(x)-\mathrm{Im}\,\tilde\phi(\delta_h)(x)\right|\le\left\|\delta\tilde\phi\right\|.\tag{3.6}$$

Since every measure μ in $M(G)$ is the so-lim of a net $\{\mu_\alpha\}$ with $\|\mu_\alpha\|\le 1$, where every μ_α is a linear combination of point masses [4, 1.1.3], then by (3.5), (3.6) and Proposition 3.1 we have

$$\left|\mu\cdot\psi(x)-\psi\cdot\mu(x)-\left[\overline{\lim}\,\mathrm{Re}\,\tilde\phi(\mu_\alpha)(x)+i\,\overline{\lim}\,\mathrm{Im}\,\tilde\phi(\mu_\alpha)(x)\right]\right|\le 2\left\|\delta\tilde\phi\right\|$$

where $\psi = \psi_r + i\,\psi_i$. Now by Proposition 3.2 (ii) there exists $\widehat{\phi} \in \mathcal{X}'$ such that

$$\begin{aligned}&\left|\mu \cdot \psi(x) - \psi \cdot \mu(x) - \delta\widehat{\phi}(\mu)(x) - \tilde{\phi}(\mu)(x)\right| \\ &\qquad \leq \left|\mu \cdot \psi(x) - \psi \cdot \mu(x) - \left[\overline{\lim}\,\mathrm{Re}\,\tilde{\phi}(\mu_\alpha)(x) + i\,\overline{\lim}\,\mathrm{Im}\,\tilde{\phi}(\mu_\alpha)(x)\right]\right| \\ &\qquad\quad + \left|\overline{\lim}\,\mathrm{Re}\,\tilde{\phi}(\mu_\alpha)(x) + i\,\overline{\lim}\,\mathrm{Im}\,\tilde{\phi}(\mu_\alpha)(x) - \delta\widehat{\phi}(\mu)(x) - \tilde{\phi}(\mu)(x)\right| \\ &\qquad \leq 17\left\|\delta\tilde{\phi}\right\|.\end{aligned}$$

Define

$$\bar{\psi}(\mu)(x) = -\delta\psi(\mu)(x) - \delta\widehat{\phi}(\mu)(x) + \tilde{\phi}(\mu)(x).$$

Then $\delta\bar{\psi} = \delta\tilde{\phi}$ and $\left|\bar{\psi}(\mu)(x)\right| \leq 17\left\|\delta\tilde{\phi}\right\|$ for every $\mu \in M(G)$ with $\|\mu\| \leq 1$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$. So $\left\|\bar{\psi}\right\| \leq 17\left\|\delta\tilde{\phi}\right\|$ and by Proposition 3.2 (i), this completes the proof. □

References

[1] W. G. Bade, P. C. Curtis, Jr. and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. **55** (1987) 359-377. MR **88f:**46098

[2] H. G. Dales, F. Ghahramani, and N. Grønbæk, *Derivations into iterated duals of Banach algebras*, Studia Math. **128** (1998), 19-54. MR **99g:**46064

[3] M. Despic and F. Ghahramani, *Weak amenability of group algebras of locally compact groups*, Canad. Math. Bull. **37** (1994), 165-167. MR **95c:**43003

[4] F. P. Greenleaf, *Norm decreasing homomorphisms of group algebras*, Pacific J. Math. **15** (1965), 1187-1219. MR **33:**3117

[5] R. I. Grigorchuk, *Some results on bounded cohomology*, London Math. Soc. Lecture Note Series **204**, Cambridge Univ. Press, Cambridge, 1995, pp. 111-163. MR **96j:**20073

[6] N. Grønbæk, *Some concepts from group cohomology in the Banach algebra context.* Proc. Banach Algebras '97 Conference, Blaubeuren, 1998, pp. 205-222. MR **2000d:**46087

[7] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Mathematics and its Applications **41**, Kluwer Academic Publishers, Dordrecht, 1989. MR **92d:**46178

[8] N. V. Ivanov, *Second bounded cohomology group*, J. Soviet Math., **167** (1988), 117-120. MR **90a:**55015

[9] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972), 96 pp. MR **51:**11130

[10] B. E. Johnson, *Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$*, Lecture Notes in Math. **1359**, Springer-Verlag, Berlin, 1988, pp. 191-198. MR **90a:**46122

[11] B. E. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc. **23** (1991), 281-284. MR **92k:**43004

[12] S. Matsumoto and S. Morita, *Bounded cohomology of certain groups of homeomorphisms*, Proc. Amer. Math. Soc. **94** (1985), 539-544. MR **87e:**55006

[13] A. Pourabbas and M. C. White, *Second Bounded Group Cohomology of Group Algebras*, to appear.

[14] H. H. Schaefer, *Banach lattices and positive operators*, Die Grundlehren der mathematischen Wissenschaften, Band 215, Springer-Verlag, Berlin, 1974. MR **54:**11023

[15] A. M. Sinclair and R. R. Smith, *Hochschild cohomology of von Neumann algebras*, London Math. Soc. Lecture Note Series **203** (1995), 196 pp. MR **96d:**46094

Faculty of Mathematics and Computer Science, Amirkabir University, 424 Hafez Avenue, Tehran 15914, Iran

E-mail address: `arpabbas@aut.ac.ir`