

POLYNOMIAL APPROXIMATION ON REAL-ANALYTIC VARIETIES IN \mathbf{C}^n

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ABSTRACT. We prove: Let Σ be a compact real-analytic variety in \mathbf{C}^n . Assume (i) Σ is polynomially convex and (ii) every point of Σ is a peak point for $P(\Sigma)$. Then $P(\Sigma) = C(\Sigma)$. This generalizes a previous result of the authors on polynomial approximation on three-dimensional real-analytic submanifolds of \mathbf{C}^n .

1. INTRODUCTION

We consider the problem of approximating arbitrary continuous functions on a compact subset X of n -dimensional complex Euclidean space \mathbf{C}^n by polynomials in the coordinate functions z_1, \dots, z_n . We let $C(X)$ denote the space of all continuous complex-valued functions on X , with norm $\|g\|_X = \max\{|g(z)| : z \in X\}$, and we let $P(X)$ denote the closure of the set of polynomials in $C(X)$. The polynomially convex hull of X will be denoted \widehat{X} . That is,

$$\widehat{X} = \{z \in \mathbf{C}^n : |Q(z)| \leq \|Q\|_X \text{ for every polynomial } Q\}.$$

Two necessary conditions for $P(X) = C(X)$ are:

- (i) X is polynomially convex, i.e., $X = \widehat{X}$;
- (ii) every point of X is a peak point for $P(X)$, i.e., given $p \in X$, there exists $f \in P(X)$ with $f(p) = 1$ and $|f| < 1$ on $X \setminus \{p\}$.

With a general uniform algebra A on a compact metric space X replacing $P(X)$, and with (i) replaced by the condition that the maximal ideal space of A coincides with X , it was once conjectured that together these two necessary conditions for $A = C(X)$ were also sufficient to imply $A = C(X)$. However, a counterexample to this “peak-point conjecture” was produced by Brian Cole in his 1968 thesis (see the appendix to [5], or [12], section 19). Additional counterexamples to the peak point conjecture have been given in the context of polynomial and rational approximation in several complex variables. In particular, these counterexamples show that conditions (i) and (ii) above are not sufficient for $P(X) = C(X)$. For more on this, see [2], [3], [4] and [9].

In [2] the first two authors established a peak-point result for two-manifolds:

Theorem 1.1. *Let M be a compact two-dimensional differentiable manifold, and A a uniform algebra on M generated by continuously differentiable functions. Assume*

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that the maximal ideal space of A is M , and that each point of M is a peak point for A . Then $A = C(M)$.

An example of Izzo [9] shows that Theorem 1.1 fails for uniform algebras on smooth three manifolds. However, in [3] the authors established the following:

Theorem 1.2. *Let Σ be a real-analytic three-dimensional submanifold of \mathbf{C}^n . Let X be a compact subset of Σ such that ∂X (the boundary of X relative to Σ), is a two-dimensional submanifold of class C^1 . If X satisfies conditions (i) and (ii) above, then $P(X) = C(X)$.*

Our purpose in this note is to extend Theorem 1.2 to real-analytic varieties of arbitrary dimension in \mathbf{C}^n . We prove:

Theorem 1.3. *Let Σ be a compact real-analytic variety in \mathbf{C}^n . Assume (i) $\Sigma = \widehat{\Sigma}$ and (ii) every point of Σ is a peak point for $P(\Sigma)$. Then $P(\Sigma) = C(\Sigma)$.*

If ∂B_n denotes the boundary of the unit ball in \mathbf{C}^n , then every point of ∂B_n is a peak point for polynomials, and so as an immediate consequence of Theorem 1.3 we have the following:

Corollary 1.4. *If $\Sigma \subset \partial B_n$ is a compact polynomially convex real-analytic variety, then $P(\Sigma) = C(\Sigma)$.*

In order to explain the idea of the proof of Theorem 1.3, we recall the proof of Theorem 1.2. The main tool is a result of Hörmander and Wermer [8] (proved by them for sufficiently smooth manifolds, and later generalized by O'Farrell, Preskenis and Walsh [11] to C^1 manifolds) that, in essence, reduces questions of approximation on subsets of real submanifolds of \mathbf{C}^n to approximation on the points where the tangent space to the manifold contains a complex line. The following dual formulation suffices (see the discussion following Proposition 2.3 of [3]):

Theorem 1.5. *Let X be a polynomially convex compact subset of \mathbf{C}^n , and let X_0 be a compact subset of X such that $X \setminus X_0$ is a totally real submanifold of \mathbf{C}^n , of class C^1 . If μ is a measure on X such that $\int_X f d\mu = 0$ for all $f \in P(X)$, then μ is supported on X_0 . In particular, $P(X) = C(X)$ if and only if $P(X_0) = C(X_0)$.*

To establish Theorem 1.2, one shows, under assumptions (i) and (ii), that the set of points in X where Σ has a complex tangent is a real-analytic variety V of dimension at most two. To show that $P(X) = C(X)$, it is enough, by Theorem 1.5, to show that $P(\partial X \cup V) = C(\partial X \cup V)$. Applying Theorem 1.5 again, one reduces the problem to showing that $P(\partial X \cup V^*) = C(\partial X \cup V^*)$, where V^* is the union of the singular set of the variety V together with the set of regular points of V at which V has a complex tangent. Theorem 1.1 implies that $P(\partial X) = C(\partial X)$. Using the peak-point property (ii), V^* can be shown to have two-dimensional Hausdorff measure zero. This suffices to show (see Lemma 3.1 of [3]) that $P(\partial X \cup V^*) = C(\partial X \cup V^*)$, and completes the proof.

The fact that the two-dimensional Hausdorff measure of V^* is zero is essential to the proof we have just described; this allows one to apply the Hartogs-Rosenthal theorem on rational approximation in the plane to certain projections. In attempting to generalize Theorem 1.2 to real-analytic manifolds of dimension greater than three, one would like to proceed in a similar way: reduce the problem of approximation on Σ to approximation on V , where V is the set of points in Σ at which

Σ has a complex tangent. Assumptions (i) and (ii) imply, as before, that V is a variety whose dimension is strictly less than the dimension of Σ , and so one hopes to proceed by induction on dimension, repeatedly applying Theorem 1.5 as before, eventually reducing to approximation on a sufficiently small set (two-dimensional Hausdorff measure zero). The fact that at each stage one has to consider approximation on varieties makes it desirable, for purposes of the induction, to prove Theorem 1.3 in the category of varieties, rather than restricting to real-analytic manifolds. However, a problem arises: the singular set of a real-analytic variety (unlike the case of complex-analytic varieties) need not itself be a variety. This problem forces a slightly different approach. We show that locally the union of the set of complex tangent points and the singular set of each variety in question is contained in a variety of smaller dimension, and this allows the induction to proceed.

2. REAL-ANALYTIC VARIETIES

In this section we review the basic facts about real-analytic varieties necessary for the proof of Theorem 1.3.

If U is an open subset of \mathbf{R}^m , $C^\omega(U)$ will denote the set of all real-valued functions that are real-analytic in U . If \mathcal{F} is a subset of $C^\omega(U)$, we set

$$V_U(\mathcal{F}) = \{x \in U : f(x) = 0, \forall f \in \mathcal{F}\}.$$

A subset Σ of an open set $\Omega \subset \mathbf{R}^m$ is said to be a (real-analytic) *variety in* Ω if for each $p \in \Omega$ there exists an open neighborhood $U \subset \Omega$ of p and a finite set $\mathcal{F} \subset C^\omega(U)$ with $\Sigma \cap U = V_U(\mathcal{F})$. It follows that Σ is a relatively closed subset of Ω ; we say that Σ is a *compact variety in* \mathbf{C}^n if it is a compact subset of \mathbf{C}^n . The class of varieties is closed under finite unions and intersections.

If Σ is a real-analytic variety, $p \in \Sigma$ is a *regular point* of Σ if there exists a neighborhood U_p of p in \mathbf{C}^n such that $\Sigma \cap U_p$ is a real-analytic submanifold of U_p . The *dimension* of Σ is the maximum dimension of these submanifolds, as p ranges over the set of regular points. The set of regular points of Σ is denoted Σ_{reg} . Its complement in Σ , the set of *singular points*, is denoted Σ_{sing} .

A variety Σ in Ω is said to be *irreducible* if whenever $\Sigma = \Sigma_1 \cup \Sigma_2$ with Σ_1, Σ_2 varieties in Ω , then either $\Sigma = \Sigma_1$ or $\Sigma = \Sigma_2$. If Σ is an irreducible variety in Ω , and Σ' is a proper subset of Σ that is also a variety in Ω , then $\dim(\Sigma) > \dim(\Sigma')$ (see [6], Theorem 3.4.8, assertion (15)).

Unlike the case of complex analytic varieties, the singular set of a real-analytic variety need not itself be a variety. Moreover, a compact real-analytic variety is not necessarily the union of a finite number of irreducible varieties. However, we have the following local properties (see [10], pp. 31–42, especially Proposition 5, and [6], section 3.4.10):

- (a) If Σ is irreducible, then for each $p \in \Sigma_{sing}$ there exist a neighborhood U of p and a function $\delta \in C^\omega(U)$ not vanishing identically on $\Sigma \cap U$ so that

$$\Sigma_{sing} \cap U \subset V_U(\{\delta\}) \cap \Sigma \equiv \Delta.$$

Moreover, there is a finite set $\mathcal{G} = \{g_1, \dots, g_t\} \subset C^\omega(U)$ so that

$$(\Sigma \cap U) \setminus \Delta = V_{U \setminus \Delta}(\mathcal{G})$$

with dg_1, \dots, dg_t linearly independent on $U \setminus \Delta$.

- (b) If Σ is a variety, then for each $p \in \Sigma$ there are a neighborhood U of p and a finite number of irreducible varieties Z_1, \dots, Z_s in U so that $\Sigma \cap U = \bigcup_{k=1}^s Z_k$.
- (c) Let H^r denote r -dimensional Hausdorff measure. If Σ is a variety of dimension d in Ω , then for each compact subset X of Ω , $H^d(\Sigma \cap X) < \infty$ and $H^{d-1}(\Sigma_{sing} \cap X) < \infty$.

We now consider real-analytic varieties in an open subset Ω of \mathbf{C}^n , identified with \mathbf{R}^{2n} . If M is a real submanifold of Ω , of class C^1 , we say that M has a complex tangent at $p \in M$ if the real tangent space $T_p M$ of M at p contains a nontrivial complex subspace of \mathbf{C}^n . This is equivalent to the condition that the restriction to M of any $(m, 0)$ form vanishes at p , where m is the real dimension of M (see [3], Lemma 2.5). The manifold M is said to be *totally real* if it has no complex tangents. If Σ is a variety in Ω , Σ_c will denote the set of points $p \in \Sigma_{reg}$ such that Σ_{reg} has a complex tangent at p . By expressing the vanishing of each $(m, 0)$ form as the vanishing of an $(m \times m)$ minor of the matrix $J = ((\partial z_l / \partial u_k))$, where (u_1, \dots, u_m) are local real-analytic coordinates on M , we see that Σ_c is a variety in $\Omega \setminus \Sigma_{sing}$. Set $\Sigma^* = \Sigma_{sing} \cup \Sigma_c$. Note that $\Sigma \setminus \Sigma^*$ is a totally real, real-analytic submanifold of $\Omega \setminus \Sigma_{sing}$.

Lemma 2.1. *Let Σ be a d -dimensional variety in an open set $\Omega \subset \mathbf{C}^n$. Assume $\dim(\Sigma_c) < d$. Then for each $p \in \Sigma^*$, there exists an open neighborhood $U \subset \Omega$ of p and a variety Y in U with $\dim(Y) < d$ and*

$$\Sigma^* \cap U \subset Y \subset \Sigma \cap U.$$

Proof. Fix $p \in \Sigma$, and take U and Z_1, \dots, Z_s as in (b). Let $d_j = \dim(Z_j)$, and set $J = \{j : 1 \leq j \leq s \text{ and } d_j = d\}$. For each $j \in J$, we proceed as follows: shrinking U if necessary, using (a) we obtain a function $\delta_j \in C^\omega(U)$, not vanishing identically on $Z_j \cap U$, with

$$(Z_j)_{sing} \subset \Delta_j \equiv V_U(\{\delta_j\}) \cap Z_j$$

and a finite set $\mathcal{G}^{(j)} \subset C^\omega(U)$ of functions whose differentials are linearly independent on $U \setminus \Delta_j$, so that

$$Z_j \setminus \Delta_j = V_{U \setminus \Delta_j}(\mathcal{G}^{(j)}).$$

Using the functions in $\mathcal{G}^{(j)}$ and the remarks on the complex tangent set preceding the statement of Lemma 2.1, we may construct a family $\Phi^{(j)}$ of functions in $C^\omega(U)$ so that

$$(Z_j)_c = V_{U \setminus \Delta_j}(\Phi^{(j)} \cup \mathcal{G}^{(j)}).$$

Set $X_j = \Delta_j \cup V_U(\Phi^{(j)} \cup \mathcal{G}^{(j)})$. Then X_j is a variety in U containing Z_j^* . Our assumption on $\dim(\Sigma_c)$ together with the irreducibility of Z_j implies that $\dim(X_j) < d$. Let X be the union of the X_j , $j \in J$. Let Z be the union of all pairwise intersections of the Z_j , together with the union of all Z_j , $j \notin J$. Set $Y = X \cup Z$. Then $\Sigma^* \cap U \subset Y \subset \Sigma \cap U$, and $\dim(Y) < d$. □

3. PROOF OF THEOREM 1.3

We let $B(p, r)$ denote the open ball of radius r centered at p . We will use repeatedly the fact (Lemma 2.1 of [3]) that properties (i) and (ii) are inherited by compact subsets of Σ .

Lemma 3.1. *Let Σ be a compact d -dimensional variety in an open set $\Omega \subset \mathbf{C}^n$. Assume that Σ is polynomially convex and that each point of Σ is a peak point for $P(\Sigma)$. Then for each $p \in \Sigma$, there exist $r > 0$ and varieties Y_1, \dots, Y_d in $B(p, r)$ so that if $Y_0 = \Sigma \cap B(p, r)$, then for $1 \leq j \leq d$ we have*

- (1) $Y_{j-1}^* \subset Y_j \subset Y_{j-1}$;
- (2) $\dim(Y_j) \leq d - j$.

Proof. The proof is by induction on j , $1 \leq j \leq d$. Note that the hypotheses on Σ imply that Σ_c has no interior relative to Σ (see [3], Lemma 3.2). Hence $\dim(\Sigma_c) < d$. We choose r_1, Y_1 as follows: in case $p \in \Sigma \setminus \Sigma^*$, take $r_1 > 0$ so that $B(p, r_1) \cap \Sigma^* = \emptyset$, and take $Y_1 = \emptyset$. If $p \in \Sigma^*$, Lemma 2.1 implies that there exists $r_1 > 0$ and a variety Y_1 in $B(p, r_1)$ so that $\dim(Y_1) \leq d - 1$ and

$$B(p, r_1) \cap \Sigma^* \subset Y_1 \subset B(p, r_1) \cap \Sigma.$$

Note that in either case, both (1) and (2) hold for $j = 1$. Now assume by induction that for some k with $1 \leq k < d$, $r_1, \dots, r_k > 0$ and Y_1, \dots, Y_k have been chosen with $r_1 \geq r_2 \geq \dots \geq r_k$, and Y_1, \dots, Y_k varieties in $B(p, r_k)$, so that (1) and (2) hold for all $j \leq k$. We obtain r_{k+1}, Y_{k+1} as follows: if $p \in Y_k \setminus Y_k^*$, take $Y_{k+1} = \emptyset$ and $r_{k+1} = r_k$. If $p \in Y_k^*$, use Lemma 2.1 to produce r_{k+1} , $0 < r_{k+1} \leq r_k$ and a variety Y_{k+1} with

$$B(p, r_{k+1}) \cap Y_k^* \subset Y_{k+1} \subset B(p, r_{k+1}) \cap Y_k$$

and $\dim(Y_{k+1}) < \dim(Y_k) \leq d - k$ (here we again use Lemma 3.2 of [3]). Then (1) and (2) will hold for $j \leq k + 1$. By induction, we can thus choose $r_1, \dots, r_d > 0$ and Y_1, \dots, Y_d so that (1) and (2) hold for $j \leq d$; take $r = r_d$. \square

We now turn to the proof of Theorem 1.3. By duality, it suffices to show that any measure μ on Σ with the property that

$$(*) \quad \int_{\Sigma} g \, d\mu = 0$$

for all $g \in P(\Sigma)$ must be the zero measure. Fix a measure μ satisfying (*). We will show that for each point $p \in \Sigma$, there is a neighborhood of p lying outside the support of μ . Fix $p \in \Sigma$, and choose $r > 0$, and Y_0, \dots, Y_d as constructed in Lemma 3.1. For $j < d$,

$$[(\Sigma \setminus B(p, r)) \cup Y_j] \setminus [(\Sigma \setminus B(p, r)) \cup Y_{j+1}] = Y_j \setminus Y_{j+1}$$

is a totally real submanifold of \mathbf{C}^n . By Theorem 1.5, if μ is supported on $(\Sigma \setminus B(p, r)) \cup Y_j$, then μ is supported on $(\Sigma \setminus B(p, r)) \cup Y_{j+1}$. By induction, μ is supported on $(\Sigma \setminus B(p, r)) \cup Y_d$. Note that $\dim(Y_d) = 0$; so the variety Y_d is a discrete point set, and therefore countable. Since every point of Σ is a peak point for $P(\Sigma)$, μ cannot have nonzero mass at any single point, and hence $|\mu|(Y_d) = 0$. Thus μ is supported on $\Sigma \setminus B(p, r)$, and the proof is complete. \square

Remarks. A referee has raised the following two interesting questions:

1. Let X be a compact real-analytic variety in \mathbf{C}^n . Assume X is rationally convex and has the peak-point property for $R(X)$. Is $R(X) = C(X)$?
2. Let X be a compact real-analytic variety in \mathbf{C}^n that is polynomially convex. Does the peak-point property hold for $P(X)$?

We believe that the arguments we have presented for the polynomially convex case can be modified to prove an analogous theorem in a more general context which, in particular, would give a positive answer to question 1. We plan to present this in a subsequent paper. We do not know the answer to question 2.

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