

**LOCAL UNIQUENESS FOR
THE FIXED ENERGY FIXED ANGLE INVERSE PROBLEM
IN OBSTACLE SCATTERING**

PLAMEN STEFANOV AND GUNTHER UHLMANN

(Communicated by David S. Tartakoff)

ABSTRACT. We prove local uniqueness for the inverse problem in obstacle scattering at a fixed energy and fixed incident angle.

We consider the inverse problem of determining a sound-soft obstacle in \mathbf{R}^n , $n \geq 2$, from its scattering amplitude at a fixed incident direction $\theta \in S^{n-1}$ and a fixed energy $k > 0$. This is a formally determined inverse problem, since the data depends on the same number of variables, $n - 1$, as does the object we want to recover.

The purpose of this note is to give a simple proof of local uniqueness for this problem. Roughly speaking, we show that if two domains are close to a given obstacle, in a precise sense described below, and have the same scattering amplitude at a fixed angle and fixed energy, then they must be the same. Previously, it was shown in [CS] that local uniqueness holds for small obstacles. The Fréchet derivative of the nonlinear map from the domain to the scattering amplitude at fixed energy and angle was computed in [P], and one can easily show that it is injective. However, this does not imply a local result, since we cannot directly apply the implicit function theorem.

The proof of our result follows by using the arguments of Schiffer's well-known proof, presented in [LP], of uniqueness when given all incident directions, and the Poincaré inequality. By obstacles, we mean compact subsets of \mathbf{R}^n with C^2 boundary and connected complement.

The scattering amplitude $A_{\mathcal{O}}(k, \theta, \omega)$ related to an obstacle \mathcal{O} is defined as follows. For $k > 0$, $\theta \in S^{n-1}$, we define the scattering solution $u(x, \theta, k)$ as the solution to the boundary value problem (see, e.g., [CK])

$$\begin{cases} (-\Delta - k^2)u = 0, & \text{in } \mathbf{R}^n \setminus \mathcal{O}, \\ u|_{\partial\mathcal{O}} = 0, \end{cases}$$

Received by the editors August 19, 2002.

2000 *Mathematics Subject Classification.* Primary 35R30; Secondary 81U40, 35P25.

The first author was partly supported by NSF Grant DMS-0196440 and MSRI.

The second author was partly supported by NSF Grant DMS-007048 and a John Simon Guggenheim fellowship. Both authors would like to thank the hospitality of the Mathematical Sciences Research Institute where part of this work was done.

such that $u = e^{ik\theta \cdot x} + v$, with v satisfying the Sommerfeld outgoing condition at infinity: $(\partial/\partial r - ik)v = O(r^{-(n+1)/2})$, as $r = |x| \rightarrow \infty$. Then

$$v(x, \theta, k) = e^{ik\theta \cdot x} + \frac{e^{ikr}}{r^{(n-1)/2}} A_{\mathcal{O}}\left(k, \theta, \frac{x}{r}\right) + O\left(\frac{1}{r^{(n+1)/2}}\right), \quad \text{as } r = |x| \rightarrow \infty.$$

The function $A_{\mathcal{O}}(k, \theta, \omega)$ is the scattering amplitude related to \mathcal{O} .

It is known that Schiffer’s proof implies uniqueness if $A_{\mathcal{O}}$ is known for all ω , fixed $k_0 > 0$, and N incident directions θ ; or for all ω , fixed θ_0 , and N frequencies $k \leq k_0$, where N is greater than the number of the Dirichlet eigenvalues $k^2 \leq k_0^2$ of the Laplacian in a ball containing the obstacles. In particular, as mentioned above, this implies uniqueness at a fixed θ_0 and a fixed k_0 for all obstacles contained in a ball with sufficiently small radius R . In the 3D case, the condition is given by $k_0 R < \pi$. We refer to [CK], [I], [KK], [CS] for details and references.

In what follows, ω_n is the volume of the unit ball in \mathbf{R}^n (not to be confused with the outgoing direction ω); in particular, $\omega_3 = 4\pi/3$.

Our main result is the following.

Theorem 1. *Fix $k_0 > 0$, $\theta_0 \in S^{n-1}$. Let $\mathcal{O}_- \subset \mathcal{O}_+$ be two obstacles and assume that $\text{Vol}(\mathcal{O}_+ \setminus \mathcal{O}_-) < \omega_n k_0^{-n}$. Let $\mathcal{O}_- \subset \mathcal{O}_j \subset \mathcal{O}_+$, $j = 1, 2$, be two other obstacles and assume that $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$. Then $\mathcal{O}_1 = \mathcal{O}_2$.*

In particular, for any fixed obstacle \mathcal{O} , and fixed $k_0 > 0$, θ_0 , any small enough perturbation of the boundary gives an obstacle with different scattering amplitude.

More precisely, there exists $\varepsilon = \varepsilon(\mathcal{O}, k_0, \theta_0) > 0$ such that if $\partial\mathcal{O}_1$ is given in boundary normal coordinates $(x', x_n) \in \partial\mathcal{O} \times (-\delta, \delta)$ by $x_n = f(x')$ with $|f(x')| \leq \varepsilon$, $\forall x'$, then $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}}(k_0, \theta_0, \omega)$ implies $\mathcal{O}_1 = \mathcal{O}$. We would like to emphasize that this is different from the uniqueness for obstacles with small diameters mentioned above.

In Theorem 1 and Proposition 1 below, we do not impose smallness assumptions on k_0 or on the diameters of the obstacles. We prove unconditional local uniqueness at fixed k_0, θ_0 near any obstacle.

Let Ω_{ext} be the connected unbounded component of $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$. Set $\Omega_{\text{int}} = \mathbf{R}^n \setminus \overline{\Omega}_{\text{ext}}$. Then $\overline{\Omega}_{\text{int}} \supset \mathcal{O}_1 \cup \mathcal{O}_2$. Note that Ω_{int} is an open set that contains the interior of $\mathcal{O}_1 \cup \mathcal{O}_2$ as well as all components of $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$ disconnected from infinity.

Theorem 1 follows from the following.

Proposition 1. *Let \mathcal{O}_1 and \mathcal{O}_2 be two obstacles. Assume that for the corresponding scattering amplitudes we have*

$$A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$$

for a fixed $\theta_0 \in S^{n-1}$, fixed $k_0 > 0$ and all $\omega \in S^{n-1}$. If

$$(1) \quad \text{Vol}(\Omega_{\text{int}} \setminus \mathcal{O}_i) < \omega_n k_0^{-n}, \quad i = 1, 2,$$

then $\mathcal{O}_1 = \mathcal{O}_2$.

Our argument is based on an estimate of the first eigenvalue of the Dirichlet Laplacian in a bounded domain.

Lemma 1. *Let k^2 be a Dirichlet eigenvalue of $-\Delta$ in the bounded domain G . Then*

$$\omega_n \leq k^n \text{Vol}(G).$$

Proof. We use the Poincaré inequality in the form presented in [GT]:

$$(2) \quad \|u\| \leq \left(\frac{\text{Vol}(G)}{\omega_n}\right)^{1/n} \|\nabla u\|, \quad \text{for any } u \in H_0^1(G).$$

Let u be a normalized eigenfunction corresponding to k^2 . Then $\|\nabla u\| = k$ and $u \in H_0^1(G)$. Applying (2), we get

$$1 \leq k \left(\frac{\text{Vol}(G)}{\omega_n}\right)^{1/n},$$

which implies the lemma. □

Proof of Proposition 1. The proof is a combination of Schiffer’s idea and Lemma 1.

Let $u_j(x, \theta, k)$ be the scattering solution related to \mathcal{O}_j , $j = 1, 2$. By a well-known argument based on Rellich’s lemma, $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$ implies that $u_1(x, \theta_0, k_0) = u_2(x, \theta_0, k_0)$ for all x outside a ball containing $\mathcal{O}_1 \cup \mathcal{O}_2$. We know that u_1 and u_2 solve

$$\begin{cases} (-\Delta - k_0^2)u_j = 0, & \text{in } \mathbf{R}^n \setminus \mathcal{O}_j, \\ u_j|_{\partial\mathcal{O}_j} = 0. \end{cases}$$

Then by analytic continuation, $u_1 = u_2$ on $\partial\Omega_{\text{ext}}$.

Suppose that $\mathcal{O}_1 \neq \mathcal{O}_2$. Then for $j = 1$ or $j = 2$, $\Omega_{\text{int}} \setminus \mathcal{O}_j$ is an open nonempty set. Suppose that this happens for $j = 1$. Let G be any connected component of $\Omega_{\text{int}} \setminus \mathcal{O}_1$. Then $u_1 = 0$ on ∂G , and therefore $u_1|_G \in H_0^1(G)$. Since ∂G may not be smooth, the latter needs some justification. This was done in [CK] by approximating u_1 by a sequence $u_{1,n} \in C_0^\infty(G)$; see [CK, Theorem 5.1 and Lemma 3.8]. Therefore, u_1 solves the problem

$$\begin{cases} (-\Delta - k_0^2)u_1 = 0, & \text{in } G, \\ u_1|_G \in H_0^1(G). \end{cases}$$

Moreover, u_1 is not identically equal to zero in G , because it is a real analytic function in the domain $\mathbf{R}^n \setminus \mathcal{O}_1$ not vanishing for large x . Thus k_0^2 is a Dirichlet eigenvalue of the Laplacian in G .

By Lemma 1, $\omega_n k_0^{-n} \leq \text{Vol}(G) \leq \text{Vol}(\Omega_{\text{int}} \setminus \mathcal{O}_1)$. This contradicts our assumption (1), which proves the proposition. Note that in (1), we can actually replace $\Omega_{\text{int}} \setminus \mathcal{O}_j$ by the biggest (in terms of volume) connected component of this set. □

Proof of Theorem 1. We claim that the open sets $\Omega_{\text{int}} \setminus \mathcal{O}_j$ are included in $\mathcal{O}_+ \setminus \mathcal{O}_-$.

To prove that, note that $\Omega_{\text{int}} \setminus \mathcal{O}_1$, for example, is a union of the interior of $\mathcal{O}_2 \setminus \mathcal{O}_1$ and all bounded components of $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$. We only need to show that any such component is in $\mathcal{O}_+ \setminus \mathcal{O}_-$. Assume that there is a point x_0 in such a component with $x_0 \notin \mathcal{O}_+ \setminus \mathcal{O}_-$. Clearly, $x_0 \notin \mathcal{O}_+$. Then we can connect x_0 and infinity with a continuous curve lying outside \mathcal{O}_+ , because \mathcal{O}_+ is an obstacle. This curve is in $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$, and this contradicts the assumption that x_0 is in a bounded component of this set. This proves the inclusion, and the theorem now follows from Proposition 1. □

REFERENCES

[CK] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd edition, Applied Mathematical Sciences, Vol. 93, Springer-Verlag, Berlin, 1998. MR 99c:35181
 [CS] D. Colton and B. D. Sleeman, *Uniqueness theorems for the inverse problem of acoustic scattering*, IMA J. Appl. Math. 31(3) (1983), 253–259. MR 85e:76044

- [GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224, Springer-Verlag, Berlin, 1977. MR **57**:13109
- [I] V. Isakov, *Inverse problems for partial differential equations*, Appl. Math. Sci., **127**, Springer-Verlag, New York, 1998. MR **99b**:35211
- [KK] A. Kirsch and R. Kress, *Uniqueness in inverse obstacle scattering*, Inverse Problems **9** (1993), 285–299. MR **94e**:35143
- [LP] P. Lax and R. Phillips, *Scattering Theory*, Academic Press, 1967. MR **36**:530
- [P] R. Potthast, *Fréchet differentiability of boundary integral operators in inverse acoustic scattering*, Inverse Problems **10** (1994), 431–447. MR **95c**:35268

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
E-mail address: stefanov@math.purdue.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195
E-mail address: gunther@math.washington.edu