PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 5, Pages 1333–1344
S 0002-9939(03)07348-9
Article electronically published on December 22, 2003

# GLOBAL HÖLDER REGULARITY FOR DISCONTINUOUS ELLIPTIC EQUATIONS IN THE PLANE

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(Communicated by David S. Tartakoff)

ABSTRACT. $C^{1,\mu}$-regularity up to the boundary is proved for solutions of boundary value problems for elliptic equations with discontinuous coefficients in the plane.

In particular, we deal with the Dirichlet boundary condition

$$u = g(x) \quad \text{ on } \partial\Omega$$

where $g(x) \in W^{2-\frac{1}{r},r}(\partial\Omega)$, $r > 2$, or with the following normal derivative boundary conditions:

$$\frac{\partial u}{\partial n} = h(x) \quad \text{ or } \quad \frac{\partial u}{\partial n} + \sigma u = h(x) \quad \text{ on } \partial\Omega$$

where $h(x) \in W^{1-\frac{1}{r},r}(\partial\Omega)$, $r > 2$, $\sigma > 0$ and $n$ is the unit outward normal to the boundary $\partial\Omega$.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper we are concerned with the study of $C^{1,\mu}$-regularity up to the boundary for solutions of boundary value problems for elliptic equations with discontinuous coefficients in the plane.

In the literature there are several results of a priori bounds for solutions of elliptic equations with continuous coefficients (for example, Shauder-Caccioppoli estimates or a priori bounds in the space $W^{2,p}$ for solutions of equations with Hölder continuous coefficients).

Since it concerns elliptic equations with only measurable and bounded coefficients, the first results are due to C. B. Morrey [10], L. Nirenberg [11], L. Bers and L. Nirenberg [2]. Later, R. Finn and J. Serrin [4], P. Hartman [7] improved some results of L. Nirenberg [11]. In the papers of Nirenberg, Finn-Serrin, and Hartman, various regularity properties have been shown, which in particular imply Hölder continuity of the first derivatives of the solution under the assumption of essential boundedness of the right-hand side of the equation. In these papers, the Hölder exponent of the first derivatives is expressed in terms of the constant that appears in the Bernstein inequality (see (1.5)).

Received by the editors April 1, 2002.
2000 *Mathematics Subject Classification.* Primary 35J25; Secondary 35J65.
*Key words and phrases.* Regularity up to the boundary, elliptic equations, boundary value problems.

Indeed, by S. Campanato [3] and E. Giusti [6], further regularity results hold.

In order to present these results carefully, let $\Omega \subset \mathbb{R}^2$ be a bounded and open set with $C^2$-smooth boundary $\partial\Omega$. We assume that $\partial\Omega$ is a closed curve and let $x_1 = x_1(\varphi), x_2 = x_2(\varphi)$ be the normal parameterization of $\partial\Omega$, with $\varphi$ being a curvilinear parameter, $\varphi \in [0, L]$.

We consider the following equation:

$$\mathcal{L}u = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) \quad \text{a.e. in } \Omega \tag{1.1}$$

where

$$\begin{gathered} a_{ij}(x), b_i(x), c(x) \in L^\infty(\Omega),\ a_{ij} = a_{ji},\ i,j = 1,2, \\ f(x) \in L^r(\Omega),\ r > 2, \end{gathered} \tag{1.2}$$

and we assume the following uniform ellipticity condition:

$$0 < \lambda_1 \|\zeta\|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\zeta_i\zeta_j \le \lambda_2\|\zeta\|^2, \quad x \in \Omega, \ \ \zeta \in \mathbb{R}^2. \tag{1.3}$$

Let us note that ellipticity condition (1.3) implies

$$a_{11}a_{22} - a_{12}^2 > 0 \tag{1.4}$$

and, under (1.4), the Bernstein inequality holds:

$$\begin{aligned} &(\frac{\partial^2 u}{\partial x_1^2})^2 + 2(\frac{\partial^2 u}{\partial x_1 \partial x_2})^2 + (\frac{\partial^2 u}{\partial x_2^2})^2 + 2A \left(\frac{\partial^2 u}{\partial x_1^2}\ \frac{\partial^2 u}{\partial x_2^2} - (\frac{\partial^2 u}{\partial x_1 \partial x_2})^2\right) \\ &\qquad \le B\left(a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2}\right)^2 \end{aligned} \tag{1.5}$$

with

$$A > \frac{1}{2}\frac{a_{11}^2 + 2a_{12}^2 + a_{22}^2}{a_{11}a_{22} - a_{12}^2},$$

$$B \ge \frac{A^2 - 1}{2A(a_{11}a_{22} - a_{12}^2) - a_{11}^2 - 2a_{12}^2 - a_{22}^2}$$

(see [8], [15]).

S. Campanato's result is the following (see [3]):

**Proposition 1.1.** *Suppose conditions* (1.2), (1.3) *are fulfilled. Then there exist two real numbers $p_0$ and $p_1$, $1 < p_0 < 2 < p_1$, such that for each $r \in (p_0, p_1)$ and for each $f \in L^r(\Omega)$, the Dirichlet problem*

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \quad \text{a.e. in } \Omega, \\ u \in W_0^{2,r}(\Omega) \end{cases} \tag{1.6}$$

*admits a unique solution $u(x)$ satisfying*

$$\|u\|_{W_0^{2,r}(\Omega)} \le \frac{\operatorname{ess\,sup}_\Omega \alpha(x)\ \|\Delta^{-1}(r)\|}{1 - \sqrt{1-\varepsilon}\|\Delta^{-1}(r)\|}\ \|f\|_{L^r(\Omega)}$$

*where* $0 < \varepsilon < \dfrac{2\lambda_1\lambda_2}{{\lambda_1}^2 + {\lambda_2}^2}$, $\alpha(x) = \dfrac{\sum_{i=1}^2 a_{ii}(x)}{\sum_{i,j=1}^2 a_{ij}^2(x)}$ *and* $\Delta^{-1}(r)$ *is the inverse operator of the Laplacian* $\Delta$, $\Delta^{-1}(r) : L^r(\Omega) \to W_0^{2,r}(\Omega)$.

E. Giusti's result is the following (see [6]). Let $L^{p,\lambda}(\Omega)$, $p > 1$, be the usual Morrey space, i.e., the set of functions $u \in L^p(\Omega)$ such that

$$\|u\|^p_{L^{p,\lambda}(\Omega)} = \sup_{x_0\in\Omega, 0<\rho\leq diam\Omega} \rho^{-\lambda} \int_{\Omega\cap B(x_0,\rho)} |u(x)|^p dx < \infty.$$

Then we have

**Proposition 1.2.** *Under the same hypotheses of Proposition* 1.1*, let* $q$ *be the positive number*

$$q = \sup_{0<x<\frac{1-\sqrt{1-\varepsilon}}{1+\sqrt{1+\varepsilon}}} \frac{\log((1+\sqrt{1-\varepsilon})x + \sqrt{1-\varepsilon})}{\log x}.$$

*Then, for each* $f \in L^{2,\lambda}(\Omega)$, $\lambda < 2q$, *the solution to the Dirichlet problem* (1.6) *satisfies the estimate*

$$\sum_{i,j=1}^{2} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^{2,\lambda}(\Omega)} \leq C\|f\|_{L^{2,\lambda}(\Omega)}.$$

It follows from Proposition 1.1 that $Du \in C^{0,1-\frac{2}{r}}(\overline{\Omega})$ for each $r \in (2, p_1)$ and

$$[Du]_{C^{0,1-\frac{2}{r}}(\overline{\Omega})} \leq C\|u\|_{W_0^{2,r}(\Omega)} \leq C\frac{\operatorname{ess\,sup}_\Omega \alpha(x)\, \|\Delta^{-1}(r)\|}{1-\sqrt{1-\varepsilon}\|\Delta^{-1}(r)\|}\, \|f\|_{L^r(\Omega)}.$$

However, what is known about $p_1$ is only that it is greater than 2. That is, $Du$ is Hölder continuous with some positive exponent, but that exponent cannot be estimated from above.

On the other hand, Proposition 1.2 provides finer information. In fact, $L^r(\Omega) \subset L^{2,\frac{2(r-2)}{r}}(\Omega)$ for $2 < r < \frac{2}{1-q}$, and it follows that $D^2u \in L^{2,\frac{2(r-2)}{r}}(\Omega)$, whence $Du \in \mathcal{L}^{2,2+\frac{2(r-2)}{r}}(\Omega) \equiv C^{0,1-\frac{2}{r}}(\overline{\Omega})$ (see [8], page 24). Thus we have an estimate for the Hölder exponent of $Du$ in terms of $\frac{2}{1-q}$.

A more precise result is due to G. Talenti [15], who derived interior Hölder continuity of the first derivatives of the solution of elliptic equations in the plane with only bounded and measurable coefficients and expressed the Hölder exponent in terms of the eigenvalues $\lambda_1$, $\lambda_2$ of the linear operator and in terms of the summability $r$ of the right-hand side of the equation. Moreover, this exponent is better than the one obtained from Proposition 1.2.

Then G. Talenti's result is the following (see [15]):

**Theorem 1.1.** *Let* $\Omega \subset \mathbb{R}^2$ *be an open set. Let conditions* (1.2) *and* (1.3) *be satisfied, and let* $u \in W^{2,2}(\Omega)$ *be a solution of equation* (1.1)*. Then, if* $f \in L^r(\Omega)$, $r > 2$, $u \in C^{1,\mu}(\Omega)$ *for each* $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1-\frac{2}{r}\}$*. Moreover, for each compact set* $K \subset \Omega$ *and for each open set* $\Omega'$ *with the cone property such that* $K \subset \Omega' \subseteq \Omega$, *there exists a constant* $C = C(\Omega', K, \lambda_1, \lambda_2, r, \mu)$ *such that*

$$|Du(z_1) - Du(z_2)| \leq C(M\|u\|_{W^{1,r}(\Omega')} + \|f\|_{L^r(\Omega)})\, |z_1 - z_2|^\mu, \tag{1.7}$$

$$\forall z_1, z_2 \in K, \quad |z_1 - z_2| \leq \frac{\operatorname{dist}(K, \partial\Omega')}{2}.$$

Our goal is to extend Talenti's result, considering the Dirichlet boundary condition or the boundary normal derivative condition, obtaining regularity up to the boundary with Hölder exponent depending on $\lambda_1$, $\lambda_2$, $r$.

Precisely, denoting by $n = (X_1(\varphi), X_2(\varphi))$ the unit outward normal to $\partial\Omega$, we obtain the following results:

**Theorem 1.2.** *Let conditions* (1.2), (1.3) *be satisfied, and let us suppose that* $c \leq 0$. *Let* $u \in W^{2,2}(\Omega)$ *be the solution of the problem*

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega, \\ u = g(x) & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

*Then, if* $f \in L^r(\Omega)$ *and* $g \in W^{2-\frac{1}{r},r}(\partial\Omega)$, $r > 2$, $u \in C^{1,\mu}(\overline{\Omega})$ *for each* $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$, *and the following estimate holds:*

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \leq C(\|f\|_{L^r(\Omega)} + \|g\|_{W^{2-\frac{1}{r},r}(\partial\Omega)}), \tag{1.9}$$

*where* $C$ *is a constant depending on* $\Omega$, $\partial\Omega$, $\lambda_1$, $\lambda_2$, $r$, $\mu$.

**Theorem 1.3.** *Let conditions* (1.2), (1.3) *be satisfied, and let us suppose that* $c \leq 0$. *Let* $u \in W^{2,2}(\Omega)$ *be the solution of the problem*

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n} + \sigma u = h(x) & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

*where* $\sigma \geq 0$. *Then, if* $f \in L^r(\Omega)$ *and* $h \in W^{1-\frac{1}{r},r}(\partial\Omega)$, $r > 2$, $u \in C^{1,\mu}(\overline{\Omega})$, *for each* $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$, *and the following estimate holds:*

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \leq C(\|f\|_{L^r(\Omega)} + \|h\|_{W^{1-\frac{1}{r},r}(\partial\Omega)}), \tag{1.11}$$

*where* $C$ *is a constant depending on* $\Omega$, $\partial\Omega$, $\lambda_1$, $\lambda_2$, $r$, $\mu$, $\sigma$.

These results also hold for nonlinear operators of the type $\mathcal{A}(x, u, Du, D^2u)$, where $D^2u$ denotes the Hessian matrix of $u$, with Dirichlet or normal derivative boundary conditions (see section 3).

## 2. Proof of the theorems

*Proof of Theorem 1.2.* First we consider problem (1.8) with homogeneous boundary condition:

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Interior regularity for the solution to problem (2.1) follows from Theorem 1.1. Let us prove boundary regularity.

Let $x_0$ be a point of $\partial\Omega$, and let us flatten $\partial\Omega$ in a neighborhood $U$ of $x_0$.

We introduce locally a transformation of coordinates $\tilde{x} = Tx$ with components

$$\begin{cases} \tilde{x}_1 = T_1(x_1, x_2), \\ \tilde{x}_2 = T_2(x_1, x_2) \end{cases}$$

such that $T(x_0) = 0$ and

$$T(U \cap \Omega) = \tilde{B}_R = \{\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2 : \; |\tilde{x}| < R, \; \tilde{x}_2 > 0\},$$
$$T(U \cap \partial\Omega) = \tilde{\Gamma}_R = \{\tilde{x} \in \mathbb{R}^2 : \; |\tilde{x}| < R, \; \tilde{x}_2 = 0\}.$$

It is possible to normalize $T$ in such a way that on $\partial\Omega$,

$$\text{(2.2)}\qquad \begin{cases} \dfrac{\partial T_2}{\partial x_1} = -X_1 \\ \dfrac{\partial T_2}{\partial x_2} = -X_2 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial T_1}{\partial x_1} = -X_2 \\ \dfrac{\partial T_1}{\partial x_2} = X_1. \end{cases}$$

In fact, by virtue of the assumptions on $\partial\Omega$, in the neighborhood $U$, $\partial\Omega$ can be represented by the graph of the function $x_2 = \psi(x_1)$, where $\psi$ is of class $C^1$ with its inverse and $x_2 > \psi(x_1)$ in $U \cap \Omega$. Then we may assume

$$\tilde{x}_1 = \int_0^{x_1} \frac{dt}{\sqrt{1+\psi'^2(t)}} + \int_0^{x_2} \frac{\psi'(\psi^{-1}(t))}{\sqrt{1+\psi'^2(\psi^{-1}(t))}}\,dt,$$
$$\tilde{x}_2 = -\int_0^{x_1} \frac{\psi'(t)}{\sqrt{1+\psi'^2(t)}}dt + \int_0^{x_2} \frac{dt}{\sqrt{1+\psi'^2(\psi^{-1}(t))}}\,dt$$

and (2.2) holds.

Let us remark that $\tilde{x}_2 \equiv 0$ if $x_2 = \psi(x_1)$ and relationships (2.2) imply that the Jacobian $J$ of the transformation $T$ is $J = 1$ on $\partial\Omega$.

If $u(x)$ is the solution to problem (2.1), let us set

$$\tilde{u}(\tilde{x}) = (u \circ T^{-1})(\tilde{x}) = u(x),$$
$$\tilde{f}(\tilde{x}) = (f \circ T^{-1})(\tilde{x}) = f(x).$$

It follows that

$$\text{(2.3)}\qquad \begin{cases} \tilde{\mathcal{L}}\tilde{u}(\tilde{x}) = \displaystyle\sum_{i,j=1}^{2} \tilde{a}_{ij}(\tilde{x}) \frac{\partial^2 \tilde{u}}{\partial \tilde{x}_i \partial \tilde{x}_j} + \sum_{i=1}^{2} \tilde{b}_i(\tilde{x}) \frac{\partial \tilde{u}}{\partial \tilde{x}_i} + \tilde{c}(\tilde{x})\tilde{u} = \tilde{f}(\tilde{x}) & \text{a.e. in } \tilde{\mathrm{B}}_{\mathrm{R}}, \\ \tilde{u}(\tilde{x}_1, 0) = 0 & \text{on } \tilde{\Gamma}_{\mathrm{R}} \end{cases}$$

where

$$\text{(2.4)}\qquad \begin{cases} \tilde{a}_{ij}(\tilde{x}) = \displaystyle\sum_{k,l=1}^{2} a_{kl}(x) \frac{\partial T_i}{\partial x_k} \frac{\partial T_j}{\partial x_l}, \\ \tilde{b}_i(\tilde{x}) = \displaystyle\sum_{k,l=1}^{2} \left( a_{kl}(x) \frac{\partial^2 T_i}{\partial x_k \partial x_l} + b_k(x) \frac{\partial T_i}{\partial x_k} \right), \\ \tilde{c}(\tilde{x}) = c(x). \end{cases}$$

Let us note that $\tilde{\mathcal{L}}$ is such that in $\tilde{B}_R$,

$$\text{(2.5)}\qquad \begin{gathered} \tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2 = a_{11}a_{22} - a_{12}^2 + \varepsilon_1(\tilde{x}), \\ \tilde{a}_{11} + \tilde{a}_{22} = a_{11} + a_{22} + \varepsilon_2(\tilde{x}), \\ \sum_{i,j=1}^{2} \tilde{a}_{ij}^2 = \sum_{i,j=1}^{2} a_{ij}^2 + \varepsilon_3(\tilde{x}), \end{gathered}$$

with $\varepsilon_1(\tilde{x})$, $\varepsilon_2(\tilde{x})$, $\varepsilon_3(\tilde{x})$ such that $\lim\limits_{\tilde{x}\to O} \varepsilon_i(\tilde{x}) = 0$, $i = 1,2,3$, and that $\tilde{\mathcal{L}}$ still satisfies conditions (1.2), (1.3) with constants $0 < \tilde{\lambda}_1 \le \tilde{\lambda}_2$.

If $\tilde{u} \in W^{2,2}(\tilde{B}_R)$ is the solution to problem (2.3), we denote by

$$\text{(2.6)}\qquad \overline{u}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{u}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \ge 0, \\ -\tilde{u}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0. \end{cases}$$

Let us prove that $\overline{u}(\tilde{x}_1, \tilde{x}_2)$ belongs to $W^{2,2}(B_R)$, where $B_R = \{\tilde{x} \in \mathbb{R}^2 : \ |\tilde{x}| < R\}$.

At first we observe that $\overline{u}$ is continuous in $\tilde{x}_2 = 0$; in fact, from the boundary condition we have

$$\lim_{\tilde{x}_2 \to 0^+} \overline{u}(\tilde{x}_1, \tilde{x}_2) = \tilde{u}(\tilde{x}_1, 0) = 0,$$
$$\lim_{\tilde{x}_2 \to 0^-} \overline{u}(\tilde{x}_1, \tilde{x}_2) = -\tilde{u}(\tilde{x}_1, 0) = 0.$$

Moreover, from (2.6) it is easily seen that $\overline{u}$, $\dfrac{\partial \overline{u}}{\partial \tilde{x}_i}$, $\dfrac{\partial^2 \overline{u}}{\partial \tilde{x}_i \partial \tilde{x}_j}$ belong to $L^2(B_R)$. If we put

$$(2.7) \qquad \begin{cases} \overline{a}_{11}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{a}_{11}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ \tilde{a}_{11}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases} \\ \overline{a}_{12}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{a}_{12}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ -\tilde{a}_{12}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases} \\ \overline{a}_{22}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{a}_{22}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ \tilde{a}_{22}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases} \\ \overline{b}_1(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{b}_1(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ \tilde{b}_1(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases} \\ \overline{b}_2(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{b}_2(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ -\tilde{b}_2(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases} \\ \overline{c}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{c}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ \tilde{c}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases} \end{cases}$$

$$\overline{f}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{f}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ -\tilde{f}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases}$$

we obtain that $\overline{u}(\tilde{x}_1, \tilde{x}_2)$ satisfies the equation

$$\overline{\mathcal{L}}\overline{u}(\tilde{x}) = \sum_{i,j=1}^{2} \overline{a}_{ij}(\tilde{x}) \frac{\partial^2 \overline{u}}{\partial \tilde{x}_i \partial \tilde{x}_j} + \sum_{i=1}^{2} \overline{b}_i(\tilde{x}) \frac{\partial \overline{u}}{\partial \tilde{x}_i} + \overline{c}(\tilde{x})\overline{u} = \overline{f}(\tilde{x}) \qquad \text{a.e. in } \mathrm{B_R}$$

with $\overline{f} \in L^r(B_R)$.

It is easily seen that the coefficients of the operator $\overline{\mathcal{L}}$ satisfy condition (1.2). Moreover, $\overline{\mathcal{L}}$ has the same eigenvalues as $\tilde{\mathcal{L}}$, and then condition (1.3) is still verified with constants $\tilde{\lambda}_1$, $\tilde{\lambda}_2$.

Then from Theorem 1.1 it follows that $\overline{u} \in C^{1,\mu}(\overline{B}_{R^*})$ with $R^* < R$ and $\mu < \min\{\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}, 1 - \frac{2}{r}\}$.

We may prove that $\overline{u} \in C^{1,\mu}(\overline{B}_\rho)$ with $\rho < R^*$ and $\mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$.

In fact, from (2.5) it follows that $\tilde{\lambda}_1 \geq \lambda_1 + \omega_1(\tilde{x})$, $\tilde{\lambda}_2 \leq \lambda_2 + \omega_2(\tilde{x})$, with $\omega_1(\tilde{x})$, $\omega_2(\tilde{x}) \to 0$ as $\tilde{x} \to O$. If $\mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$, then there exists $\rho$, $0 < \rho < R^*$, such that in $\overline{B}_\rho$, $\mu < \min\{\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}, 1 - \frac{2}{r}\}$ and then, for what we have proved, $\overline{u} \in C^{1,\mu}(\overline{B}_\rho)$.

Moreover, the following estimate holds:

$$[D\overline{u}]_{C^{0,\mu}(\overline{\tilde{B}_\rho})} = [D\tilde{u}]_{C^{0,\mu}(\overline{\tilde{B}_\rho})} \leq C(\|\overline{f}\|_{L^r(B_R)} + M\|\overline{u}\|_{W^{1,r}(B_R)}).$$

Taking into account that

$$\|\overline{f}\|_{L^r(B_R)} = 2\|\tilde{f}\|_{L^r(\tilde{B}_R)},$$
$$\|\overline{u}\|_{W^{1,r}(B_R)} = 2\|\tilde{u}\|_{W^{1,r}(\tilde{B}_R)}$$

we may conclude

$$[D\tilde{u}]_{C^{0,\mu}(\overline{\tilde{B}_\rho})} \le C(\|\tilde{f}\|_{L^r(\tilde{B}_R)} + M\|\tilde{u}\|_{W^{1,r}(\tilde{B}_R)}). \tag{2.8}$$

Since $\partial\Omega$ is compact, only a finite number of such neighborhoods are needed to cover it, and then we obtain boundary Hölder regularity of $Du$ with exponent $\mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$.

Finally, from interior and boundary regularity, global regularity follows.

Moreover, since, by virtue of the assumptions, for the unique solution to problem (2.1) we have

$$\|u\|_{W^{2,2}(\Omega)} \le C\|f\|_{L^r(\Omega)}$$

by (1.7), (2.8) and the above estimate, we get

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \le C(\|u\|_{W^{1,r}(\Omega)} + \|f\|_{L^r(\Omega)}) \le C\|f\|_{L^r(\Omega)}. \tag{2.9}$$

Now we may study the nonhomogeneous case.

Let $u$ be the solution to problem (1.8) with $g \in W^{2-\frac{1}{r},r}(\partial\Omega)$, and let $w$ be the solution to the problem

$$\begin{cases} \Delta w = 0 & \text{a.e. in } \Omega, \\ w = g(x) & \text{on } \partial\Omega. \end{cases}$$

By classic results it follows that $w \in W^{2,r}(\Omega)$ and

$$\|w\|_{W^{2,r}(\Omega)} \le c\|g\|_{W^{2-\frac{1}{r},r}(\partial\Omega)}.$$

By means of the Sobolev Theorem we get $w \in C^{1,\lambda}(\overline{\Omega})$, for each $\lambda$, $0 < \lambda \le 1 - \frac{2}{r}$, and

$$\|w\|_{C^{1,\lambda}(\overline{\Omega})} \le c\|w\|_{W^{2,r}(\Omega)} \le c\|g\|_{W^{2-\frac{1}{r},r}(\partial\Omega)}. \tag{2.10}$$

The function $v = u - w$ is the solution of the problem

$$\begin{cases} \mathcal{L}v = f(x) - \mathcal{L}w & \text{a.e. in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Since the right-hand side of the equation in (2.11) belongs to $L^r(\Omega)$, for what we have proved in the homogeneous case, it follows that $v \in C^{1,\mu}(\overline{\Omega})$, for each $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$ and then $u = v + w$ belongs to $C^{1,\mu}(\overline{\Omega})$, for each $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$. Moreover by (2.9), (2.10),

$$\begin{aligned} [Du]_{C^{0,\mu}(\overline{\Omega})} \le [Dv]_{C^{0,\mu}(\overline{\Omega})} + [Dw]_{C^{0,\mu}(\overline{\Omega})} &\le C(\|f - \mathcal{L}w\|_{L^r(\Omega)} + [Dw]_{C^{0,\mu}(\overline{\Omega})}) \\ &\le C(\|f\|_{L^r(\Omega)} + \|w\|_{W^{2,r}(\Omega)}) \\ &\le C(\|f\|_{L^r(\Omega)} + \|g\|_{W^{2-\frac{1}{r},r}(\partial\Omega)}). \end{aligned}$$

□

*Proof of Theorem 1.3.* As in the Dirichlet case, we prove the assertion for problem (1.10) with homogeneous boundary condition, and then we obtain the result for the nonhomogeneous case.

First we study problem (1.10) with $\sigma = 0$:

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

If $u \in W^{2,2}(\Omega)$ is the solution to problem (2.12), interior regularity follows from Theorem 1.1.

In order to obtain boundary regularity, we repeat the same arguments as above, introducing the transformation $\tilde{x} = Tx$ with the same properties.

The function

$$\tilde{u}(\tilde{x}) = (u \circ T^{-1})(\tilde{x})$$

satisfies the boundary condition

$$\frac{\partial \tilde{u}}{\partial \tilde{x}_2}(\tilde{x}_1, 0) = 0 \quad \text{on } \tilde{\Gamma}_{\mathrm{R}}.$$

In fact, on $\tilde{\Gamma}_R$,

$$\begin{aligned}\frac{\partial u}{\partial n} &= \frac{\partial u}{\partial x_1} X_1 + \frac{\partial u}{\partial x_2} X_2 \\ &= \frac{\partial \tilde{u}}{\partial \tilde{x}_1}(\tilde{x}_1, 0)\,\Big(\frac{\partial T_1}{\partial x_1} X_1 + \frac{\partial T_1}{\partial x_2} X_2\Big) + \frac{\partial \tilde{u}}{\partial \tilde{x}_2}(\tilde{x}_1, 0)\,\Big(\frac{\partial T_2}{\partial x_1} X_1 + \frac{\partial T_2}{\partial x_2} X_2\Big).\end{aligned}$$

From properties (2.2) of transformation $T$, it follows that

$$\frac{\partial T_2}{\partial x_1} X_1 + \frac{\partial T_2}{\partial x_2} X_2 = -1$$

and

$$\frac{\partial T_1}{\partial x_1} X_1 + \frac{\partial T_1}{\partial x_2} X_2 = 0.$$

Then $\tilde{u}$ satisfies

$$\begin{cases} \tilde{\mathcal{L}}\tilde{u}(\tilde{x}) = \displaystyle\sum_{i,j=1}^{2} \tilde{a}_{ij}(\tilde{x}) \frac{\partial^2 \tilde{u}}{\partial \tilde{x}_i \partial \tilde{x}_j} + \sum_{i=1}^{2} \tilde{b}_i(\tilde{x}) \frac{\partial \tilde{u}}{\partial \tilde{x}_i} + \tilde{c}(\tilde{x})\tilde{u} = \tilde{f}(\tilde{x}) & \text{a.e. in } \tilde{\mathrm{B}}_{\mathrm{R}}, \\ \dfrac{\partial \tilde{u}}{\partial \tilde{x}_2}(\tilde{x}_1, 0) = 0 & \text{on } \tilde{\Gamma}_{\mathrm{R}} \end{cases}$$

where $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}$ are prescribed in (2.4) and $\tilde{f}(\tilde{x}) = (f \circ T^{-1})(\tilde{x}) = f(x)$.

Now if we denote by

$$\overline{u}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{u}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ \tilde{u}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases}$$

we obtain that $\overline{u}(\tilde{x}_1, \tilde{x}_2)$ belongs to $W^{2,2}(B_R)$.

If we put $\overline{a}_{ij}$, $\overline{b}_i$, $\overline{c}$ as in (2.7) and

$$\overline{f}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \tilde{f}(\tilde{x}_1, \tilde{x}_2) & \text{if } \tilde{x}_2 \geq 0, \\ \tilde{f}(\tilde{x}_1, -\tilde{x}_2) & \text{if } \tilde{x}_2 < 0, \end{cases}$$

we obtain that $\overline{u}(\tilde{x}_1, \tilde{x}_2)$ satisfies the equation

$$\overline{\mathcal{L}}\overline{u}(\tilde{x}) = \sum_{i,j=1}^{2} \overline{a}_{ij}(\tilde{x}) \frac{\partial^2 \overline{u}}{\partial \tilde{x}_i \partial \tilde{x}_j} + \sum_{i=1}^{2} \overline{b}_i(\tilde{x}) \frac{\partial \overline{u}}{\partial \tilde{x}_i} + \overline{c}(\tilde{x})\overline{u} = \overline{f}(\tilde{x}) \qquad \text{a.e. in } \mathrm{B}_{\mathrm{R}}$$

with $\overline{f} \in L^r(B_R)$.

Repeating the same arguments as above, we obtain that $u \in C^{1,\mu}(\overline{\Omega})$ for each $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$ and the following estimate holds:

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \leq C \|f\|_{L^r(\Omega)}. \tag{2.13}$$

Now we may consider problem (1.10) with $\sigma > 0$ and the homogeneous boundary condition:

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n} + \sigma\, u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Let $u \in W^{2,2}(\Omega)$ be the solution of (2.14), and let $w$ be the solution to the problem

$$\begin{cases} \Delta w - w = 0 & \text{a.e. in } \Omega, \\ \dfrac{\partial w}{\partial n} = \sigma u \ \in W^{\frac{3}{2},2}(\partial\Omega) & \text{on } \partial\Omega. \end{cases}$$

From classic results it follows that $w \in C^{2,1}(\overline{\Omega})$ and

$$\|w\|_{W^{2,r}(\Omega)} \le c\|u\|_{W^{\frac{3}{2},2}(\partial\Omega)} \le c\|u\|_{W^{2,2}(\Omega)} \le c\|f\|_{L^r(\Omega)}. \tag{2.15}$$

If we set $v = u + w$, it follows that

$$\begin{cases} \mathcal{L}v = f + \mathcal{L}w & \text{a.e. in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Since the right-hand side of the equation in (2.16) belongs to $L^r(\Omega)$, for what we proved in the case $\sigma = 0$, $v \in C^{1,\mu}(\overline{\Omega})$, for each $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$ and then $u = v - w$ belongs to $C^{1,\mu}(\overline{\Omega})$ for each $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1 - \frac{2}{r}\}$. Moreover, by (2.13) and (2.15) we get the following estimate:

$$\begin{aligned} [Du]_{C^{0,\mu}(\overline{\Omega})} &\le [Dv]_{C^{0,\mu}(\overline{\Omega})} + [Dw]_{C^{0,\mu}(\overline{\Omega})} \\ &\le C(\|f + \mathcal{L}w\|_{L^r(\Omega)} + [Dw]_{C^{0,\mu}(\overline{\Omega})}) \\ &\le C(\|f\|_{L^r(\Omega)} + \|w\|_{W^{2,r}(\Omega)}) \le C\|f\|_{L^r(\Omega)}. \end{aligned} \tag{2.17}$$

Finally it remains to study problem (1.10) with nonhomogeneous boundary condition, with $h \in W^{1-\frac{1}{r},r}(\partial\Omega)$.

Repeating the same arguments as in the Dirichlet problem, we obtain that the solution $u$ to problem (1.10) belongs to $C^{1,\mu}(\overline{\Omega})$, for each $\mu$, $0 < \mu < \min\{\frac{\lambda_1}{\lambda_2}, 1-\frac{2}{r}\}$, and estimate (1.11) holds. $\square$

## 3. Global regularity for the nonlinear problem

Let $\mathcal{A}(x,z,p,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \to \mathbb{R}$, $f(x,z,p) : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be real-valued functions satisfying Carathéodory's condition, i.e., they are measurable in $x$ for all $(z,p,\xi) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ and continuous in the other variables for almost all $x \in \Omega$.

We will prove global Hölder regularity for solutions to Dirichlet or normal derivative problems associated to the nonlinear operator $\mathcal{A}\left(x,u,Du,D^2u\right)$ under the following ellipticity condition:

*there exist $\alpha$, $\gamma$, $\delta > 0$, $\gamma + \delta < 1$, such that, for almost all $x \in \Omega$, for all $z \in \mathbb{R}$, $p \in \mathbb{R}^2$, $\xi$, $\tau \in \mathbb{R}^{2\times 2}$, one has*

$$\left|\sum_{i=1}^{2} \xi_{ii} - \alpha\left[\mathcal{A}(x,z,p,\xi+\tau) - \mathcal{A}(x,z,p,\tau)\right]\right| \le \gamma\,\|\xi\| + \delta\left|\sum_{i=1}^{2}\xi_{ii}\right| \tag{A}$$

where $\xi = \{\xi_{ij}\}_{i,j=1,2}$, $\|\xi\| = \left(\sum_{ij=1}^{2} \xi_{ij}^2\right)^{\frac{1}{2}}$, and

$$\mathcal{A}(x,z,p,0) = 0.$$

Let us observe that condition (A) does not imply continuity of the operator with respect to the independent variable $x$, but, in view of Carathéodory's character, only measurability with respect to $x$ is required (see [8] for more details).

Since it concerns the function $f(x,z,p)$, we assume that

$$\text{(B)} \qquad |f(x,z,p)| \le f_1(|z|)\left[f_2(x) + |p|^2\right]$$

for almost all $x \in \Omega$, for all $(z,p) \in \mathbb{R} \times \mathbb{R}^2$, where $f_1 \in C^0(\mathbb{R}^+)$ is a positive, monotone nondecreasing function and $f_2 \in L^r(\Omega)$, $r > 2$, is positive; and

$$\text{(C)} \qquad (\operatorname{sign} z) f(x,z,p) \le 2\sqrt{\frac{G(x)\det|a_{ij}|}{H(p)}}$$

for a.a. $x \in \Omega$, $|z| \ge N = \mathit{const} > 0$, $p \in \mathbb{R}^2$, where $a_{ij}(x,z,p,\xi) = \int_0^1 \frac{\partial \mathcal{A}}{\partial \xi_{ij}}(x,z,p,s\xi)ds \in L^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2})$, $G(x) \in L^1(\Omega)$ and $H \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ are positive functions such that $\int_\Omega G(x)dx < \int_{R^2} H(p)dp$.

Under conditions (A), (B), (C) the results by [5], [14] ensure the existence of a solution $u \in W^{2,r}(\Omega)$, $r \in ]2, r_0[$, to the following Dirichlet and normal derivative problems:

$$\text{(3.1)} \qquad \begin{cases} \mathcal{A}\left(x,u,Du,D^2u\right) = f(x,u,Du) & \text{a.e. in } \Omega, \\ u = g(x) & \text{on } \partial\Omega \end{cases}$$

and

$$\text{(3.2)} \qquad \begin{cases} \mathcal{A}\left(x,u,Du,D^2u\right) = f(x,u,Du) & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n} + \sigma u = h(x) & \text{on } \partial\Omega \end{cases}$$

with $\sigma \ge 0$. Moreover, the following estimates hold (see [5], [14], [17]):

$$\text{(3.3)} \qquad \|u\|_{L^\infty(\Omega)} \le \max\{N, \|g\|_{W^{2-\frac{1}{r},r}(\partial\Omega)}\} + R\operatorname{diam}\Omega,$$

$$\text{(3.4)} \qquad \|u\|_{L^\infty(\Omega)} \le \max\{N, \frac{1}{\sigma}\|h\|_{W^{1-\frac{1}{r},r}(\partial\Omega)}\} + R\left\{\operatorname{diam}\Omega + \frac{1}{\sigma}\right\}$$

where $R$ is such that

$$\int_{B_R(0)} H(p)dp = \int_\Omega G(x)dx$$

and $B_R(0)$ is a ball with center at the origin and radius $R$ and

$$\text{(3.5)} \qquad \|Du\|_{L^{2r}(\Omega)} \le C$$

with $r \in ]2, r_0[$ and $C = C(\alpha,\gamma,\delta,r,\partial\Omega,f_1,f_2,N,R,\operatorname{diam}\,\Omega)$ for problem (3.1) and $C$ also depends on $\sigma$ for problem (3.2).

We may prove the following $C^{1,\mu}(\overline{\Omega})$-regularity results.

**Theorem 3.1.** *Let conditions* (A), (B), (C) *be fulfilled. Let* $g \in W^{2-\frac{1}{r},r}(\partial\Omega)$, $r \in ]2, r_0[$, *and let* $u \in W^{2,r}(\Omega)$ *be a solution of problem* (3.1). *Then* $u \in C^{1,\mu}(\overline{\Omega})$, *for each* $\mu$, $0 < \mu < \min\{\frac{1-(\gamma+\delta)^2}{4(\gamma+\sqrt{2}(1+\delta))}, 1 - \frac{2}{r}\}$, *and the following estimate holds:*

$$\text{(3.6)} \qquad [Du]_{C^{0,\mu}(\overline{\Omega})} \le C_1(1 + \|g\|_{W^{2-\frac{1}{r},r}(\partial\Omega)}),$$

*where* $C_1 = C_1(\alpha,\gamma,\delta,r,\partial\Omega,f_1,f_2,N,R,\operatorname{diam}\Omega,\mu)$.

**Theorem 3.2.** *Let conditions* (A), (B), (C) *be satisfied. Let* $h \in W^{1-\frac{1}{r},r}(\partial\Omega)$, $r \in ]2, r_0[$, *and let* $u \in W^{2,r}(\Omega)$ *be a solution of problem* (3.2). *Then* $u \in C^{1,\mu}(\overline{\Omega})$, *for each* $\mu$, $0 < \mu < \min\{\frac{1-(\gamma+\delta)^2}{4(\gamma+\sqrt{2}(1+\delta))}, 1-\frac{2}{r}\}$ *and the following estimate holds:*

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \le C_2(1 + \|h\|_{W^{1-\frac{1}{r},r}(\partial\Omega)}), \tag{3.7}$$

*where* $C_2 = C_2(\alpha, \gamma, \delta, r, \partial\Omega, f_1, f_2, N, R, \operatorname{diam}\Omega, \sigma, \mu)$.

*Proof of Theorems 3.1 and 3.2.* We start by recalling the following result, which shows that condition (A) implies uniform ellipticity of our nonlinear operator (see [8], [13]).

**Lemma 3.3.** *Assume* $\mathcal{A}(x, u, Du, D^2u)$ *satisfies condition* (A). *Then the function* $\xi \to \mathcal{A}(x, z, p, \xi)$ *is a.e. differentiable in* $\mathbb{R}^{2\times 2}$ *for a.a.* $x \in \Omega$, $\forall (z,p) \in \mathbb{R}\times\mathbb{R}^2$.

*Moreover, if we set* $A_{ij}(x,z,p,\xi) = \dfrac{\partial \mathcal{A}(x,z,p,\xi)}{\partial \xi_{ij}}$, *it follows that* $A_{ij}(x,z,p,\xi) \in L^\infty(\Omega\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^{2\times 2})$ *and the following ellipticity condition holds:*

$$\frac{1-(\gamma+\delta)^2}{2\alpha}\sum_{i=1}^{2}\zeta_i^2 \le \sum_{i,j=1}^{2}\frac{\partial\mathcal{A}}{\partial\xi_{ij}}(x,z,p,\xi)\zeta_i\zeta_j \le 2\frac{\gamma+\sqrt{2}(1+\delta)}{\alpha}\sum_{i=1}^{2}\zeta_i^2 \tag{3.8}$$

*for a.a.* $x \in \Omega$, $\forall z \in \mathbb{R}$, $p \in \mathbb{R}^2$, $\forall \xi \in \mathbb{R}^{2\times 2}$, $\forall \zeta \in \mathbb{R}^2$.

Using Lemma 3.3, we may rewrite the equation in problem (3.1) as an equation with linear structure with $L^\infty$ principal coefficients in the following way:

$$\sum_{i,j=1}^{2} A^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x,u,Du) \tag{3.9}$$

where

$$A^{ij}(x) = \int_0^1 \frac{\partial\mathcal{A}(x,u(x),Du(x),tD^2u)}{\partial\xi_{ij}}\,dt \quad \in L^\infty(\Omega)$$

and $\{A^{ij}(x)\}_{i,j=1,2}$ is a positive definite matrix with eigenvalues less than or equal to $\frac{2(\gamma+\sqrt{2}(1+\delta))}{\alpha}$ and greater than or equal to $\frac{1-(\gamma+\delta)^2}{2\alpha}$. Then problem (3.1) is equivalent to

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} A^{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} = f(x,u,Du) & \text{a.e. in } \Omega, \\ u = g(x) & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

If $u \in W^{2,r}(\Omega)$ is a solution of problem (3.1), $f(x,u,Du) \in L^r(\Omega)$. In fact,

$$|f(x,u,Du)| \le f_1(|u|)\left[f_2(x) + |Du|^2\right] \le f_1(\|u\|_{L^\infty(\Omega)})\left[f_2(x)+|Du|^2\right]. \tag{3.11}$$

Taking into account the equivalence between problems (3.1) and (3.10), applying Theorem 1.2, we get that $u \in C^{1,\mu}(\overline{\Omega})$, for each $\mu$, $0 < \mu < \min\{\frac{1-(\gamma+\delta)^2}{4(\gamma+\sqrt{2}(1+\delta))}, 1-\frac{2}{r}\}$ and the following estimate holds:

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \le C(\|f(x,u,Du)\|_{L^r(\Omega)} + \|g\|_{W^{2-\frac{1}{r},r}(\partial\Omega)}). \tag{3.12}$$

Taking into account (3.3), (3.5) and (3.11) we obtain (3.6).

Repeating the same arguments for problem (3.2), applying Theorem 1.3, we get the assertion of Theorem 3.2. $\square$

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