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STABLE MINIMAL SURFACES IN $\mathbf{R}^4$ WITH DEGENERATE GAUSS MAP

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ABSTRACT. A complete oriented stable minimal surface in $\mathbf{R}^3$ is a plane, but in $\mathbf{R}^4$, there are many non-flat examples such as holomorphic curves. The Gauss map plays an important role in the theory of minimal surfaces. In this paper, we prove that a complete oriented stable minimal surface in $\mathbf{R}^4$ with α-degenerate Gauss map (for $\alpha > 1/4$) is a plane.

1. INTRODUCTION

A minimal surface is called stable if (and only if) the second variation of the area functional is nonnegative for all compactly supported deformations. A classification theorem for complete stable minimal surfaces in three-dimensional Riemannian manifolds of nonnegative scalar curvature has been obtained by Fischer-Colbrie and Schoen [3]. As a corollary they showed that all complete oriented stable minimal surfaces in $\mathbf{R}^3$ are planes. This was also proved by do Carmo and Peng [2]. In the higher-codimensional case, Wirtinger showed that a holomorphic curve in $\mathbf{C}^n$ is absolutely area minimizing [7]. Micallef considered the converse problem [6] and proved that any complete oriented parabolic stable minimal surface in $\mathbf{R}^4$ is holomorphic with respect to some orthogonal complex structure on $\mathbf{R}^4$. He also proved that any complete oriented stable minimal surface with at least 1/3-degenerate Gauss map in $\mathbf{R}^4$ is a plane. On the other hand, Lawson gave a characterization of the Gauss map for holomorphic curves in $\mathbf{C}^n$ [4]. According to this, the 0-degeneracy of the Gauss map is equivalent to the property that the surface is a holomorphic curve in $\mathbf{C}^2$ $(= \mathbf{R}^4)$. In view of this, it is reasonable to ask whether this 1/3-degeneracy is sharp, or more strongly, whether any complete oriented stable minimal surface with degenerate Gauss map is a holomorphic curve or a plane. We consider this problem and prove that any complete oriented stable minimal surface with α-degenerate Gauss map (for $\alpha > 1/4$) in $\mathbf{R}^4$ is a plane (Main Theorem).

The paper is divided into four sections. In §2 we review Fischer-Colbrie and Schoen's results on the operator $\triangle - q$ on a complete Riemannian manifold M of arbitrary dimension, and we prove the nonexistence of a positive solution u of $\triangle u - aKu = 0$ for $a > 1/2$ on the unit disk with complete metric (Theorem 2.3). This essentially improves Fischer-Colbrie and Schoen's nonexistence theorem for $a \geq 1$ (Theorem 2.2). In §3 we introduce the generalized Gauss map and the

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degeneracy given by Hoffman and Osserman [5]. In §4 we prove our main theorem. For the proof, there are two key ingredients: the nonexistence theorem given by Theorem 2.3, and a deformation of the stability inequality obtained in §4.

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2. Fundamental facts for the operator $\triangle - q$

Let (M, ds^2) be an n-dimensional complete noncompact Riemannian manifold, and let q be a smooth function on M. Given any bounded domain $D \subset M$, we let $\lambda_1(D) < \lambda_2(D) \leq \lambda_3(D) \leq \ldots$ be the sequence of eigenvalues of $\triangle - q$ acting on functions vanishing on ∂D. The usual variational characterization of $\lambda_1(D)$ is

$$\lambda_1(D) = \inf \left\{ \int_M (|\nabla f|^2 + qf^2) dv \,|\, \operatorname{supp} f \subset D, \int_M f^2 dv = 1 \right\}, \tag{1}$$

where dv denotes the volume element of M with respect to ds^2.

Fischer-Colbrie and Schoen proved the following results.

Theorem 2.1. [3] *The following conditions are equivalent:*
(1) $\lambda_1(D) \geq 0$ *for every bounded domain* $D \subset M$*;*
(2) $\lambda_1(D) > 0$ *for every bounded domain* $D \subset M$*;*
(3) *there exists a positive solution* u *satisfying the equation* $\triangle u - qu = 0$ *on* M*.*

Theorem 2.2. [3] *Let* (M, ds^2) *be the unit disk endowed with the complete metric* $ds^2 = \lambda(z)\,|dz|^2$*. Let* K *denote the Gaussian curvature of* M*. For* $a \geq 1$*, there is no positive* u *satisfying* $\triangle u - aKu = 0$ *on* M*.*

Next we prove our main nonexistence theorem.

Theorem 2.3. *Let* (M, ds^2) *be the unit disk with complete metric. Then there exists no positive* u *satisfying* $\triangle u - aKu = 0$ *for* $a > 1/2$*.*

Proof. Since the case $a \geq 1$ is given in Theorem 2.2, we may assume $1/2 < a < 1$. Denote the metric by $ds^2 = \lambda(z)|dz|^2$ in a local complex coordinate $z = x + iy$ on M. Put $h = \lambda^{-1/2}$. As is well known, the Gaussian curvature is given by $K = \triangle \log \lambda^{-1/2}$. Thus,

$$K = \triangle \log h = \frac{\triangle h}{h} - \frac{|\nabla h|^2}{h^2}. \tag{2}$$

Let $D \subset M$ be a bounded domain, and let ζ be a smooth function on M with a compact support in D. We now calculate

$$
\begin{aligned}
(3)\qquad & \int_M (|\nabla(\zeta h)|^2 + aK(\zeta h)^2)dv \\
&= \int_M (|\nabla\zeta|^2 h^2 + 2\zeta h(\nabla\zeta\cdot\nabla h) + \zeta^2|\nabla h|^2 + a(\zeta^2 h\triangle h - \zeta^2|\nabla h|^2)dv \\
&= \int_M (|\nabla\zeta|^2 h^2 + 2\zeta h(\nabla\zeta\cdot\nabla h) + \zeta^2|\nabla h|^2 \\
&\qquad\qquad - 2a\zeta h(\nabla\zeta\cdot\nabla h) - a\zeta^2|\nabla h|^2 - a\zeta^2|\nabla h|^2)dv \\
&= \int_M (|\nabla\zeta|^2 h^2 + (1-2a)\zeta^2|\nabla h|^2 + 2(1-a)\zeta h(\nabla\zeta\cdot\nabla h))dv \\
&\le \int_M (|\nabla\zeta|^2 h^2 + (1-2a)\zeta^2|\nabla h|^2 + (1-a)(\varepsilon\zeta^2|\nabla h|^2 + \frac{1}{\varepsilon}h^2|\nabla\zeta|^2))dv \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(for any } \varepsilon > 0) \\
&= (1 + \frac{1-a}{\varepsilon})\int_M |\nabla\zeta|^2 h^2 dv + (1 - 2a + (1-a)\varepsilon)\int_M \zeta^2|\nabla h|^2 dv,
\end{aligned}
$$

where the first equality is due to (2), the second equality is by the integration by parts and the inequality follows from the Schwarz inequality and the arithmetic-geometric mean inequality. Because of (1) and (3), we obtain

$$
\begin{aligned}
(4)\qquad \lambda_1(D)\int_M (\zeta h)^2 dv &+ (2a - 1 + (a-1)\varepsilon)\int_M \zeta^2|\nabla h|^2 dv \\
&\le (1 + \frac{1-a}{\varepsilon})\int_M |\nabla\zeta|^2 h^2 dv.
\end{aligned}
$$

Now we can define a smooth function $\zeta(r)$ for $r \in \mathbf{R}$ that satisfies

$$
\begin{aligned}
(5)\qquad \zeta(r) &\equiv 1 \quad \text{for} \quad r \le \frac{1}{2}R, \\
\zeta(r) &\equiv 0 \quad \text{for} \quad r \ge R, \\
|\zeta'| &\le \frac{C}{R} \quad \text{for all} \quad r,
\end{aligned}
$$

where r measures the metric distance to any $P \in M$, R is any positive number and C is a constant independent of R.

Since $a \in (1/2, 1)$, we can choose $\varepsilon > 0$ arbitrarily small so that

$$(a-1)\varepsilon + 2a - 1 > 0, \quad 1 + \frac{1-a}{\varepsilon} > 0,$$

By (4) and (5), we obtain

$$
\begin{aligned}
(6)\qquad \lambda_1(B_R(P))\int_M (\zeta h)^2 dv &+ (2a - 1 + (a-1)\varepsilon)\int_M \zeta^2|\nabla h|^2 dv \\
&\le (1 + \frac{1-a}{\varepsilon})\int_M |\nabla\zeta|^2 dx\,dy \le (1 + \frac{1-a}{\varepsilon})\frac{C^2}{R^2}\pi,
\end{aligned}
$$

where $B_R(P)$ is the geodesic ball of radius R, center at P, and we use $\int_M dx\,dy = \pi$. Since ds^2 is the complete metric, $|\nabla h|$ is not identically zero on M. Hence we conclude that $\lambda_1(B_R(P)) < 0$ by choosing R sufficiently large in (6). By Theorem

2.1, this implies that there is no positive solution u of $\triangle u - aKu = 0$ for $1/2 < a < 1$ on M. □

3. The generalized Gauss map and the degeneracy

Let $G_{n,m}$ denote the Grassmannian of oriented m-planes in $\mathbf{R}^n$. Let $F : M \longrightarrow \mathbf{R}^n$ be an isometric immersion of a real m-dimensional oriented manifold into the Euclidean space $\mathbf{R}^n$, $2 \leq m \leq n-1$. The generalized Gauss map $G : M^m \longrightarrow G_{n,m}$ is defined by $G(P) = F_*(T_P M)$, which is obtained by a parallel translation of the tangent space $T_P M$ to the origin of $\mathbf{R}^n$.

We now focus on the surface case $m = 2$ and recall that $G_{n,2}$ is identified with the quadric $Q_{n-2} \subset \mathbf{C}P^{n-1}$ defined by $\{[w] \in \mathbf{C}P^{n-1} \mid w \cdot w = \sum_i (w^i)^2 = 0\}$, where " $\cdot$ " is the complex bilinear inner product. If z is a local complex coordinate on M, then $F_z(P)$ is a homogeneous coordinate for $G(P)$. If the Gauss image lies in a hyperplane of $\mathbf{C}P^{n-1}$, that is, if there exists a nonzero vector $A \in \mathbf{C}^n$ (and can be considered as $[A] \in \mathbf{C}P^{n-1}$) such that $A \cdot F_z \equiv 0$, we call the Gauss map degenerate. In this case, we can define $\alpha := |A \cdot A|/|A|^2 \in [0,1]$ and call it an α-degenerate Gauss map. We can normalize this nonzero vector $[A] \in \mathbf{C}P^{n-1}$ as follows.

Lemma 3.1. ([5, p. 28 Proposition 2.4]) *To each point $[A] = [a_1 : \cdots : a_n] \in \boldsymbol{C}P^{n-1}, n \geq 3$ one may assign a real number t lying in the interval $0 \leq t \leq 1$ with the following properties:*
(1) *$[A]$ is equivalent under the action of $SO(n)$ to $[t : i : 0 : \cdots : 0]$;*
(2) *$t = 0 \Longleftrightarrow [A]$ is a real vector (i.e., $[a_1 : \cdots : a_n] = \lambda[r_1 : \cdots : r_n]$, $\lambda \in \boldsymbol{C}$, $r_i \in \boldsymbol{R}$, $i = 1, \cdots, n$);*
(3) *$t = 1 \Longleftrightarrow [A] \in Q_{n-2}$;*
(4) *if t, t' correspond to vectors $[A], [A']$, then $[A]$ and $[A']$ are equivalent under $SO(n)$ if and only if $t = t'$.*

The minimal surface lies fully in $\mathbf{R}^n$ if the image $F(M)$ does not lie in any proper affine subspace of $\mathbf{R}^n$. In $\mathbf{R}^3$, we know the following:

Theorem 3.1. ([5, p. 52 Proposition 4.2]) *Let $F : M \longrightarrow \boldsymbol{R}^3$ be an isometric minimal immersion of an oriented surface M. The following are equivalent:*
(1) *the Gauss map is degenerate;*
(2) *$F(M)$ does not lie fully in $\boldsymbol{R}^3$;*
(3) *$F(M)$ lies on a plane.*

By [3] Corollary 4 or [2], the only complete oriented stable minimal surface in $\mathbf{R}^3$ is a plane. Therefore the stability and the degeneracy in $\mathbf{R}^3$ are equivalent. On the other hand, the relation between the stable regions on a minimal surface M in $\mathbf{R}^3$ and the area of their Gauss image has been studied by Barbosa and do Carmo [1]. They proved that if the area of the Gauss image is smaller than 2π, then the domain is stable. In the general case, there are few results on the relation between the stability and the degeneracy.

4. The main result

In this section, we consider the problem: Is a complete oriented stable minimal surface in $\mathbf{R}^4$ with degenerate Gauss map a holomorphic curve or a plane? Let $F : M \longrightarrow \mathbf{R}^4$ be an isometric immersion of an oriented surface M into Euclidean

4-space. Let $\{e_1, e_2, e_3, e_4\}$ be a local oriented orthonormal frame for $\mathbf{R}^4$ on M such that $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are local oriented orthonormal frames for the tangent and normal bundles of M, respectively. In this case, TM and NM can each be given a complex structure, namely, rotation by $\pi/2$ in an anticlockwise direction. We obtain the decomposition $N_{\mathbf{C}}M = N_{\mathbf{C}}M^{(1,0)} \oplus N_{\mathbf{C}}M^{(0,1)}$ with respect to the complex structure just mentioned. If v is any vector in $\mathbf{C}^4$, let $v^{1,0}$ and $v^{0,1}$ denote the orthogonal projection of v onto $N_{\mathbf{C}}M^{(1,0)}$ and $N_{\mathbf{C}}M^{(0,1)}$, respectively. Let D denote the covariant differentiation in $N_{\mathbf{C}}M$, $z = x + iy$ be a local complex coordinate, and put $\partial_z = (1/2)(\partial/\partial x - i\partial/\partial y)$. Let $\varepsilon = (e_3 - ie_4)/\sqrt{2}$. Micallef proved the following:

Theorem 4.1. ([6] Theorem III) *Let $F : M \longrightarrow \boldsymbol{R}^4$ be an isometric stable minimal immersion of a complete oriented surface M. If the Gauss map for F is at least $1/3$-degenerate, then the image of F is a plane.*

Lawson gave the following characterization of the Gauss map for holomorphic curves in $\mathbf{C}^n$:

Theorem 4.2. ([4, p. 165, Proposition 16]) *Let $F : M \longrightarrow \boldsymbol{R}^{2n}$ be a minimal immersion of an oriented surface M with associated Gauss map $G : M \longrightarrow Q_{2n-2}$. Then there exists an orthogonal complex structure on $\boldsymbol{R}^{2n}$ with respect to which F is holomorphic if and only if the Gauss image $G(M)$ lies in a linear subspace of Q_{2n-2}.*

Let $F : M \longrightarrow \mathbf{R}^4$ be an isometric stable minimal immersion of a complete oriented surface M with α-degenerate Gauss map. If $\alpha = 0$, then $t = 1$ in Theorem 3.1 and thus $[A] \in Q_2$. Therefore F is holomorphic with respect to some orthogonal complex structure on $\mathbf{R}^4$ without the stability condition, by Theorem 4.2. If $\alpha \geq 1/3$, then $F(M)$ is a plane by Theorem 4.1. In view of this, it is natural to ask what happens for $\alpha \in (0, 1/3)$. Our result is:

Main Theorem. *Let $F : M \longrightarrow \boldsymbol{R}^4$ be an isometric stable minimal immersion of a complete oriented surface M. If the Gauss map is α-degenerate (for $\alpha > 1/4$), then the image of F is a plane.*

In this case, by the stability inequality [6], we obtain

$$(7) \qquad 2\int_M f^2 \frac{|\sigma \cdot F_{zz}|^2}{|F_z|^4} dv + 2\int_M \frac{f^2}{|F_z|^2} \mathrm{Re}(\bar{\sigma} \cdot D_z D_{\bar{z}} \sigma) dv \leq \int_M |df|^2 |\sigma|^2 dv,$$

where f is a smooth real-valued function with a compact support and σ is a complex-valued normal section that needs not have a compact support.

We assume the Gauss map is α-degenerate ($\alpha \neq 0$) and put $s := A^{1,0}$, $t := A^{0,1}$ and $\sigma = |t|(s - \bar{s})$. Because $s = (A \cdot \bar{\varepsilon})\,\varepsilon$ and $t = (A \cdot \varepsilon)\,\bar{\varepsilon}$, putting $De_3 = \omega_3^4 \otimes e_4$ and $D\varepsilon = i\omega_3^4 \otimes \varepsilon$, we have

$$(8) \qquad \begin{aligned} D_{\bar{z}} s &= (A \cdot \partial_{\bar{z}} \bar{\varepsilon}^T)\,\varepsilon + (A \cdot (-i\omega_3^4(\partial_{\bar{z}})\bar{\varepsilon}))\varepsilon + (A \cdot \bar{\varepsilon})\, i\omega_3^4(\partial_{\bar{z}})\varepsilon \\ &= (A \cdot \partial_{\bar{z}} \bar{\varepsilon}^T)\varepsilon. \end{aligned}$$

It is easy to verify from minimality that

$$(9) \qquad \partial_{\bar{z}} \bar{\varepsilon}^T = -\frac{1}{|F_z|^2} (F_{\bar{z}\bar{z}} \cdot \bar{\varepsilon})\, F_z,$$

$$(10) \qquad \partial_z \bar{\varepsilon}^T = -\frac{1}{|F_z|^2} (F_{zz} \cdot \bar{\varepsilon})\, F_{\bar{z}}.$$

By (8), (9) and the degeneracy $A \cdot F_z \equiv 0$, we obtain

$$D_{\bar{z}} s = 0, \tag{11}$$

and similarly,

$$D_{\bar{z}} t = 0. \tag{12}$$

Thus s and t are holomorphic. Moreover, from

$$A = \frac{1}{|F_z|^2}(A \cdot F_{\bar{z}})\, F_z + (A \cdot \bar{\varepsilon})\, \varepsilon + (A \cdot \varepsilon)\, \bar{\varepsilon}, \tag{13}$$

we obtain

$$A \cdot A = 2(A \cdot \bar{\varepsilon})(A \cdot \varepsilon) = 2s \cdot t, \tag{14}$$

$$\alpha |A|^2 = |A \cdot A| = 2|s||t|, \tag{15}$$

and

$$\begin{aligned} |A|^2 &= \frac{1}{|F_z|^2}|A \cdot F_{\bar{z}}|^2 + |s|^2 + |t|^2 \\ &= \frac{1}{|F_z|^2}|A \cdot F_{\bar{z}}|^2 + |s|^2 + \frac{|A \cdot A|^2}{4|s|^2}. \end{aligned} \tag{16}$$

In particular, $A \cdot \bar{\varepsilon}$ and $A \cdot \varepsilon$ never vanish, that is, s and t never vanish since $\alpha \neq 0$. Because $\sigma = |t|(s - \bar{s}) = |t|((A \cdot \bar{\varepsilon})\, \varepsilon - (\bar{A} \cdot \varepsilon)\, \bar{\varepsilon})$, we obtain from (15),

$$|\sigma|^2 = 2|s|^2|t|^2 = \frac{|A \cdot A|^2}{2} \tag{17}$$

and

$$\begin{aligned} |\sigma \cdot F_{zz}|^2 &= |t|^2 |(A \cdot \bar{\varepsilon})(F_{zz} \cdot \varepsilon) - (\bar{A} \cdot \varepsilon)(F_{zz} \cdot \bar{\varepsilon})|^2 \\ &= |t|^2 \{(A \cdot \bar{\varepsilon})(F_{zz} \cdot \varepsilon) - (\bar{A} \cdot \varepsilon)(F_{zz} \cdot \bar{\varepsilon})\} \\ &\qquad \times \{(\bar{A} \cdot \varepsilon)(F_{\bar{z}\bar{z}} \cdot \bar{\varepsilon}) - (A \cdot \bar{\varepsilon})(F_{\bar{z}\bar{z}} \cdot \varepsilon)\} \\ &= |t|^2 \{|s|^2 |F_{zz} \cdot \varepsilon|^2 + |s|^2 |F_{zz} \cdot \bar{\varepsilon}|^2 \\ &\qquad - 2\mathrm{Re}\{(A \cdot \bar{\varepsilon})(A \cdot \bar{\varepsilon})(F_{zz} \cdot \varepsilon)(F_{\bar{z}\bar{z}} \cdot \varepsilon)\}\}. \end{aligned} \tag{18}$$

On the other hand, $A \cdot F_z \equiv 0$ implies $A \cdot F_{zz} \equiv 0$; hence from

$$F_{zz} 3D F_{zz}^N + \frac{1}{|F_z|^2}(F_{zz} \cdot F_{\bar{z}})\, F_z,$$

follows $A \cdot F_{zz}^N \equiv 0$, and we obtain

$$(A \cdot \bar{\varepsilon})(F_{zz} \cdot \varepsilon) + (A \cdot \varepsilon)(F_{zz} \cdot \bar{\varepsilon}) \equiv 0. \tag{19}$$

From these we get

$$\begin{aligned} |\sigma \cdot F_{zz}|^2 &= |t|^2 \{|s|^2 (|F_{zz}^{1,0}|^2 + |F_{zz}^{0,1}|^2) + 2\mathrm{Re}\{(A \cdot \varepsilon)(A \cdot \bar{\varepsilon})\}|F_{zz}^{1,0}|^2\} \\ &= |t|^2 |s|^2 |F_{zz}^N|^2 + |t|^2 \mathrm{Re}(A \cdot A)|F_{zz}^{1,0}|^2 \\ &= \frac{|A \cdot A|^2 |F_z|^4 (-K)}{4} + |t|^2 \mathrm{Re}(A \cdot A)|F_{zz}^{1,0}|^2, \end{aligned} \tag{20}$$

where we use the Gauss equation, $|F_{zz}^N|^2 = |F_z|^4(-K)$, (14) and (15). By (11) and (12), we can express $D_z s$ and $D_z t$ as follows:

$$D_z s = \frac{D_z s \cdot \bar{s}}{|s|^2} s = (\log |s|^2)_z s, \tag{21}$$

$$D_z t = \frac{D_z t \cdot \bar{t}}{|t|^2} t = (\log |t|^2)_z t. \tag{22}$$

Thus, we obtain

$$\begin{aligned} D_z D_{\bar{z}} \sigma &= D_z\{|t|_{\bar{z}}(s - \bar{s}) - |t|(\log |s|^2)_{\bar{z}} \bar{s}\} \\ &= |t|_{\bar{z}z}(s - \bar{s}) + |t|_{\bar{z}}(\log |s|^2)_z s - |t|_z(\log |s|^2)_{\bar{z}} \bar{s} - |t|(\log |s|^2)_{\bar{z}z} \bar{s} \end{aligned}$$

and

$$\begin{aligned} \bar{\sigma} \cdot D_z D_{\bar{z}} \sigma &= |t|(\bar{s} - s) \cdot D_z D_{\bar{z}} \sigma \\ &= 2|t||s|^2|t|_{\bar{z}z} + 2|t||s||t|_{\bar{z}}|s|_z + 2|t||s||s|_{\bar{z}}|t|_z \\ &\quad + 2|t|^2|s||s|_{\bar{z}z} - 2|t|^2|s|_{\bar{z}}|s|_z. \end{aligned} \tag{23}$$

On the other hand, from (15) follows $|s|_z|t| + |s||t|_z = 0$ and

$$|s|_{\bar{z}z}|t| + |s|_z|t|_{\bar{z}} + |s|_{\bar{z}}|t|_z + |s||t|_{z\bar{z}} = 0. \tag{24}$$

Therefore, (23) can be reduced to

$$\bar{\sigma} \cdot D_z D_{\bar{z}} \sigma = -2|t|^2|s|_{\bar{z}}|s|_z. \tag{25}$$

Combining

$$\begin{aligned} 2|s||s|_z = (|s|^2)_z &= \partial_z\{(A \cdot \bar{\varepsilon})(\bar{A} \cdot \varepsilon)\} \\ &= (A \cdot \partial_z \bar{\varepsilon}^T)(\bar{A} \cdot \varepsilon) + (A \cdot \bar{\varepsilon})(\bar{A} \cdot \partial_z \varepsilon^T) \end{aligned}$$

with (9), (10) and by the degeneracy, we obtain

$$|s|_z = -\frac{(F_{zz} \cdot \bar{\varepsilon})(A \cdot F_{\bar{z}})(\bar{A} \cdot \varepsilon)}{2|s||F_z|^2}. \tag{26}$$

By (16) and (26), (25) is deformed into

$$\bar{\sigma} \cdot D_z D_{\bar{z}} \sigma = -\frac{|t|^2|F_{zz}^{1,0}|^2}{2|F_z|^2}(|A|^2 - |s|^2 - \frac{|A \cdot A|^2}{4|s|^2}). \tag{27}$$

Therefore, combining (20), (27) and (17), we can rewrite (7) as

$$\begin{aligned} 2\int_M f^2 \Bigg\{ \frac{|A \cdot A|^2(-K)}{4} &+ \frac{|t|^2|F_{zz}^{1,0}|^2}{|F_z|^4} \mathrm{Re}(A \cdot A) \\ &- \frac{|t|^2|F_{zz}^{1,0}|^2}{2|F_z|^4}(|A|^2 - |s|^2 - \frac{|A \cdot A|^2}{4|s|^2}) \Bigg\} dv \le \frac{|A \cdot A|^2}{2} \int_M |df|^2 dv, \end{aligned}$$

and moreover by (15), we obtain

$$\begin{aligned} \int_M f^2(-K) dv + \int_M \frac{f^2|F_{zz}^{1,0}|^2}{|F_z|^4|s|^2}\{\mathrm{Re}(A \cdot A) - \frac{|A|^2}{2} + \frac{|s|^2}{2} + \frac{|A \cdot A|^2}{8|s|^2}\} dv \\ \le \int_M |df|^2 dv. \end{aligned} \tag{28}$$

On the other hand, from (19) follows

$$|s|^2|F_{zz}^N|^2 = |s|^2|F_{zz}^{1,0}|^2 + |s|^2|F_{zz}^{0,1}|^2 = (|s|^2 + |t|^2)|F_{zz}^{1,0}|^2,$$

and hence

$$\frac{|F_{zz}^{1,0}|^2}{|F_z|^4|s|^2} = \frac{1}{|s|^2+|t|^2}\frac{|F_{zz}^N|^2}{|F_z|^4} = \frac{-K}{|s|^2+|t|^2}. \tag{29}$$

Thus we obtain from (28) and (29),

$$\int_M f^2\{1+\frac{|A\cdot A| - \frac{|A|^2}{2} + \frac{|s|^2}{2} + \frac{|A\cdot A|^2}{8|s|^2}}{|s|^2+|t|^2}\}(-K)dv \le \int_M |df|^2 dv, \tag{30}$$

where we choose $A \in \mathbf{C}^4$ such that $\mathrm{Re}(A\cdot A) = |A\cdot A|$ by Lemma 3.1.

Proof of Main Theorem. Since (30) holds for all real-valued functions f with compact support, there exists a positive solution u satisfying the equation

$$\triangle u + \left\{1+\frac{|A\cdot A| - \frac{|A|^2}{2} + \frac{|s|^2}{2} + \frac{|A\cdot A|^2}{8|s|^2}}{|s|^2+|t|^2}\right\}(-K)u = 0, \tag{31}$$

by Theorem 2.1. The lift of u to the universal covering $\overline{M}$ of M satisfies the same equation as u. Suppose that $\overline{M}$ is the unit disk. In order to compare the coefficient of $-Ku$ with $\frac{1}{2}$, we calculate

$$\begin{aligned}
&\left\{1+\frac{|A\cdot A| - \frac{|A|^2}{2} + \frac{|s|^2}{2} + \frac{|A\cdot A|^2}{8|s|^2}}{|s|^2+|t|^2}\right\} - \frac{1}{2}\\
&= \frac{1}{|s|^2+|t|^2}\{(1-\frac{1}{2})(|s|^2+|t|^2) + |A\cdot A| - \frac{|A|^2}{2} + \frac{|s|^2}{2} + \frac{|A\cdot A|^2}{8|s|^2}\}\\
&= \frac{1}{|s|^2+|t|^2}\{\frac{1}{2}(|s|^2 + \frac{|A\cdot A|^2}{4|s|^2}) + |A\cdot A| - \frac{|A|^2}{2} + \frac{|s|^2}{2} + \frac{|A\cdot A|^2}{8|s|^2}\}\\
&= \frac{1}{|s|^2+|t|^2}\{|s|^2 + \frac{|A\cdot A|^2}{4|s|^2} + |A\cdot A| - \frac{|A|^2}{2}\}\\
&> \frac{1}{|s|^2+|t|^2}\{|s|^2 + \frac{|A|^4}{4|s|^2}\frac{1}{16} + \frac{|A|^2}{4} - \frac{|A|^2}{2}\}\\
&= \frac{1}{|s|^2+|t|^2}\{|s|^2 - \frac{|A|^2}{4} + \frac{|A|^4}{64|s|^2}\}\\
&= \frac{1}{|s|^2+|t|^2}(|s| - \frac{|A|^2}{8|s|})^2 \ \ge\ 0,
\end{aligned}$$

where we use (15) in the second equality and the assumption $|A\cdot A| > |A|^2/4$ in the first inequality. By Theorem 2.3, there is no positive solution u satisfying (31). Therefore $\overline{M}$ is the complex plane $\mathbf{C}$. But since the coefficient of u in (31) is nonnegative, u is a positive superharmonic function on $\mathbf{C}$. Thus by the parabolicity of $\mathbf{C}$, u must be constant and therefore $K \equiv 0$. This completes the proof. □

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