PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 5, Pages 1517–1524
S 0002-9939(03)07258-7
Article electronically published on October 3, 2003

ADJOINT RESTRICTION ESTIMATES AND SCALING ON SPHERES

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(Communicated by Andreas Seeger)

ABSTRACT. We test the restriction conjecture, in its adjoint form, against a class of measures $\phi_\delta d\sigma$ on the sphere $\mathbf{S}^{n-1}$. The densities ϕ_δ are smoothed out characteristic functions of $\delta^{a_2} \times \delta^{a_3} \times \cdots \times \delta^{a_n}$ rectangular caps on $\mathbf{S}^{n-1}$, where $a_2, a_3, \ldots, a_n$ are fixed nonnegative numbers.

1. INTRODUCTION

Let $n \geq 2$, $1 \leq p \leq \infty$, $2n/(n-1) < q < \infty$, and consider the adjoint restriction estimate

$$\|\widehat{f d\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{S}^{n-1})} \tag{1}$$

for $f \in L^p(\mathbb{S}^{n-1})$, where σ is the surface measure on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. The restriction conjecture asserts that (1) holds whenever

$$\frac{n-1}{p} + \frac{n+1}{q} \leq n-1. \tag{2}$$

Fix nonnegative numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ and a C^∞ function ϕ supported in a ball in $\mathbb{R}^n$ of center 0 and small[1] radius c_0. For $\theta \in \mathbb{S}^{n-1}$ and $0 < \delta \leq 1$, define

$$\phi_\delta(\theta) = \phi((\theta_1 - 1)/\delta^{a_1}, \theta_2/\delta^{a_2}, \ldots, \theta_n/\delta^{a_n}).$$

Then ϕ_δ is supported in a rectangular cap on $\mathbb{S}^{n-1}$ of dimensions $c_0\delta^{a_2} \times c_0\delta^{a_3} \times \cdots \times c_0\delta^{a_n}$ and center $e_1 = (1, 0, \ldots, 0)$, $\|\phi_\delta\|_{L^p(\mathbb{S}^{n-1})} \lesssim \delta^{a/p}$ where $a = a_2 + a_3 + \cdots + a_n$, and (1) implies

$$\|\widehat{\phi_\delta d\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \delta^{a/p} \tag{3}$$

with the implicit constant[2] independent of δ.

The purpose of this paper is to find necessary and sufficient conditions on p and q for which the estimate (3) holds.

In the case $a_1 = \cdots = a_n = 1$, a well-known argument, due to A. Knapp, shows that (2) is a necessary condition for (1) to hold by showing that (2) is necessary for the weaker estimate (3) when ϕ is nonnegative and satisfies $\phi(x) \gtrsim 1$ for $|x| \lesssim 1$. The argument goes as follows. $|\widehat{\phi_\delta d\sigma}| \approx \delta^{n-1}$ on a $C\delta^{-2} \times C\delta^{-1} \times \cdots \times C\delta^{-1}$ box

Received by the editors May 20, 2002 and, in revised form, January 28, 2003.
2000 *Mathematics Subject Classification.* Primary 42B10, 42B15.

[1]It suffices for our purposes to take $c_0 \leq 1/(100n)^2$.
[2]All constants in this paper are positive and independent of δ.

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(for an appropriate constant C) centered at the origin in $\mathbb{R}^n$ with the long side parallel to e_1. Inserting this into (3) one obtains

$$\frac{\delta^{n-1}}{\delta^{(n+1)/q}} \lesssim \delta^{\frac{n-1}{p}}$$

and (2) follows by letting $\delta \to 0$. For further details we refer to [1].

Repeating Knapp's argument in the general case, we see that $|\widehat{\phi_\delta d\sigma}| \approx \delta^a$ on a $C\delta^{-2a_2} \times C\delta^{-a_2} \times \cdots \times C\delta^{-a_n}$ box centered at the origin with the δ^{-2a_2} side parallel to e_1; so for (3) to hold, p and q must satisfy

$$\frac{a}{p} + \frac{a + 2a_2}{q} \leq a. \tag{4}$$

Our first result is the following improvement on (4).

Theorem 1. *Assume ϕ is nonnegative and $\phi(x) \gtrsim 1$ for $|x| \lesssim 1$. Suppose $1 \leq p \leq \infty$ and $1 \leq q < \infty$. If (3) holds, then $q > 2n/(n-1)$, (p, q) satisfies (4), and*

$$\frac{a}{p} + \frac{2na_n - a}{q} \leq (n-1)a_n. \tag{5}$$

We then turn matters around and try to find a range of (p, q) for which (3) holds. Of course, one would like to prove that (3) holds for the full range of (p, q) in Theorem 1, which is larger than that of (2), but we shall be able to do this (up to the sharp line) only in the case $a_2 = \cdots = a_n$ (when Theorem 1 and (2) agree). In the general case we obtain a range of (p, q) smaller than that in (2), but, at any rate, larger than what is already known for the restriction problem.

Theorem 2. *Suppose that $1 \leq p \leq \infty$ and $2n/(n-1) < q < \infty$. Then (3) holds whenever*

$$\frac{n-1}{p} + \frac{n+1}{q} \leq n-1 \quad \text{and} \quad \frac{a}{p} + \frac{(n+1)a_n}{q} < \frac{a + (n-1)a_n}{2}. \tag{6}$$

In particular, if $a_2 = \cdots = a_n$, then (3) holds whenever

$$\frac{n-1}{p} + \frac{n+1}{q} < n-1. \tag{7}$$

The proofs of Theorems 1 and 2 depend heavily on the smoothness of the function ϕ that will be used to obtain an asymptotic expansion formula for $\widehat{\phi_\delta d\sigma}$. This will be done in Section 2. In Section 3 we prove Theorem 1, and in Section 4, Theorem 2.

Acknowledgment

The author wishes to thank the referee for many helpful suggestions that led, in particular, to strengthening Proposition 1 to its present form.

2. Main estimate

In this section we use the method of stationary phase to produce an asymptotic expansion formula for $\widehat{\phi_\delta d\sigma}$ with the dependence of the various terms on δ explicitly described.

Recall that ϕ is supported in a ball in $\mathbb{R}^n$ of center 0 and small radius c_0.

Proposition 1. *Let* $c = 2c_0/\sqrt{1-c_0^2}$ *and*

$$\Omega_\delta = \{\theta \in \mathbb{S}^{n-1} : |\theta_j| \le c\,\delta^{a_j} \text{ for } j = 2,\dots,n\}.$$

Then:

(i) *For* $\xi = |\xi|\xi'$ *with* $|\xi| > 0$ *and* $\xi' \in \Omega_\delta$, *we have*

$$\widehat{\phi_\delta d\sigma}(\xi) = \frac{1}{|\xi|^{\frac{n-1}{2}}}\left(\phi_\delta(\xi')e^{-2\pi i\left(|\xi| - \frac{n-1}{8}\right)} + \phi_\delta(-\xi')e^{2\pi i\left(|\xi| - \frac{n-1}{8}\right)} + O\!\left(\frac{1}{\delta^{2a_n}|\xi|}\right)\right).$$

(ii) *For* $\xi = |\xi|\xi'$ *with* $|\xi| > 0$ *and* $\xi' \notin \Omega_\delta$, *we have*

$$|\widehat{\phi_\delta d\sigma}(\xi)| \lesssim \frac{\delta^a}{(\delta^{2a_n}|\xi|)^N}$$

for any positive integer N.

Proof. A point $x \in \mathbb{R}^{n-1}$ will be written as $x = (x_2,\dots,x_n)$.

Let $B = B(0,1/2) = \{x \in \mathbb{R}^{n-1} : |x| < 1/2\}$ and consider the local coordinate $F : B \to \mathbb{S}^{n-1}$ given by

$$F(x) = (\sqrt{1-|x|^2}, x).$$

Then

$$\widehat{\phi_\delta d\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-2\pi i\xi\cdot\theta}\phi_\delta(\theta)d\sigma(\theta) = \int_B e^{-2\pi i\xi\cdot F(x)}\phi_\delta(F(x))q(x)dx,$$

where q is a real-valued C^∞ function supported in B with $q(0) = 1$. Since the phase function in the above integral has no critical points in Supp $\phi_\delta \circ F$ when $\xi' \notin (\text{Supp }\phi_\delta) \cup (-\text{Supp }\phi_\delta)$, part (ii) of the proposition follows from an easy argument of integration by parts. So we only need to consider the situation when $\xi' \in \Omega_\delta$, i.e., when

$$|\xi'_j| \le c\,\delta^{a_j} \quad \text{for} \quad j = 2,\dots,n.$$

Clearly, we can also assume that $\xi'_1 > 0$.

Set

$$C_\delta = \{z \in \mathbb{R}^n : \tfrac{1}{2} < z_1 < 2 \text{ and } |z_j| \le c\,\delta^{a_j} \text{ for } j = 2,\dots,n\},$$

$\lambda = |\xi|$, and let T be a rotation on $\mathbb{R}^n$ such that $T\xi' = e_1$. Then $\xi' \in C_\delta \cap \mathbb{S}^{n-1}$ and

$$\widehat{\phi_\delta d\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-2\pi i\lambda e_1\cdot\theta}\psi(\theta)d\sigma(\theta), \tag{8}$$

where $\psi(\theta) = \phi_\delta(T^{-1}\theta)$. Now consider the dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ given by

$$A_t z = (t^{a_1}z_1,\dots,t^{a_n}z_n),$$

and notice that

$$\psi(\theta) = \phi'_\delta(\theta) = \phi'(A_{1/\delta}(\theta - e_1)),$$

where

$$\phi'(\theta) = \phi(A_{1/\delta}T^{-1}A_\delta\theta + A_{1/\delta}(\xi' - e_1)).$$

To estimate the integral in (8), we shall require of the rotation T, in addition to $T\xi' = e_1$, the following two properties:

- there is a constant c' such that

  $$T(C_\delta) \subset \{z \in \mathbb{R}^n : 0 < z_1 < 3 \text{ and } |z_j| \le c'\delta^{a_j} \text{ for } j = 2,\dots,n\};$$

- $\|A_{1/\delta}T^{-1}A_\delta\| \lesssim 1$.

The first property tells us that Supp $\phi' \subset \{z \in \mathbb{R}^n : |z_j| \leq c' \text{ for } j = 2, \ldots, n\}$. The second guarantees that the partial derivatives of ϕ' are bounded uniformly in δ. To find such a rotation, put $\beta_{-1} = 1$ and

$$\beta_j = \sqrt{1 - (\xi'_{n-j})^2 - (\xi'_{n-j+1})^2 - \cdots - (\xi'_n)^2} \qquad (j = 0, \ldots, n-2),$$

define rotations S_j on $\mathbb{R}^n$ by

$$S_j(e_k) = \begin{cases} (\beta_j/\beta_{j-1})\, e_1 + (\xi'_{n-j}/\beta_{j-1})\, e_{n-j} & \text{if } k = 1, \\ -(\xi'_{n-j}/\beta_{j-1})\, e_1 + (\beta_j/\beta_{j-1})\, e_{n-j} & \text{if } k = n-j, \\ e_k & \text{if } k \neq 1, n-j \end{cases}$$

(of course, $\{e_1, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^n$), and put

$$T^{-1} = S_{n-2} S_{n-3} \ldots S_1 S_0.$$

Then

$$T^{-1} e_k = S_{n-2} \ldots S_{n-k}(e_k) \qquad (k = 2, \ldots, n),$$

and since $S_{n-k}, \ldots, S_{n-2}$ do not change the last $n-k$ coordinates of any point in $\mathbb{R}^n$, it follows that the last $n-k$ coordinates of $T^{-1}e_k$ are all zeroes. On the other hand, since $\beta_j \geq \sqrt{1-(n-1)c^2}$ for $j = 0, \ldots, n-2$, each of the first $k-1$ coordinates of $T^{-1}e_k$ is $\leq |\xi'_k|/(1-(n-1)c^2) \leq c\,\delta^{a_k}/(1-(n-1)c^2)$. It is now an easy matter to verify that T satisfies the required properties.

Going back to (8), we now have

$$\widehat{\phi_\delta d\sigma}(\xi) = \int_B e^{-2\pi i \lambda \sqrt{1-|x|^2}} \psi(F(x)) q(x) dx.$$

Making the change of variables $x = G(u) = (G_2(u), \ldots, G_n(u))$ with

$$G_j(u) = u_j \sqrt{1 - \frac{u_j^2}{4} - \frac{1}{2}(u_{j+1}^2 + \cdots + u_n^2)} \qquad (j = 2, \ldots, n),$$

we get

$$\widehat{\phi_\delta d\sigma}(\xi) = \int_U e^{-2\pi i \lambda \sqrt{1-|G(u)|^2}} \Phi(u) du = e^{-2\pi i \lambda} \int_U e^{\pi i \lambda |u|^2} \Phi(u) du,$$

where $U = G^{-1}B$ and $\Phi = (\psi \circ F \circ G)(q \circ G)|J_G|$. The observations we made before about Supp ϕ' and its derivatives tell us that

$$\text{Supp } \Phi \subset \{u \in \mathbb{R}^{n-1} : |u_j| \leq 2c'\delta^{a_j} \text{ for } j = 2, \ldots, n\} \tag{9}$$

and

$$\begin{aligned} |D^m \Phi(u)| &\lesssim \sum_{m' \leq m} |D^{m'}(\psi \circ F \circ G)(u)| \\ &\lesssim \sum_{m' \leq m} \frac{1}{\delta^{m' \cdot (a_2, \ldots, a_n)}} \\ &\lesssim \frac{1}{\delta^{m \cdot (a_2, \ldots, a_n)}} \end{aligned} \tag{10}$$

for all $u \in U$ and every multi-index $m = (m_2, \ldots, m_n)$. Also,

$$\Phi(0) = \psi(e_1) q(0) |J_G(0)| = \phi_\delta(\xi').$$

Since the distributional Fourier transform of $e^{\pi i \lambda |u|^2}$ is

$$e^{\pi i \frac{n-1}{4}} \lambda^{-\frac{n-1}{2}} e^{-\pi i \lambda^{-1} |y|^2},$$

it follows that

$$\begin{aligned}\widehat{\phi_\delta d\sigma}(\xi) &= e^{-2\pi i\left(\lambda-\frac{n-1}{8}\right)}\,\lambda^{-\frac{n-1}{2}}\int e^{-\pi i\lambda^{-1}|y|^2}\widehat{\Phi}(y)dy\\ &= e^{-2\pi i\left(\lambda-\frac{n-1}{8}\right)}\,\lambda^{-\frac{n-1}{2}}\,\Phi(0)+\lambda^{-\frac{n-1}{2}-1}\,E(\lambda)\end{aligned}$$

with

$$E(\lambda)=e^{-2\pi i\left(\lambda-\frac{n-1}{8}\right)}\,\lambda\int\left(e^{-\pi i\lambda^{-1}|y|^2}-1\right)\widehat{\Phi}(y)dy.$$

To estimate this last integral, set $\Phi^\delta(u)=\Phi(A_\delta u)$, let $N>n$ be an integer, and observe that

$$\begin{aligned}|E(\lambda)| &\leq \int |y|^2|\widehat{\Phi}(y)|dy\\ &= \int |y|^2|\widehat{\Phi^\delta}(A_\delta y)|\delta^a dy\\ &= \int |A_{1/\delta}v|^2|\widehat{\Phi^\delta}(v)|dv\\ &\leq \delta^{-2a_n}\int |v|^2|\widehat{\Phi^\delta}(v)|dv\\ &\leq \delta^{-2a_n}\int \frac{(1+|v|)^{N+2}}{(1+|v|)^N}|\widehat{\Phi^\delta}(v)|dv\\ &\lesssim \delta^{-2a_n}\sum_{|m|\leq N+2}\int\frac{1}{(1+|v|)^N}|v^m\widehat{\Phi^\delta}(v)|dv\\ &\approx \delta^{-2a_n}\sum_{|m|\leq N+2}\int\frac{1}{(1+|v|)^N}|\widehat{D^m\Phi^\delta}(v)|dv\\ &\lesssim \delta^{-2a_n}\sum_{|m|\leq N+2}\|\widehat{D^m\Phi^\delta}\|_{L^\infty(\mathbb{R}^{n-1})}\\ &\leq \delta^{-2a_n}\sum_{|m|\leq N+2}\|D^m\Phi^\delta\|_{L^1(\mathbb{R}^{n-1})}\\ &\lesssim \delta^{-2a_n}\sum_{|m|\leq N+2}\|D^m\Phi^\delta\|_{L^\infty(\mathbb{R}^{n-1})}\\ &\lesssim \delta^{-2a_n}\end{aligned}$$

where in the last two lines we have used (9) and (10). □

3. Proof of Theorem 1

Choose a constant c' such that $\phi_\delta\gtrsim 1$ on the spherical cap

$$C_\delta=\{\theta\in\mathbb{S}^{n-1}:\theta_1>0,|\theta_2|\leq c'\delta^{a_2},\ldots,|\theta_n|\leq c'\delta^{a_n}\}.$$

By Proposition 1, there is a constant c'' such that

$$|\widehat{\phi_\delta d\sigma}(\xi)|\gtrsim\frac{1}{|\xi|^{\frac{n-1}{2}}}$$

for $\xi = |\xi|\xi'$, $\xi' \in C_\delta$, $|\xi| \geq c''\delta^{-2a_n}$. Thus

$$\begin{aligned} \|\widehat{\phi_\delta d\sigma}\|^q_{L^q(\mathbb{R}^n)} &\geq \int_{c''\delta^{-2a_n}}^\infty \int_{C_\delta} |\widehat{\phi_\delta d\sigma}(r\theta)|^q r^{n-1} d\sigma(\theta) dr \\ &\gtrsim \sigma(C_\delta) \int_{c''\delta^{-2a_n}}^\infty r^{n-1-(n-1)(q/2)} dr. \end{aligned}$$

If $q \leq 2n/(n-1)$, the above integral equals ∞, so that $\|\widehat{\phi_\delta d\sigma}\|_{L^q(\mathbb{R}^n)} = \infty$. If $q > 2n/(n-1)$, we get

$$\|\widehat{\phi_\delta d\sigma}\|_{L^q(\mathbb{R}^n)} \gtrsim \delta^{a/q} \delta^{(n-1)a_n - 2na_n/q}.$$

Inserting this estimate into (3) we obtain

$$\frac{a}{p} + \frac{2na_n - a}{q} \leq (n-1)a_n.$$

4. Proof of Theorem 2

Let $\epsilon > 0$ and put $R = \delta^{-2a_n - \epsilon}$. Also, let $B = B(0, R)$ be the ball in $\mathbb{R}^n$ of center 0 and radius R. Proposition 1 tells us that

$$\begin{aligned} &\|\widehat{\phi_\delta d\sigma}\|_{L^q(B^c)} \\ &\lesssim \left(\sigma(\Omega_\delta) \int_R^\infty \frac{r^{n-1}}{r^{\frac{n-1}{2}q}} dr\right)^{1/q} + \delta^{-2Na_n} \left(\sigma(\mathbb{S}^{n-1} - \Omega_\delta) \int_R^\infty \frac{r^{n-1}}{r^{Nq}} dr\right)^{1/q} \\ &\lesssim \left(\delta^a \int_{\delta^{-2a_n}}^\infty r^{n - \frac{n-1}{2}q - 1} dr\right)^{1/q} + \delta^{-2Na_n} \left(R^{n-Nq}\right)^{1/q} \\ &\lesssim \delta^{(a-2na_n)/q + (n-1)a_n} + \delta^{N\epsilon - (n\epsilon + 2na_n)/q} \\ &\lesssim \delta^{(a-2na_n)/q + (n-1)a_n}. \end{aligned}$$

It follows that

$$\|\widehat{\phi_\delta d\sigma}\|_{L^q(B^c)} \lesssim \delta^{a/p}$$

whenever

$$\frac{a}{p} + \frac{2na_n - a}{q} \leq (n-1)a_n,$$

which is larger than the range of exponents in (6). So we only need to worry about $\|\widehat{\phi_\delta d\sigma}\|_{L^q(B)}$.

We are going to use three basic estimates. The first is the Tomas-Stein estimate (see [2] and [3])

$$\|\widehat{f d\sigma}\|_{L^{(2n+2)/(n-1)}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{S}^{n-1})} \tag{11}$$

for $f \in L^2(\mathbb{S}^{n-1})$. The second is the local estimate

$$\|\widehat{f d\sigma}\|_{L^q(B)} \lesssim R^{\frac{1}{2}\left(\frac{n+1}{q} - \frac{n+1}{2} + 1\right)} \|f\|_{L^2(\mathbb{S}^{n-1})} \tag{12}$$

for $f \in L^2(\mathbb{S}^{n-1})$ and $2 \leq q \leq (2n+2)/(n-1)$, which is obtained by interpolating between (11) and the trace estimate

$$\|\widehat{f d\sigma}\|_{L^2(B)} \lesssim R^{1/2} \|f\|_{L^2(\mathbb{S}^{n-1})}.$$

To obtain the third estimate, we use a standard uncertainty principle argument.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $|\widehat{\psi}(\xi)| \geq 1$ for $|\xi| \leq 1$, and set

$$\psi_R(x) = R^n \psi(Rx) \quad \text{and} \quad \Psi = R^{-1} \psi_R * (\phi_\delta d\sigma).$$

Then Ψ is supported in a box of center e_1 and dimensions $C\delta^{2a_2} \times C\delta^{a_2} \times \cdots \times C\delta^{a_n}$ with the δ^{2a_2} side parallel to e_1, $\|\Psi\|_{L^\infty(\mathbb{R}^n)} \lesssim 1$, and

$$\|\widehat{\phi_\delta d\sigma}\|_{L^q(B)} \lesssim R\,\|\widehat{\Psi}\|_{L^q(\mathbb{R}^n)}.$$

Cover $\mathbb{R}^n$ by a finitely overlapping sequence $\{T_l\}$ of boxes such that each T_l has dimensions $C\delta^{-2a_2} \times C\delta^{-a_2} \times \cdots \times C\delta^{-a_n}$ with the δ^{-2a_2} side parallel to e_1, and

$$\sum_{l=1}^{\infty} \|\widehat{\Psi}\|^2_{L^\infty(T_l)} |T_l| \lesssim \|\Psi\|^2_{L^2(\mathbb{R}^n)}.$$

(This is a straightforward application of the inequality

$$|f(x)| \lesssim \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_{L^1(B(x,1))}$$

and Plancherel's theorem.) Then

$$\sum_{l=1}^{\infty} \|\widehat{\Psi}\|^2_{L^\infty(T_l)} \lesssim \left(\delta^{a+2a_2}\right)^2.$$

Letting $c_l = \delta^{-2(a+2a_2)}\|\widehat{\Psi}\|^2_{L^\infty(T_l)}$, we get

$$|\widehat{\Psi}(\xi)|^2 \lesssim \delta^{2(a+2a_2)} \sum_{l=1}^{\infty} c_l\, \chi_{T_l}(\xi)$$

with $\sum_{l=1}^{\infty} c_l \lesssim 1$. Thus

$$\|\widehat{\phi_\delta d\sigma}\|_{L^q(B)} \lesssim R\,\delta^{a+2a_2-(a+2a_2)/q} = \delta^{a+2a_2-2a_n-\epsilon-(a+2a_2)/q} \tag{13}$$

provided $q \geq 2$.

Interpolating between (11) and the trivial estimate $\|\widehat{f d\sigma}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{S}^{n-1})}$, we get (3) for $q \geq (2n+2)/(n-1)$ and

$$\frac{n-1}{p} + \frac{n+1}{q} \leq n-1.$$

Applying (12) with $f = \phi_\delta$, we obtain

$$\|\widehat{\phi_\delta d\sigma}\|_{L^q(B)} \lesssim \delta^{-(a_n+\frac{\epsilon}{2})(\frac{n+1}{q}-\frac{n+1}{2}+1)}\delta^{a/2}$$

for $2 \leq q \leq (2n+2)/(n-1)$. So to get

$$\|\widehat{\phi_\delta d\sigma}\|_{L^q(B)} \lesssim \delta^{a/p}, \tag{14}$$

we need

$$\frac{a}{p} + \frac{(n+1)(a_n+\epsilon/2)}{q} \leq \frac{a+(n-1)(a_n+\epsilon/2)}{2}. \tag{15}$$

Now if (p,q) is such that

$$\frac{a}{p} + \frac{(n+1)a_n}{q} < \frac{a+(n-1)a_n}{2},$$

we choose ϵ small enough for (p,q) to satisfy (15), and (14) follows. Therefore, (3) holds whenever (p,q) satisfies (6).

Finally, we notice that (13) tells us that (3) holds whenever

$$\frac{a}{p} + \frac{a+2a_2}{q} < a + 2a_2 - 2a_n$$

(compare with (4)). When $a_2 = \cdots = a_n$, this becomes

$$\frac{n-1}{p} + \frac{n+1}{q} < n-1.$$

This shows that one can get (7) without having to appeal to Tomas and Stein.

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