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VISCOSITY AND RELAXATION APPROXIMATIONS FOR A HYPERBOLIC-ELLIPTIC MIXED TYPE SYSTEM

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Abstract. To a given system of conservation laws

$$\begin{cases} u_t + f(u,v,h(u,v))_x = 0 \\ v_t + g(u,v,h(u,v))_x = 0 \end{cases}$$

we associate the system

$$\begin{cases} u_t + f(u,v,s)_x = \epsilon u_{xx} \\ v_t + g(u,v,s)_x = \epsilon v_{xx} \\ s_t + \frac{s-h(u,v)}{\tau} = \epsilon s_{xx}, \end{cases}$$

which is of mixed type. Under certain conditions, convergence of this latter system for $\epsilon \to 0$ with $\tau = o(\epsilon)$ is established without the need of stability criteria or hyperbolicity of the left-hand sides of the equations.

1. Introduction

In this paper we study singular limits of stiff relaxation and dominant diffusion for a general hyperbolic-elliptic mixed type system of three equations:

$$\begin{cases} u_t + f(u,v,s)_x = \epsilon u_{xx} \\ v_t + g(u,v,s)_x = \epsilon v_{xx} \\ s_t + \frac{s-h(u,v)}{\tau} = \epsilon s_{xx} \end{cases} \tag{1.1}$$

with initial data

$$\big(u,v,s\big)_{|t=0} = \big(u_0, v_0, s_0\big). \tag{1.2}$$

The third equation in (1.1) contains a relaxation mechanism with $h(u,v)$ as the equilibrium value for s, τ the relaxation time, and ϵ the viscous parameter. That is, the relaxation time τ tends to zero faster than the diffusion parameter ϵ, $\tau = o(\epsilon), \epsilon \to 0$. We establish the following general framework: if there exists an a priori L^∞ bound uniform with respect to ϵ and τ for the solutions of the Cauchy problem (1.1) and (1.2), then the solution sequence converges to the corresponding equilibrium solution of this system:

$$\begin{cases} u_t + f(u,v,h(u,v))_x = 0 \\ v_t + g(u,v,h(u,v))_x = 0 \end{cases} \tag{1.3}$$

where system (1.3) is strictly hyperbolic and genuinely nonlinear in the sense of Lax. Our results indicate that the convergent behavior of such a limit is independent

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of either the stability criterion or the hyperbolicity of the corresponding inviscid quasilinear system

$$\begin{cases} u_t + f(u,v,s)_x = 0 \\ v_t + g(u,v,s)_x = 0 \\ s_t + \frac{s-h(u,v)}{\tau} = 0, \end{cases} \tag{1.4}$$

which is not the case for other types of limits considered in [1] and [3]. In other words, suppose system (1.4) is of mixed type and does not allow for a stability criterion (which would be available if (1.4) were hyperbolic). Then our method would still allow for convergence to the equilibrium system (1.3).

More precisely, we have the following.

Theorem 1.1. *Suppose that the solution sequence* $(u^{\epsilon,\tau}, v^{\epsilon,\tau}, s^{\epsilon,\tau})$ *of the Cauchy problem* (1.1), (1.2) *is uniformly bounded in* L^∞ *with respect to the parameters* ϵ, τ. *If system* (1.3) *is strictly hyperbolic, genuinely nonlinear and the function* $h(u,v)^2$ *is an entropy of system* (1.3)*, then there exists a subsequence (still denoted by* $(u^{\epsilon,\tau}, v^{\epsilon,\tau}, s^{\epsilon,\tau})$*) such that*

$$(u^{\epsilon,\tau}, v^{\epsilon,\tau}, s^{\epsilon,\tau}) \to (u,v,s), a.e. \tag{1.5}$$

as ϵ *and* τ *go to zero, where*

$$s = h(u,v), a.e. \tag{1.6}$$

and (u,v) satisfies system (1.3) *in the sense of distributions.*

2. Proof of Theorem 1.1

Since system (1.3) is strictly hyperbolic and genuinely nonlinear in the sense of Lax, by using DiPerna's framework [2] and the theory of compensated compactness, we could prove that $(u^{\epsilon,\tau}, v^{\epsilon,\tau}) \to (u,v)$, a.e. as ϵ and τ go to zero if we could prove that

$$\eta(u^{\epsilon,\tau}, v^{\epsilon,\tau})_t + q(u^{\epsilon,\tau}, v^{\epsilon,\tau})_x \text{ are compact in } H^{-1}_{loc}(R \times R^+), \tag{2.1}$$

where $(\eta(u,v), q(u,v))$ is any pair of entropy-entropy flux of system (1.3).

We can use the estimates in Lemma 2.1 to get the proof of (2.1).

Lemma 2.1. *If the conditions in Theorem* 1.1 *are satisfied, then* $\epsilon(u^{\epsilon,\tau})^2, \epsilon(v^{\epsilon,\tau})^2$, $\epsilon(s^{\epsilon,\tau})^2$ *and* $(s^{\epsilon,\tau} - h(u^{\epsilon,\tau}, v^{\epsilon,\tau}))^2/\tau$ *are uniformly bounded in* $L^1_{loc}(R \times R^+)$ *with respect to* ϵ, τ.

In fact, if the conclusions in Lemma 2.1 are true, we can rewrite the first two equations in (1.1) by

$$\begin{cases} u_t + f(u,v,h(u,v))_x = (f(u,v,h(u,v)) - f(u,v,s))_x + \epsilon u_{xx} \\ v_t + g(u,v,h(u,v))_x = (g(u,v,h(u,v)) - g(u,v,s))_x + \epsilon v_{xx}. \end{cases} \tag{2.2}$$

Then for any pair of entropy-entropy flux $(\eta(u,v),q(u,v))$ of system (1.3), we have

$$\begin{aligned}\eta(u,v)_t + q(u,v)_x &\\ &= \eta_u(u,v)(f(u,v,h(u,v)) - f(u,v,s))_x\\ &\quad + \eta_v(u,v)(g(u,v,h(u,v)) - g(u,v,s))_x\\ &\quad + \epsilon\eta_u(u,v)u_{xx} + \epsilon\eta_v(u,v)v_{xx}\\ &= \epsilon\eta(u,v)_{xx} - \epsilon(\eta_{uu}u_x^2 + 2\eta_{uv}u_xv_x + \eta_{vv}v_x^2)\\ &\quad + [\eta_u(u,v)(f(u,v,h(u,v)) - f(u,v,s))]_x\\ &\quad + [\eta_v(u,v)(g(u,v,h(u,v)) - g(u,v,s))]_x\\ &\quad + \eta_{ux}(u,v)f_s(u,v,\beta_1)(s-h(u,v))\\ &\quad + \eta_{vx}g_s(u,v,\beta_2)(s-h(u,v))\\ &= I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned} \tag{2.3}$$

where $\beta_i, i = 1,2$, are some values between s and $h(u,v)$. From the estimates in Lemma 2, we can easily prove that I_1, I_3 are compact in $H_{loc}^{-1}(R\times R^+)$, and I_2, I_4 and I_5 are bounded in $L_{loc}^1(R\times R^+)$. Thus the right-hand side of (2.3) is compact in $W_{loc}^{-1,q}(R\times R^+)$ for some exponent q, where $1 < q < 2$. But the left-hand side of (2.3) is bounded in $W^{-1,\infty}$. Therefore, by using Murat's Lemma, we get the proof of (2.1). Since $(u^{\epsilon,\tau}, v^{\epsilon,\tau}) \to (u,v)$, a.e., using the last estimate in Lemma 2.1, we have the convergence $s^{\epsilon,\tau} \to s$, a.e. as ϵ goes to zero. So the proof of Theorem 1.1 is ended.

Proof of Lemma 2.1. Since system (1.3) is strictly hyperbolic and genuinely nonlinear, then there exists a strictly convex entropy $\eta_1(u,v)$. Since (u,v,s) is bounded, we can choose a large constant C_1 such that the function

$$p(u,v,s) = \frac{s^2}{2} - h(u,v)s + C_1\eta_1(u,v)$$

satisfies

$$\begin{aligned}&p_{uu}(u,v,s)u_x^2 + p_{vv}(u,v,s)v_x^2 + p_{ss}(u,v,s)s_x^2\\ &\qquad + 2p_{uv}(u,v,s)u_xv_x + 2p_{us}(u,v,s)u_xs_x + 2p_{vs}(u,v,s)v_xs_x\\ &\quad \geq C_2(u_x^2 + v_x^2 + s_x^2),\end{aligned} \tag{2.4}$$

for some constant $C_2 > 0$.

Multiplying system (1.1) by (p_u, p_v, p_s), we have from (2.4) that

$$\begin{aligned}&p(u,v,s)_t + p_u(u,v,s)f(u,v,s)_x + p_v(u,v,s)g(u,v,s)_x\\ &\quad + \frac{(s-h(u,v))^2}{\tau} + \epsilon C_2(u_x^2 + v_x^2 + s_x^2)\\ &\leq \epsilon p_{xx}(u,v,s).\end{aligned} \tag{2.5}$$

Since

$$
\begin{aligned}
& p_u(u,v,s)f(u,v,s)_x + p_v(u,v,s)g(u,v,s)_x \\
= \;& (C_1\eta_{1u} - h_u(u,v)s)f(u,v,s)_x + (C_1\eta_{1v} - h_u(u,v)s)g(u,v,s)_x \\
= \;& (C_1\eta_{1u} - h_u(u,v)h(u,v))(f(u,v,s) - f(u,v,h(u,v)))_x \\
+ \;& (C_1\eta_{1u} - h_u(u,v)h(u,v))f(u,v,h(u,v))_x \\
+ \;& (C_1\eta_{1v} - h_v(u,v)h(u,v))(g(u,v,s) - g(u,v,h(u,v)))_x \\
+ \;& (C_1\eta_{1v} - h_v(u,v)h(u,v))g(u,v,h(u,v))_x \\
- \;& h_u(u,v)(s-h(u,v))f(u,v,s)_x - h_v(u,v)(s-h(u,v))g(u,v,s)_x \\
= \;& [(C_1\eta_{1u} - h_u(u,v)h(u,v))(f(u,v,s) - f(u,v,h(u,v)))]_x \\
+ \;& [(C_1\eta_{1v} - h_v(u,v)h(u,v))(g(u,v,s) - g(u,v,h(u,v)))]_x \\
- \;& (C_1\eta_{1u} - h_u(u,v)h(u,v))_x f_s(u,v,\beta_3)(s-h(u,v)) \\
- \;& (C_1\eta_{1v} - h_v(u,v)h(u,v))_x g_s(u,v,\beta_4)(s-h(u,v)) \\
+ \;& (C_1 q_1(u,v) - q_2(u,v))_x,
\end{aligned} \tag{2.6}
$$

where q_1, q_2 are two entropy fluxes of system (1.3) corresponding to the entropies η_1 and $h^2(u,v)/2$, and $\beta_i, i = 3, 4$ are some values between s and $h(u,v)$. If τ is much smaller than ϵ, then we have from (2.5) and (2.6) that

$$
\begin{aligned}
& p(u,v,s)_t + F(u,v,s)_x + C_3 \tfrac{(s-h(u,v))^2}{\tau} \\
& + \epsilon C_4(u_x^2 + v_x^2 + s_x^2) \leq \epsilon p_{xx}(u,v,s),
\end{aligned} \tag{2.7}
$$

for a function $F(u,v,s)$ and positive constants C_3 and C_4 depending on the bounds of the second derivatives of p and the first derivatives of f and g.

Multiplying (2.7) by a suitable nonnegative test function and then integrating by parts on $R \times R^+$, we get the estimates in Lemma 2.1.

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