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ON A SPECTRAL PROPERTY OF JACOBI MATRICES

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ABSTRACT. Let J be a Jacobi matrix with elements b_k on the main diagonal and elements a_k on the auxiliary ones. We suppose that J is a compact perturbation of the free Jacobi matrix. In this case the essential spectrum of J coincides with $[-2,2]$, and its discrete spectrum is a union of two sequences $\{x_j^\pm\}, x_j^+ > 2, x_j^- < -2$, tending to ± 2. We denote sequences $\{a_{k+1}-a_k\}$ and $\{a_{k+1}+a_{k-1}-2a_k\}$ by ∂a and $\partial^2 a$, respectively.

The main result of the note is the following theorem.

Theorem. Let J be a Jacobi matrix described above and σ be its spectral measure. Then $a-1, b \in l^4,\ \partial^2 a, \partial^2 b \in l^2$ if and only if

$$\text{i)} \int_{-2}^{2} \log \sigma'(x)(4-x^2)^{5/2}\,dx > -\infty, \qquad \text{ii)} \sum_j (x_j^\pm \mp 2)^{7/2} < \infty.$$

INTRODUCTION

It has been observed recently [4] that the sum rules [1], [2] for a Jacobi matrix J lead to direct relations between elements of J and its spectral properties. However, the sum rules become considerably more complicated when their order increases. In this note, we carry out a complete analysis of a sum rule of order six.

Let us consider a Jacobi matrix

$$J = J(a,b) = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix},$$

defined by two sequences $a = \{a_k\},\ a_k > 0$, and $b = \{b_k\},\ b_k \in \mathbb{R}$. We suppose that J is a compact perturbation of the so-called free Jacobi matrix J_0,

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix}.$$

The (scalar) spectral measure σ of J is defined by the relation

$$((J-z)^{-1}e_0, e_0) = \int_{\mathbb{R}} \frac{d\sigma(x)}{x-z},$$

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with $z \in \mathbb{C}\backslash\mathbb{R}$. It is well known that in this case the continuous part σ_c of the measure is supported by $[-2,2]$, and its discrete part σ_d lives on the union of two sequences $\{x_j^\pm\}$ such that $x_j^+ > 2,\ x_j^- < -2$, and $x_j^\pm \to \pm 2$, as $j \to \infty$.

Let us put $\partial a = \{a_{k+1} - a_k\}$ and $\partial^2 a = \{a_{k+1} + a_{k-1} - 2a_k\}$, and denote by 1 the sequence consisting of ones. The following theorem holds.

Theorem 0.1. *Let $J = J(a,b)$ and σ be its spectral measure. Then*

$$a - 1, b \in l^4,\ \partial^2 a, \partial^2 b \in l^2 \tag{0.1}$$

if and only if

$$\text{i) } \int_{-2}^{2} \log \sigma'(x)(4 - x^2)^{5/2}\, dx > -\infty, \quad \text{ii) } \sum_j (x_j^\pm \mp 2)^{7/2} < \infty. \tag{0.2}$$

The theorem has an immediate corollary.

Corollary 0.2. *Let $J = J(a,b)$, and let σ be its spectral measure. If $a - 1, b \in l^4,\ \partial a, \partial b \in l^2$, then relations* (0.2) *hold.*

In this direction we also have (see [7]) that

$$\text{i) } \int_{-2}^{2} \log \sigma'(x)(4 - x^2)^{3/2}\, dx > -\infty, \quad \text{ii) } \sum_j (x_j^\pm \mp 2)^{5/2} < \infty,$$

when $a - 1, b \in l^3,\ \partial a, \partial b \in l^2$. Together with Corollary 0.2, this observation leads to a natural conjecture. The conjecture would yield a right discrete counterpart of results, obtained in [6] for Schrödinger operators on the half-line.

Conjecture 0.3. For an integer $k \geq 1$, conditions $a-1, b \in l^{k+1},\ \partial a, \partial b \in l^2$ imply

$$\text{i) } \int_{-2}^{2} \log \sigma'(x)(4 - x^2)^{k-1/2}\, dx > -\infty, \quad \text{ii) } \sum_j (x_j^\pm \mp 2)^{k+1/2} < \infty.$$

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1. Proof of Theorem 0.1

It is convenient to map the domain $\mathbb{C}\backslash[-2,2]$ onto the unit disk with the help of the transformation $\zeta(z) = \frac{1}{2}(z - \sqrt{z^2 - 4})$. We construct a measure μ on the unit circle by letting $d\mu(\theta) = d\sigma(x)/(2\sin\theta)$, where $x = z(e^{i\theta})$ and $z(\zeta) = \zeta + 1/\zeta$. The sequences $\{x_j^\pm\}$ are mapped to points $\{\zeta_j\}$, lying on the real diameter of the unit disk. In these new terms Theorem 0.1 says that $a - 1, b \in l^4,\ \partial^2 a, \partial^2 b \in l^2$ if and only if

$$\text{i) } \int_0^{2\pi} \log \mu'(\theta) \sin^6\theta\, d\theta > -\infty, \quad \text{ii) } \sum_j (1 - |\zeta_j|)^7 < \infty. \tag{1.1}$$

Let us assume first that $\operatorname{rank}(J - J_0) < \infty$. For $j \geq 1$, we define Chebyshev polynomials by recurrence relations $T_{j+1}(z) = zT_j(z) - T_{j-1}(z)$, and we take

$T_0(z) = 2, T_1(z) = z$. We have the so-called sum rules [1], [2], [4], [8], [9]

$$0)\qquad \frac{1}{4\pi}\int_0^{2\pi} \log\frac{2\sin\theta}{\mu'(\theta)}\,d\theta = \sum_j \log\beta_j - \sum_j \log a_j,$$

$$n)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log\frac{2\sin\theta}{\mu'(\theta)}\cos n\theta\,d\theta = \frac{1}{n}\sum_j(\beta_j^n - \beta_j^{-n}) - \frac{1}{n}\mathrm{tr}\,\{T_n(J) - T_n(J_0)\},$$

where we let $\beta_j = 1/|\zeta_j|$. Taking into account that $\sin^6\theta = (1/2^5)(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{2\sin\theta}{\mu'(\theta)}\sin^6\theta\,d\theta + \frac{1}{6}\sum_j F(\beta_j) = \frac{1}{6}\,\mathrm{tr}\,\{G(J) - G(J_0) - 120\log A\},$$

where $A = \mathrm{diag}\,\{a_k\}$, and

$$F(\beta) = \frac{1}{2^5}\left((\beta^6 - \beta^{-6}) - 9(\beta^4 - \beta^{-4}) + 45(\beta^2 - \beta^{-2}) - 120\log\beta\right),$$

$$G(J) = \frac{1}{2^5}\left(T_6(J) - 9T_4(J) + 45T_2(J)\right) = \frac{1}{2^5}(J^6 - 15J^4 + 90J^2 - 110). \tag{1.2}$$

Notice that $F(\beta) = C_0(\beta-1)^7 + O((\beta-1)^8)$, where $\beta \geq 1$ and C_0 is a positive constant.

Let

$$\Psi(a,b) = \Psi(J(a,b)) = \mathrm{tr}\,\{G(J) - G(J_0) - 120\log A\}. \tag{1.3}$$

The following lemma is a cornerstone of the proof of the theorem.

Main Lemma. *Let $a = \{a_k\}$, $b = \{b_k\}$ and $a_k \neq 1, b_k \neq 0$, for finitely many indices only. Also let norms $||a-1||_\infty$, $||b||_\infty$ be small enough. Then*

$$\Psi(a,b) \asymp (||a-1||_4^4 + ||b||_4^4 + ||\partial^2 a||_2^2 + ||\partial^2 b||_2^2). \tag{1.4}$$

In particular, $\Psi(a,b) \geq 0$ for these a and b.

The norms $||.||_p$ refer to the standard l^p-space norms. The sign "$\asymp$" means a two-sided estimate with positive constants depending on $||a-1||_\infty$, $||b||_\infty$. The lemma will be proved in the next section.

With the exception of this lemma, the proof of Theorem 0.1 follows a well-known scheme [4], [5]. We give only its sketch.

Proof of Theorem 0.1. Since the absolutely continuous spectrum of $J = J(a,b)$ is the interval $[-2,2]$, we have (see [3]) that $a_k \to 1$ and $b_k \to 0$. Consequently, discarding the first N_0 rows and columns of J (see [8], Section 3), we may make norms $||a-1||_\infty$ and $||b||_\infty$ small enough to apply the Main Lemma.

First, we show that condition (0.1) implies (1.1), which is equivalent to showing (0.2). Denote

$$\Phi(J) = \Phi(\mu) = \Phi_1(\mu) + \Phi_2(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{2\sin\theta}{\mu'(\theta)}\sin^6\theta\,d\theta + \frac{1}{6}\sum_j F\left(\frac{1}{|\zeta_j|}\right).$$

Let $a_N = \{(a_N)_k\}$ and $a'_N = \{(a'_N)_k\}$, where

$$(a_N)_k = \begin{cases} a_k, & k \leq N, \\ 1, & k > N, \end{cases} \qquad (a'_N)_k = \begin{cases} 1, & k \leq N, \\ a_k, & k > N. \end{cases}$$

We also consider b_N, b'_N, constructed in the same way (of course, with 1's replaced by 0's). Let $J_N = J(a_N, b_N)$ and μ_N be the corresponding measure on the unit circle. Obviously, we have that $a'_N - 1, b'_N \to 0$ and $\partial^2 a'_N, \partial^2 b'_N \to 0$ in corresponding norms. Since

$$\Psi(a,b) = \sum_k \psi(a_k, a_{k+1}, a_{k+2}; b_k, b_{k+1}, b_{k+2})$$

for a function ψ (see Section 2), we have that

$$|\Psi(J) - \Psi(J_N)| \leq \Psi(a'_N, b'_N) \leq C_1(||a'_N - 1||_4^4 + ||b'_N||_4^4 + ||\partial^2 a'_N||_2^2 + ||\partial^2 b'_N||_2^2) \to 0,$$

or $\Psi(J_N) \to \Psi(J)$. On the other hand, $(J_N - z(\zeta))^{-1} \to (J - z(\zeta))^{-1}$, for ζ inside the unit disk and, consequently, $\mu_N \to \mu$ weakly. Using [4], Corollary 5.3 and Theorem 6.2, we get

$$\Phi_1(\mu) \leq \liminf_N \Phi_1(\mu_N)$$

and

$$\lim_{N\to\infty} \Phi_2(\mu_N) = \Phi_2(\mu).$$

Summing up, we obtain

$$\Phi(\mu) \leq \limsup_N \Phi(\mu_N) = \limsup_N \Psi(J_N) = \lim_{N\to\infty} \Psi(J_N) = \Psi(J).$$

We now show the converse, i.e., condition (1.1) yields relations (0.1). Recall ([4], Proposition 8.4) that if $\mu = \mu(J)$ satisfies inequality $\mu \geq \delta m$, then

$$\Psi(J) \leq \Phi(J),$$

where $\delta > 0$ and m is the Lebesgue measure on the unit circle. We take a J with the property $\Psi(J) < \infty$, and we put $\mu_\delta = (1-\delta)\mu + \delta m$ for $\mu = \mu(J)$ and $\delta \in (0,1)$. The measure μ_δ defines a Jacobi matrix $J_\delta = J(a_\delta, b_\delta)$. Theorem 8.1 of [4] shows that $\Phi(J) = \lim_{\delta\to 0} \Phi(J_\delta)$. On the other hand (see [4], Proposition 8.4 and Theorem 8.1),

$$\Psi(J) \leq \liminf_\delta \Psi(J_\delta).$$

Hence,

$$0 \leq \Psi(J) \leq \liminf_\delta \Psi(J_\delta) \leq \liminf_\delta \Phi(J_\delta) = \lim_{\delta\to 0} \Phi(J_\delta) = \Phi(J).$$

The theorem is proved. □

2. Proof of the main lemma

We make a few simplifications before going into the proof. First, we drop the term $\operatorname{tr} G(J_0)$ in (1.3), since it is finite. Second, we discard factors $1/2^5$ (see (1.2)). Having two sequences $a = \{a_k\}$ and $b = \{b_k\}$, we denote the sequence $\{a_k b_k\}$ by ab. We also write $a^\pm$ for the sequence $\{a_{k\pm 1}\}$.

The starting point of the proof is the computation of $\operatorname{tr} J^n$. According to formulas from [9], Section 6.1, we have $\operatorname{tr} J^j = \sum_k (g_j)_k$, where sequences g_j are constructed as

(2.1)
$$\begin{aligned} g_{j+1} &= h_j + h_j^- + b g_j, \\ h_{j+1} &= a^2 \sum_{l=0}^{j} g_{j-l} g_l^- - \sum_{l=0}^{j} h_{j-l} h_l, \end{aligned}$$

with $g_0 = 1, h_0 = 0$.

We give the proof of the lemma in cases when $J = J(a,b)$ with $b = 0$, and $J = J(a,b)$ with $a = 1$. Since the proof in the general case follows along the same lines and is long and tedious, we omit it.

Let us consider the case when $b = 0$.

Lemma 2.1. *Let $a = \{a_k\}$ and $a_k \neq 1$ for finitely many indices. Then*

$$\begin{aligned}\Psi(a) = \Psi(J(a,0)) \quad &= \quad \textstyle\sum_k \big\{ -9\lambda_k^3 + 3\lambda_k^2\lambda_{k+1} + 3\lambda\lambda_{k+1}^2 + 3\lambda_{k-1}\lambda_k\lambda_{k+1} \\ &+ \quad 9\lambda_k^2 - 12\lambda\lambda_{k+1} + 3\lambda_{k-1}\lambda_{k+1} + \tfrac{15}{2}\lambda_k^4 + O(|\lambda_k|^5)\big\},\end{aligned}$$

where $\lambda = a^2 - 1$.

Proof. A straightforward computation using (2.1) yields $g_1 = g_3 = g_5 = 0$ and

$$\begin{aligned} g_2 &= a^2 + a^{-2}, \\ g_4 &= a^{-2}(a^{--2} + a^{-2} + a^2) + a^2(a^{-2} + a^2 + a^{+2}), \\ g_6 &= a^2\big(a^{-2}(a^{--2} + \ldots + a^{+2}) + a^2(a^{-2} + \ldots + a^{+2}) \\ &+ a^{+2}(a^2 + \ldots + a^{++2})\big) \\ &+ a^{-2}\big(a^{--2}(a^{---2} \\ &+ \ldots + a^2) + a^{-2}(a^{--2} + \ldots + a^2) + a^2(a^{-2} + \ldots + a^{+2})\big). \end{aligned}$$

We also have $\operatorname{tr}\log A = \frac{1}{2}\sum_k \log(1+\lambda_k)$. Rewriting in terms of $\lambda = a^2 - 1$ and recalling that $\operatorname{tr} J^j = \sum_k (g_j)_k$, we get the conclusion of the lemma. □

In particular, we see that

$$\Psi(a) = \sum_k \left\{\psi_1(\lambda_k, \lambda_{k+1}, \lambda_{k+2}) + O(|\lambda_k|^5)\right\},$$

where

$$\begin{aligned}\psi_1(x,y,z) \quad &= \quad \frac{1}{2}(x^3 - 20y^3 + z^3 + 6xy^2 + 6y^2z + 6xyz) \\ &+ \quad \frac{3}{2}(x^2 + 4y^2 + z^2 - 4xy - 4yz + 2xz) + \frac{5}{4}(x^4 + 4y^4 + z^4).\end{aligned}$$

Note also that $\lambda_k = 2(a_k - 1) + O((a_k - 1)^2)$.

Lemma 2.2. *We have*

$$\psi_1(x,y,z) \asymp (x^4 + y^4 + z^4 + (x + z - 2y)^2).$$

Proof. We notice that

$$\begin{aligned}\psi_1(x,y,z) \quad &= \quad \frac{1}{2}\big((x+z-2y)(x^2 + 10y^2 + z^2 + 2xy + 2yz - zx) \\ &+ \quad 3(x+z-2y)^2 + \frac{5}{2}(x^4 + 4y^4 + z^4)\big),\end{aligned} \tag{2.2}$$

and the bound from above follows from the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ for $a, b \geq 0$. To get the bound from below, we estimate the first term in (2.2) as

$$|(x+z-2y)(x^2+10y^2+z^2+2xy+2yz-zx)| \leq \frac{1}{2}(x+z-2y)^2 + \frac{3}{2}(x^4+4y^4+z^4),$$

and, consequently,

$$\psi_1(x,y,z) \geq \frac{1}{2}\left(\frac{5}{2}(x+z-2y)^2 + (x^4 + 4y^4 + z^4)\right).$$

The lemma is proved. □

Now, we turn to the second case $J = J(a, b)$ with $a = 1$.

Lemma 2.3. *Let* $b = \{b_k\}$ *and* $b_k \neq 0$ *for finitely many indices. Then*

$$\begin{aligned}\Psi(b) = \Psi(1, b) \;&=\; \textstyle\sum_k \big\{b_k^6 + (-3b_k^4 + 6b_k b_{k+1}^3 + 6b_k^3 b_{k+1} + 6b_k^2 b_{k+1}^2) \\ &+\; (18b_k^2 - 24b_k b_{k+1} + 6b_{k-1}b_{k+1})\big\}.\end{aligned}$$

Proof. The argument is exactly as in Lemma 2.1. We only quote expressions for g_5 and g_6:

$$\begin{aligned} g_5 &= b^5 + (b^{-3} + 8b^3 + b^{+3} + 2b^{-2}b + 3b^-b^2 + 3b^2b^+ + 2bb^{+2}) \\ &+ (b^{--} + 6b^- + 16b + 6b^+ + b^{++}), \end{aligned}$$

$$\begin{aligned} g_6 &= b^6 + (b^{-4} + 10b^4 + b^{+4} \\ &+ 2b^{-3}b + 3b^{-2}b^2 + 4b^-b^3 + 4b^3b^+ + 3b^2b^{+2} + 2bb^{+3}) \\ &+ (b^{--2} + 8b^{-2} + 30b^2 + 8b^{+2} + b^{++2} + 2b^{--}b^- + 2b^{--}b \\ &+ 16b^-b + 16bb^+ + 2bb^{++} + 2b^+b^{++} + 2b^+b^-) + 20. \end{aligned}$$

□

Hence, we obtain

$$\Psi(b) = \sum_k \{\psi_2(b_k, b_{k+1}, b_{k+2}) + O(b_k^6)\},$$

where

$$\psi_2(x, y, z) = -3y^4 + (6xy^3 + 6y^3z + 3y^2x^2 + 3y^2z^2) + 3(x + z - 2y)^2.$$

Lemma 2.4. *We have*

$$\psi_2(x, y, z) \asymp (y^4 + (x + z - 2y)^2).$$

Proof. One more time, the bound from above follows at once from inequality $ab \leq 1/(p\,a^p) + 1/(q\,b^q)$, where $a, b \geq 0$, and $1/p + 1/q = 1$.

Furthermore, we see that

$$\psi_2(x, y, z) = -3y^4 + y^2\left(\tfrac{3}{2}(x - z)^2 + 3(x + y + z)^2 - \tfrac{1}{2}(x + z - 2y)^2\right) + 3(x + z - 2y)^2.$$

Estimating moduli of negative terms as in Lemma 2.2 and taking into account

$$-3y^4 + 3y^2(x + y + z)^2 - \frac{1}{2}y^2(x + z - 2y)^2 + 3(x + z - 2y)^2 \geq \frac{1}{2}(y^4 + (x + z - 2y)^2),$$

finishes the proof. □

References

[1] K. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Mathematical Phys. **15** (1974), 2166–2174. MR **50:**6342

[2] K. Case, *Orthogonal polynomials, II*, J. Mathematical Phys. **16** (1975), 1435–1440. MR **56:**2008

[3] S. Denisov, *On the Nevai's conjecture and Rakhmanov's theorem for Jacobi matrices*, preprint.

[4] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Annals of Math. (2) **158** (2003), 253–321.

[5] A. Laptev, S. Naboko, and O. Safronov, *On new relations between spectral properties of Jacobi matrices and their coefficients*, to appear.

[6] S. Molchanov, M. Novitskii, and B. Vainberg, *First KdV integrals and absolutely continuous spectrum for 1-D Schrödinger operator*, Comm. Math. Phys. **216** (2001), no. 1, 195–213. MR **2001k:**35259

[7] O. Safronov, *The spectral measure of a Jacobi matrix in terms of the Fourier transform of the perturbation*, submitted.

[8] B. Simon and A. Zlatos, *Sum rules and the Szego condition for orthogonal polynomials on the real line*, submitted.

[9] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs, 72, Amer. Math. Soc., Providence, RI, 2000. MR **2001b:**39019

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