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H^1-BOUNDS FOR SPECTRAL MULTIPLIERS ON GRAPHS

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ABSTRACT. We prove that certain spectral multipliers associated with the discrete Laplacian on graphs satisfying the doubling volume property and the Poincaré inequality are bounded on the Hardy space H^1.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let Γ be a countable infinite set, and let $\sigma_{xy} = \sigma_{yx}$ be a nonnegative and symmetric weight on $\Gamma \times \Gamma$ satisfying $\sigma_{xx} > 0$, $x \in \Gamma$. This weight induces a graph structure on Γ. We call the vertices x and y neighbors and we write $x \sim y$ when $\sigma_{xy} \neq 0$. We assume that Γ is connected, i.e., any two vertices are joined by a path and that Γ is endowed with its natural distance d.

We consider the discrete measure

$$\mu(\{x\}) = \sum_{y \sim x} \sigma_{xy}, \quad x \in \Gamma.$$

For simplicity, we set $\mu_x = \mu(\{x\})$ and $L^p = L^p(\Gamma, \mu)$. We define the volume $V(A)$ of a subset A of Γ by $V(A) = \sum_{x \in A} \mu_x$ and write $V(x,r)$ or $V(B)$ for $V(B(x,r))$, where $B(x,r) = \{y : d(x,y) \leq r\}$.

We consider the kernel

$$p(x,y) = \frac{\sigma_{xy}}{\mu_x}, \quad x, y \in \Gamma.$$

Clearly, $p(x,y) \geq 0$ and $\sum_y p(x,y) = 1$, for every $x \in \Gamma$, i.e., $p(x,y)$ is a Markov kernel. This kernel is not symmetric, but it is reversible with respect to the measure μ, i.e., $p(x,y)\mu_x = p(y,x)\mu_y$.

We shall finally assume that there is an $\alpha > 0$ such that

$$p(x,y) > \alpha, \text{ whenever } x \sim y. \tag{1.1}$$

Let us now present the geometry of the graph. We shall assume that Γ satisfies the so-called "doubling volume property", i.e., there is a constant $C > 0$ such that

$$V(x,2r) \leq CV(x,r), \quad x \in \Gamma, \quad r > 0. \tag{DV}$$

This implies that there is a $D > 0$ depending only on C such that

$$V(x,r) \leq C\left(\frac{r}{s}\right)^D V(x,s), \quad \text{for } r > s > 0. \tag{1.2}$$

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A graph satisfying (DV) is a space of homogeneous type in the sense of Coifman and Weiss. Thus we can define the atomic Hardy space $H^1(\Gamma)$ and the space of functions of bounded mean oscillation, $BMO(\Gamma)$, in the standard way. See below for precise definitions. Furthermore, by Theorem B of [4], $BMO(\Gamma)$ is the dual of $H^1(\Gamma)$.

We shall also assume that Γ satisfies an L^2-Poincaré inequality, i.e., there is a constant $C>0$ such that

$$\text{(PI)} \qquad \sum_{x\in B(x_0,r)} |f(x)-f_B|^2 \mu_x \le Cr^2 \sum_{x,y\in B(x_0,2r)} |f(y)-f(x)|^2 \sigma_{xy},$$

for all functions f on Γ, $x_0\in\Gamma$ and $r>0$, where

$$f_B = \frac{1}{V(x_0,r)} \sum_{x\in B(x_0,r)} f(x)\mu_x.$$

We consider the Markov operator

$$Pf(x) = \sum_y p(x,y) f(y).$$

The operator $L=I-P$ is the discrete analogue of the Laplace-Beltrami operator on a Riemannian manifold.

The reversibility of $p(x,y)$ with respect to μ implies that P is selfadjoint on L^2. Also, it is easy to check that L satisfies

$$\langle Lf,f\rangle_{L^2} = \frac{1}{2}\sum_{x,y} (f(x)-f(y))^2 \sigma_{xy}.$$

Therefore the operator L is positive. Since it is also selfadjoint on L^2, it admits a spectral resolution

$$L=\int_0^\infty \lambda dE_\lambda.$$

Given a bounded measurable function $m(\lambda)$ we can define, by using the spectral theorem, the operator

$$m(L) = \int_0^\infty m(\lambda)\, dE_\lambda.$$

This operator is bounded on L^2. The function $m(\lambda)$ is called a multiplier.

We say that $m(\lambda)$ is of Laplace transform type if

$$m(\lambda) = \lambda \int_0^\infty M(t)\, e^{-\lambda t} dt, \tag{1.3}$$

where $M(t)$ is a bounded measurable function.

These multipliers have been studied by Stein [12]. He proved that if L is the infinitesimal generator of a strongly continuous semigroup of operators on $L^2(E,d\xi)$, with E a locally compact space and ξ a positive measure on E, the operator $m(L)$ is bounded on $L^p(E,d\xi)$ for $p\in(1,\infty)$.

In this article we shall prove the following.

Theorem 1. *Let Γ and L be as above. If $m(\lambda)$ is a multiplier satisfying* (1.3)*, then the operator $m(L)$ is bounded on $H^1(\Gamma)$ and there is $c>0$, independent of M, such that*

$$\|m(L)\|_{H^1\to H^1} \le c\,\|M\|_\infty. \tag{1.4}$$

Note that by duality, (1.4) implies that $m(L)$ is bounded on BMO and by interpolation on L^p for $p \in (1, \infty)$.

For $\gamma \in \mathbb{R}_*$, we have

$$\lambda^{i\gamma} = \lambda \int_0^\infty \frac{t^{i\gamma}}{\Gamma(i\gamma + 1)} e^{-t\lambda} dt.$$

Thus Theorem 1 has the following

Corollary 1. *Let Γ and L be as above. Then, for any $\gamma \in \mathbb{R}_*$, $L^{i\gamma}$ is bounded on $H^1(\Gamma)$ and there is $c > 0$ such that*

$$\left\| L^{i\gamma} \right\|_{H^1 \to H^1} \leq c \left| \Gamma(i\gamma + 1) \right|^{-1}.$$

It follows from (1.3) that $\lambda^k m^{(k)}(\lambda)$ is bounded for all $k \in \mathbb{N}$. This implies that Laplace transform type multipliers can be considered as Marcinkiewitz or Hörmander-Mikhlin multipliers. This class of multipliers has been extensively studied in several contexts. See, for example, [1], [2] and the references therein. Alexopoulos, [2], proved, by a different method, that Hörmander-Mikhlin spectral multipliers are $L^1 - L^1_{weak}$ bounded and L^p-bounded for $p \in (1, \infty)$.

Imaginary powers of the Laplacian and more generally of second-order differential operators, have been studied for example in [3], [11], [7]. Especially, Corollary 1 is proved in [7] in the setting of Riemannian manifolds satisfying the doubling volume property and a Poincaré inequality.

Throughout this article the different constants will always be denoted by the same letter c. When their dependence or independence is significant, it will be clearly stated.

2. Preliminaries

2.1. **The spaces H^1 and BMO.** An atom is a function a on Γ that is supported in a ball $B = B(y_0, r)$, has mean value 0, i.e. $\sum_x a(x) \mu_x = 0$, and satisfies

$$\|a\|_\infty \leq V(y_0, r)^{-1}. \tag{2.1}$$

Note that (2.1) implies that

$$\|a\|_1 \leq 1 \quad \text{and} \quad \|a\|_2 \leq V(y_0, r)^{-1/2}. \tag{2.2}$$

A function $f \in L^1$ is said to be in the atomic Hardy space H^1 if there is a sequence $(\lambda_n) \in \ell^1$ and a sequence of atoms (a_n) such that $f = \sum_n \lambda_n a_n$, where the convergence is taken in the sense of L^1. We set

$$\|f\|_{H^1} = \inf \sum_n |\lambda_n|,$$

where the infimum is taken over all such decompositions of f.

Given $f \in L^p_{loc}$, $1 \leq p < \infty$, we say that $f \in BMO_p$ if there is a $C_p > 0$ such that

$$\left(\frac{1}{V(B)} \sum_{x \in B} |f(x) - f_B|^p \mu_x \right)^p < C_p, \tag{2.3}$$

for all the balls B of Γ, where f_B is the mean value of f on B. The norm $\|f\|_{*,p}$ is defined as the smallest of the constants C_p satisfying (2.3).

As is shown in [4], Theorem B, the spaces BMO_{p_1} and BMO_{p_2}, $1 \leq p_1, p_2 < \infty$, coincide as vector spaces and the norms $\|.\|_{*,p_1}$ and $\|.\|_{*,p_2}$ are equivalent. Henceforth, we shall denote by BMO the space BMO_p and by $\|.\|_*$ one of the equivalent norms $\|.\|_{*,p}$.

Let $C_0(\Gamma)$ be the space of functions on M with finite support, and define VMO as its closure in BMO.

The dual of H^1 is BMO ([4], Theorem B, p. 593) and as a consequence, H^1 itself is the dual of VMO ([4], Theorem 4.1).

Finally, one can easily deduce from (2.3) with $p = 2$, that there is $c > 0$ such that for all $f \in BMO$, all $k \in \mathbb{N}$ and all balls $B \subset \Gamma$,

$$\frac{1}{V(2^k B)} \sum_{x \in 2^k B} |f(x) - f_B|^2 \mu_x < ck^2 \|f\|_*^2, \tag{2.4}$$

where $2^k B = B(x_0, 2^k r)$ if $B = B(x_0, r)$.

2.2. **Markov kernels estimates.** Let $p_n(x,y)$ denote the nth iterate of the kernel $p(x,y)$ defined by

$$p_n(x,y) = \sum_z p(x,z) p_{n-1}(z,y), \quad n \geq 1. \tag{2.5}$$

As usual, we set $p_0(x,y) = \delta_x(y)$, where δ_x is the Dirac mass at x.

We have that

$$P^n f(x) = \sum_y p_n(x,y) f(y). \tag{2.6}$$

Note that since $p(x,y) \neq 0$ if and only if $x \sim y$, (2.5) implies that

$$p_n(x,y) = 0, \quad \text{if } d(x,y) > n.$$

These kernels are reversible with respect to the measure μ, i.e., $p_n(x,y)\mu_x = p_n(y,x)\mu_y$. This fact and the Markov property of $p(x,y)$ implies that for all $n \in \mathbb{N}$ and $y \in \Gamma$,

$$\sum_x p_n(x,y)\mu_x = \mu_y. \tag{2.7}$$

It has been proved by Delmotte (cf. [6], [5]) that the geometric assumptions (DV), (PI) on Γ and the condition (1.1) on the kernel $p(x,y)$ are in fact equivalent with the following Gaussian estimates of $p_n(x,y)$: there are constants $c_1, c_1', c_2, c_2' > 0$ such that

$$c_1 \frac{\mu_y}{V(x,\sqrt{n})} e^{-c_1' \frac{d(x,y)^2}{n}} \leq p_n(x,y) \leq c_2 \frac{\mu_y}{V(x,\sqrt{n})} e^{-c_2' \frac{d(x,y)^2}{n}}, \tag{G}$$

for all $x, y \in \Gamma$.

We set

$$r_n(x,y) = p_n(x,y) - p_{n+1}(x,y), \quad n \in \mathbb{N}.$$

The kernel $r_n(x,y)$ is the "discrete time derivative" of $p_n(x,y)$.

We recall the following lemma from [9].

Lemma 1. *There are constants $c, c' > 0$ such that*

$$|r_n(x,y)| \leq c' \frac{\mu_y}{nV(x,\sqrt{n})} e^{-c \frac{d(x,y)^2}{n}}, \tag{2.8}$$

for all $x, y \in \Gamma$ and $n \in \mathbb{N}$.

Let us fix y and $y_0 \in \Gamma$ and set

$$q_n(x) = \frac{r_n(y,x) - r_n(y_0,x)}{\mu_x}, \quad x \in \Gamma.$$

It is easy to check that both $r_n(x,y)$ and $q_n(x)$ satisfy the discrete parabolic equation

$$u_{n+1}(x) - u_n(x) = -Lu_n(x). \tag{2.9}$$

Let us now recall that the solutions of (2.9) satisfy the following Hölder regularity property (see Proposition 4.1 in [6]).

Lemma 2. *There are $h \in (0,1)$ and $c > 0$ such that for all $x_0 \in \Gamma$, $n_0 \in \mathbb{Z}$, $R \in \mathbb{N}$, every function u satisfying (2.9) in $Q = \left(\mathbb{Z} \cap \left[n_0 - 2R^2, n_0\right]\right) \times B(x_0, 2R)$, $x_1, x_2 \in B(x_0,R)$ and $n \in \mathbb{Z} \cap \left[n_0 - R^2, n_0\right]$,*

$$|u(n,x_1) - u(n,x_2)| \leq c\left[\frac{d(x_1,x_2)}{R}\right]^h \sup_Q |u|. \tag{2.10}$$

As a consequence of (2.8) and (2.10), one can prove as in [8, Lemma 27] that $q_n(x)$ satisfies the following estimate.

Lemma 3. *There are $h \in (0,1)$ and $c_3, c_3' > 0$ such that, for every $n \in \mathbb{N}$ and $x, y, y_0 \in \Gamma$ such that $d(y,y_0) \leq \sqrt{n}$,*

$$|q_n(x)| \leq \frac{c_3'}{nV(x,\sqrt{n})}\left[\frac{d(y,y_0)}{\sqrt{n}}\right]^h e^{-c_3\frac{d(x,y_0)^2}{n}}. \tag{2.11}$$

We can deduce from (2.11), by arguing as in the proof of Lemma 11 of [8], the following.

Lemma 4. *Let c_3 be as in (2.11). Then, for every $\beta < 2c_3$, there is $c_\beta > 0$ such that, for $d(y,y_0) \leq \sqrt{n}$,*

$$\sum_x |q_n(x)|^2 e^{\beta\frac{d(x,y)^2}{n}}\mu_x \leq \frac{c_\beta}{n^2V(x,\sqrt{n})}\left[\frac{d(y,y_0)}{\sqrt{n}}\right]^h. \tag{2.12}$$

3. Proof of Theorem 1

Since VMO is the dual of H^1, in order to prove that $m(L)$ is bounded on H^1 it suffices to show that there is a $c > 0$ such that for every atom a and $\phi \in BMO$ with finite support,

$$\left|\sum_{x\in\Gamma} m(L)\,a(x)\,\phi(x)\,\mu_x\right| \leq c\,\|\phi\|_*.$$

For every $\varepsilon \in (0,1)$ we set

$$m_\varepsilon(z) = z\int_\varepsilon^{1/\varepsilon} M(t)\,e^{-zt}dt, \quad \operatorname{Re} z > 0.$$

Since $m_\varepsilon(z) \longrightarrow m(z)$, as $\varepsilon \to 0$, uniformly on compacta, one has that for all $f \in L^2(\Gamma)$,

$$\|m_\varepsilon(L)f - m(L)f\|_2 \to 0 \text{ as } \varepsilon \to 0.$$

Furthermore,

$$\begin{aligned} m_\varepsilon(L) &= L\int_\varepsilon^{1/\varepsilon} M(t)\, e^{-Lt}dt = \int_\varepsilon^{1/\varepsilon} M(t)(I-P)\, e^{-t}e^{tP}dt \\ &= \int_\varepsilon^{1/\varepsilon} M(t)\, e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(P^n - P^{n+1}\right)dt. \end{aligned}$$

Let $T_\varepsilon(x,y)$ be the kernel of $m_\varepsilon(L)$. By using (2.6) we have that

$$\begin{aligned} T_\varepsilon(x,y) &= \int_\varepsilon^{1/\varepsilon} M(t)\, e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(p_n(x,y) - p_{n+1}(x,y)\right)dt \\ &= \int_\varepsilon^{1/\varepsilon} M(t)\, e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!} r_n(x,y)\, dt. \end{aligned} \tag{3.1}$$

Proposition 1. *The operators* $m_\varepsilon(L)$, $\varepsilon\in(0,1)$, *are bounded from* H^1 *to* L^1.

Proof. Let a be an atom supported in a ball $B = B(y_0, r)$. Thus, by (3.1) and (2.7),

$$\begin{aligned} \sum_x |m_\varepsilon(L)a(x)|\,\mu_x &\le \sum_x \mu_x \left|\int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\, e^{-t}dt \sum_{n\in\mathbb{N}}\frac{t^n}{n!}\sum_y r_n(x,y)\, a(y)\right| \\ &\le \|M\|_\infty \sum_y |a(y)| \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}dt\sum_{n\in\mathbb{N}}\frac{t^n}{n!}\sum_x \left[p_n(x,y)+p_{n+1}(x,y)\right]\mu_x \\ &\le 2\|M\|_\infty \sum_y |a(y)| \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}dt \sum_{n\in\mathbb{N}}\frac{t^n}{n!}\mu_y \\ &= 2\|M\|_\infty \|a\|_1 \int_\varepsilon^{\frac{1}{\varepsilon}} dt = c(\varepsilon)\|M\|_\infty, \end{aligned}$$

since $\|a\|_1 \le 1$. ∎

The following cancellation property is crucial for the proof of Theorem 1.

Proposition 2. *For every atom* a,

$$\sum_x m_\varepsilon(L)\, a(x)\, \mu_x = 0. \tag{3.2}$$

Proof. In the proof of Proposition 1 above, we have seen that the integral

$$\sum_x m_\varepsilon(L)\, a(x)\, \mu_x = \sum_x \left(\int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\, e^{-t}\left(\sum_{n\in\mathbb{N}}\frac{t^n}{n!}\sum_y r_n(x,y)\, a(y)\right)dt\right)\mu_x$$

is absolutely convergent. Therefore, we can interchange the order of summation and by using (2.7) we get

$$\begin{aligned}\sum_x m_\varepsilon(L)\,a(x)\,\mu_x &= \int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\,e^{-t}\sum_{n\in\mathbb{N}}\frac{t^n}{n!}dt\sum_y a(y)\sum_x r_n(x,y)\,\mu_x\\ &= \int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\,e^{-t}\sum_{n\in\mathbb{N}}\frac{t^n}{n!}dt\sum_y a(y)\sum_x\left(p_n(x,y)-p_{n+1}(x,y)\right)\mu_x\\ &= \int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\,e^{-t}\sum_{n\in\mathbb{N}}\frac{t^n}{n!}dt\sum_y a(y)\left[\mu_y-\mu_y\right]=0.\end{aligned}$$

□

Let $\phi\in BMO$ and, given a ball $B(y_0,r)$, let us set $2B=B(y_0,2r)$ and denote by ϕ_{2B} the mean value of ϕ on $2B$. By the cancellation property (3.2) we can write

$$\sum_x m_\varepsilon(L)\,a(x)\,\phi(x)\,\mu_x=\sum_x m_\varepsilon(L)\,a(x)\left\{\phi(x)-\phi_{2B}\right\}\mu_x.$$

Decompose $\phi-\phi_{2B}$ as follows:

$$\phi-\phi_{2B}=(\phi-\phi_{2B})\,\chi_{2B}+(\phi-\phi_{2B})\,\chi_{(2B)^c}=\phi_1+\phi_2,$$

and write

$$\begin{aligned}\sum_x m_\varepsilon(L)\,a(x)\,\phi(x)\,\mu_x &= \sum_x m_\varepsilon(L)\,a(x)\,\phi_1(x)\,\mu_x+\sum_x m_\varepsilon(L)\,a(x)\,\phi_2(x)\,\mu_x\\ &= E_1+E_2.\end{aligned}$$

Estimation of E_1. By (2.2), the L^2-boundedness of $m_\varepsilon(L)$ and (2.4) we have that

$$\begin{aligned}|E_1| &\le \|m_\varepsilon(L)\,a\|_2\,\|\phi_1\|_2\le\|m_\varepsilon(L)\,a\|_2\,\|(\phi-\phi_{2B})\,\chi_{2B}\|_2\\ &\le \|m_\varepsilon(L)\|_{2\to2}\,\|a\|_2\,\|(\phi-\phi_{2B})\,\chi_{2B}\|_2\\ &\le c\,\|m_\varepsilon\|_\infty\,V(B)^{-1/2}\,V(2B)^{1/2}\,\|\phi\|_*\le c\,\|m_\varepsilon\|_\infty\,\|\phi\|_* \qquad (3.3)\end{aligned}$$

where in the last inequality we have used the doubling property (DV).

Estimation of E_2. Let us consider the shells $S_k=2^{k+1}B-2^kB$. We have that

$$\begin{aligned}E_2 &= \sum_{x\in\Gamma} m_\varepsilon(L)\,a(x)\,\phi_2(x)\,\mu_x\\ &= \sum_{k=1}^\infty\sum_{x\in S_k}\left(\int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\,e^{-t}\left(\sum_{n\ge0}\frac{t^n}{n!}\sum_y r_n(x,y)\,a(y)\right)dt\right)\phi_2(x)\,\mu_x\\ &= \sum_{k=1}^\infty\sum_{x\in S_k}\left(\int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\,e^{-t}\left(\sum_{n=0}^{r^2}\frac{t^n}{n!}\sum_y r_n(x,y)\,a(y)\right)dt\right)\phi_2(x)\,\mu_x\\ &+ \sum_{k=1}^\infty\sum_{x\in S_k}\left(\int_\varepsilon^{\frac{1}{\varepsilon}} M(t)\,e^{-t}\left(\sum_{n=r^2+1}^{\infty}\frac{t^n}{n!}\sum_y r_n(x,y)\,a(y)\right)dt\right)\phi_2(x)\,\mu_x\\ &:= \sum_{k=1}^\infty E_{2,k}^0+E_{2,k}^\infty.\end{aligned}$$

Estimation of $E^{\infty}_{2,k}$. Since the atom a has mean value 0, we have that

$$\sum_y r_n(x,y)\,a(y) = \sum_y \left\{\frac{r_n(x,y)}{\mu_y} - \frac{r_n(x,y_0)}{\mu_{y_0}}\right\} a(y)\,\mu_y = \sum_y q_n(x)\,a(y)\,\mu_y.$$

Therefore,

$$E^{\infty}_{2,k} = \sum_{x\in S_k}\left(\int_{\varepsilon}^{\frac{1}{\varepsilon}} M(t)\,e^{-t}\left(\sum_{n=r^2+1}^{\infty}\frac{t^n}{n!}\sum_y q_n(x)\,a(y)\,\mu_y\right)dt\right)\phi_2(x)\,\mu_x. \tag{3.4}$$

Making use of the estimate (2.12) for $q_n(x)$ and (2.4) we have

$$\begin{aligned}
&\sum_{x\in S_k}|q_n(x)\,\phi_2(x)|\,\mu_x\\
&\le \left(\sum_x |q_n(x)|^2 e^{\beta\frac{d(y,x)^2}{n}}\mu_x\right)^{\frac12}\left(\sum_{x\in S_k} e^{-\beta\frac{d(y,x)^2}{n}}|\phi_2(x)|^2\mu_x\right)^{\frac12}\\
&\le \frac{c}{n\sqrt{V(y,\sqrt{n})}}\left(\frac{d(y,y_0)}{\sqrt{n}}\right)^h e^{-\beta\frac{\left(2^k r\right)^2}{2n}}\left(\sum_{x\in 2^{k+1}B}|\phi(x)-\phi_{2B}|^2\mu_x\right)^{\frac12}
\end{aligned}$$

$$\le \frac{c}{n\sqrt{V(y,\sqrt{n})}}\left(\frac{r}{\sqrt{n}}\right)^h e^{-\beta\frac{\left(2^k r\right)^2}{2n}} kV\left(y_0,2^{k+1}r\right)^{\frac12}\|\phi\|_*. \tag{3.5}$$

Now, if $x\in S_k$ and $y\in B(y_0,r)$, then $d(x,y)\le 2^{k+2}r$ and hence $2^{k+1}B\subset B\left(y,2^{k+3}r\right)$. So, if $n\le 2^{2k+6}r^2$, then by (1.2) and (3.5) we get

$$\sum_{x\in S_k}|q_n(x)\,\phi_2(x)|\,\mu_x \le \frac{ck\|\phi\|_*}{n}\left(\frac{r}{\sqrt{n}}\right)^h e^{-\beta\frac{\left(2^k r\right)^2}{2n}}\left(\frac{2^{k+3}r}{\sqrt{n}}\right)^D$$

$$\le \frac{ck\|\phi\|_*}{n}\left(\frac{r}{\sqrt{n}}\right)^h e^{-\beta\frac{\left(2^k r\right)^2}{4n}}. \tag{3.6}$$

If $n\ge 2^{2k+6}r^2$, we have $V(y,\sqrt{n})\ge V\left(2^{k+3}r\right)$. Hence by (3.5),

$$\sum_{x\in S_k}|q_n(x)\,\phi_2(x)|\,\mu_x \le \frac{c}{n\sqrt{V(y,\sqrt{n})}}\left(\frac{r}{\sqrt{n}}\right)^h e^{-\beta\frac{\left(2^k r\right)^2}{2n}} kV\left(y_0,2^{k+1}r\right)^{\frac12}\|\phi\|_*$$

$$\le \frac{ck\|\phi\|_*}{n}\left(\frac{r}{\sqrt{n}}\right)^h e^{-\beta\frac{\left(2^k r\right)^2}{4n}}. \tag{3.7}$$

It follows from (3.4), (3.6) and (3.7) that

$$\begin{aligned}
\left|E^{\infty}_{2,k}\right| &\le \|M\|_\infty\sum_y |a(y)|\,\mu_y\sum_{n=r^2+1}^{\infty}\frac{1}{n!}\int_0^\infty t^n e^{-t}dt\sum_{x\in S_k}|q_n(x)\,\phi_2(x)|\,\mu_x\\
&\le c\,\|M\|_\infty k\,\|\phi\|_*\sum_y |a(y)|\,\mu_y\sum_{n=r^2+1}^{\infty}\frac{1}{n}\left(\frac{r}{\sqrt{n}}\right)^h e^{-\beta\frac{\left(2^k r\right)^2}{4n}}\\
&\le c\,\|M\|_\infty k\,\|\phi\|_*\,\|a\|_1\int_{r^2}^{\infty} e^{-\beta\frac{\left(2^k r\right)^2}{4s}}\left(\frac{r}{\sqrt{s}}\right)^h\frac{ds}{s}\\
&\le c\,\|M\|_\infty k\,\|\phi\|_*\,2^{-kh}.
\end{aligned}$$

This implies that

$$|E_2^\infty| \le \sum_k \left|E_{2,k}^\infty\right| \le c\,\|M\|_\infty \|\phi\|_* \sum_k \frac{k}{2^{kh}} \le c\,\|M\|_\infty \|\phi\|_* . \tag{3.8}$$

Estimation of $E_{2,k}^0$. In order to estimate $E_{2,k}^0$, the Gaussian estimate (2.8) for $r_n(x,y)$ is sufficient. In fact, it follows from (2.8) and (1.2) that

$$|r_n(x,y)| \le c \frac{\mu_y}{nV(y,\sqrt{n})} e^{-c_2' \frac{d(x,y)^2}{2n}} . \tag{3.9}$$

Proceeding as in the case of $E_{2,k}^\infty$ and using (3.9) instead of the estimate (2.12) of $q_n(x)$, we get that

$$\begin{aligned}\left|E_{2,k}^0\right| &\le c\,\|M\|_\infty k\,\|\phi\|_* \sum_y |a(y)|\,\mu_y \sum_{n=1}^{r^2} \frac{1}{n} e^{-c_2' \frac{\left(2^k r\right)^2}{n}} \left(\frac{2^{k+3}r}{\sqrt{n}}\right)^D \\ &\le c\,\|M\|_\infty k\,\|\phi\|_* \|a\|_1 \int_0^{r^2} e^{-c_2' \frac{\left(2^k r\right)^2}{s}} \left(\frac{2^{k+3}r}{\sqrt{s}}\right)^D \frac{ds}{s} \\ &\le c\,\|M\|_\infty k\,\|\phi\|_* 2^{-kD}.\end{aligned}$$

Therefore,

$$\left|E_2^0\right| \le \sum_k \left|E_{2,k}^0\right| \le c\,\|M\|_\infty \|\phi\|_* \sum_k k 2^{-kD} = c\,\|M\|_\infty \|\phi\|_* . \tag{3.10}$$

End of proof of Theorem 1. It follows from (3.3), (3.8) and (3.10) that there is $c>0$ such that for every $\varepsilon \in (0,1)$ and every atom a and ϕ in VMO,

$$\left|\sum_{x\in\Gamma} m_\varepsilon(L)\,a(x)\,\phi(x)\,\mu_x\right| \le c\left\{\|m_\varepsilon\|_\infty + \|M\|_\infty\right\}\|\phi\|_* . \tag{3.11}$$

Since $m_\varepsilon(L)\,a \to m(L)\,a$ in L^2 and $m_\varepsilon(\lambda) \to m(\lambda)$ as $\varepsilon \to 0$, it follows from (3.11) that

$$\left|\sum_{x\in\Gamma} m(L)\,a(x)\,\phi(x)\,\mu_x\right| \le c\,\|M\|_\infty \|\phi\|_*$$

and the proof of Theorem 1 is complete.

Final remarks

It is worth mentioning that our approach applies also in the context of Lie groups and discrete groups of polynomial volume growth and Riemannian manifolds satisfying the doubling volume property and a Poincaré inequality. For instance, in the Riemannian case the estimates of the heat kernel p_t and of its time derivative as well as the Hölder continuity of the solutions of the heat equation are given in [10]. Having these, one can prove the analogue of Lemma 4 and then, by the same approach as in Section 4, the H^1-boundedness of the spectral multiplier $m(L)$, where L is the Laplace operator of the manifold. Finally, we note that the H^1-boundedness of the imaginary powers of the Laplacian in the above Riemannian setting is establish in [7].

References

[1] G. Alexopoulos, *Spectral multipliers on discrete groups*, Bull. London Math. Soc., **33**, (2001), 417-424. MR **2002d:**22010

[2] G. Alexopoulos, *L^p bounds for spectral multipliers from Gaussian estimates of the transition kernels*, preprint.

[3] M. Christ, *L^p bounds for spectral multipliers on nilpotent groups*, Trans. Amer. Math. Soc., **328**, (1991), 73-81. MR **92k:**42017

[4] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83**, (1977), 569-645. MR **56:**6264

[5] T. Coulhon, *Random walks and geometry on infinite graphs*, Lectures notes on analysis on metric spaces, Trento, C.I.R.M., 1999, Luigi Ambrosio, Francesco Serra Cassano, ed., Scuola Normale Superiore di Pisa, (2000), 5-23.

[6] T. Delmotte, *Parabolic Harnack inequality and estimates of Markov chains on graphs*, Rev. Mat. Iberoamericana, **15**, (1999), 181-232. MR **2000b:**35103

[7] M. Marias and E. Russ, *H^1-boundedness of Riesz transforms and imaginary powers of the Laplacian on Riemmanian manifolds*, Ark. Mat., **41**, (2003), 115–132.

[8] E. Russ, *$H^1 - L^1$ boundedness of Riesz transforms on Riemannian manifolds and on graphs*, Potential Anal., **14**, (2001), 301-330. MR **2003b:**42029

[9] E. Russ, *Temporal regularity for random walks and Riesz transforms on graphs for* $1 \leq p \leq 2$, preprint.

[10] L. Saloff-Coste, *Parabolic Harnack inequality for divergence form second order differential operators*, Potential Anal., **4**, (1995), 429-467. MR **96m:**35031

[11] A. Sikora and J. Wright, *Imaginary powers of Laplace operators*, Proc. Amer. Math. Soc., **129**, (2001), 1745-1754. MR **2001m:**35076

[12] E. M. Stein, *Topics in Harmonic Analysis related to the Littlewood-Paley Theory*, Annals of Mathematical Studies, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970. MR **40:**6176

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