POLAROID OPERATORS AND WEYL’S THEOREM

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Abstract. “Polaroid elements” represent an attempt to abstract part of the condition, “Weyl’s theorem holds” for operators.

In the early days of operator theory Hermann Weyl [12] made an interesting observation about selfadjoint operators: “Weyl’s theorem” says that, whenever $T = T^* \in B(X)$ for a Hilbert space $X$, we have the equality

$$\sigma(T) \setminus \omega_{ess}(T) = \pi_{0}^{left}(T).$$

Here $\sigma(T)$ is the usual spectrum of the operator $T$, collecting complex numbers $\lambda$ for which $T - \lambda I$ does not have an inverse; the “Weyl spectrum” $\omega_{ess}(T)$ consists of those $\lambda \in \mathbb{C}$ for which $T - \lambda I$ fails to be Fredholm of index zero, and $\pi_{0}^{left}(T)$ denotes the isolated points of the spectrum that are eigenvalues of finite multiplicity. In a curious choice of words, (0.1) is usually described as saying that “Weyl’s theorem holds” for $T$: this has been extended to normal operators, both on Hilbert space and on Banach space, and to hyponormal, semi-hyponormal and $p$-hyponormal operators on Hilbert space.

In a felicitous misreading, the second author replaced (0.1) by

$$\sigma(T) \subseteq \omega_{ess}(T) \cup \pi_{0}^{left}(T);$$

in homage to the terminology above, this could be described [8] by saying “Browder’s theorem holds” for $T$. The permanence properties of (0.2) are (8, 9) somewhat better than those of (0.1), but it of course misses the disjointness

$$\omega_{ess}(T) \cap \pi_{0}^{left}(T) = \emptyset.$$

In the present note we wish to explore a variant of this disjointness condition; we begin with an excursion into Banach algebras.

Suppose $A$ is a semigroup, with identity $1$ and invertible group $A^{-1}$, or more generally an abstract category: then we call $a \in A$ simply polar iff (5; 6, Definition 7.5.2) it has a commuting generalized inverse $a' \in A$ for which

$$a = a a' a \text{ with } a a' = a a'.$$

This is a strong condition: for example, when $A$ is a ring and $a' \in A$ satisfies (0.4), then also

$$a'' = a' a a' + (1 - a' a)$$

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is an invertible generalized inverse for \( a \in A \). More generally, \( a \in A \) is said to be polar if there is \( n \in \mathbb{N} \) for which

\[(0.6) \quad a^n \text{ is simply polar ;}\]

more generally still, if \( A \) is a complex Banach algebra we call \( a \in A \) quasipolar if there is \( p \in A \) for which

\[(0.7) \quad p = p^2, \quad ap = pa, \quad p \in aA \cap Aa \quad \text{and} \quad \|a^n(1 - p)\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty .\]

Thus \( ap \in pAp \) is invertible (not in \( A \)) while \( a(1 - p) \) is quasinilpotent. Equivalently, with \( q = 1 - p \), we have [11]

\[(0.8) \quad q = q^2, \quad aq = qa, \quad a + q \in A^{-1} \quad \text{and} \quad \|a^nq\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty .\]

To attempt to do this in more general rings we need (7, 11) an algebraic version of “quasinilpotent”. It is familiar (5; 6, Theorem 7.5.3) that if \( a \in A \) is quasipolar, then the projection \( p = a^\bullet \) of (0.7) is unique and double commutes with \( a \), as is the relative inverse \( a^\times \in A \) for which

\[(0.9) \quad a^\bullet = a^\times a = aa^\times \quad \text{and} \quad a^\times = a^\times a^\bullet = a^\bullet a^\times .\]

We shall call the projection \( a^\bullet \) the support of \( a \in A \) and—a slight abuse of language [11] —the relative inverse \( a^\times \) the Drazin inverse. We can very slightly improve the double commutivity:

**1. Lemma.** If \( a \in A \) and \( b \in A \) are quasipolar, and if \( v \in A \) satisfies

\[(1.1) \quad bv = va ,\]

then also

\[(1.2) \quad b^\bullet v = va^\bullet \quad \text{and} \quad b^\times v = va^\times .\]

**Proof.** If \( p = a^\bullet \) and \( q = b^\bullet \), then we claim

\[(1.3) \quad qv = qvp = vp :\]

for if \( n \in \mathbb{N} \) is arbitrary,

\[qv - qvp = qv(1 - p) = b^\times nb^n v(1 - p) = b^\times nva^n(1 - p) \to 0 ,\]

and similarly for the second equality. Also,

\[b^\times v = b^\times qv = b^\times vp = b^\times vaa^\times = b^\times bva^\times = qva^\times = vpa^\times = va^\times .\]

A necessary and sufficient condition for \( a \in A \) to be quasipolar in a Banach algebra is that \( 0 \in \mathbb{C} \) is not an accumulation point of the spectrum \( \sigma(a) = \{ \lambda \in \mathbb{C} : a - \lambda \notin A^{-1} \} \):

\[(1.4) \quad 0 \notin \text{acc } \sigma(a) .\]

When (1.4) holds, then the support \( a^\bullet \) and the Drazin inverse \( a^\times \) are given by familiar Cauchy integrals.

Spectral inclusion is incorporated in the following “quasi-affine comparison” of elements: we shall write

\[(1.5) \quad a \prec_{\text{aff}} b\]

to mean that there is \( v \in A \) for which

\[(1.6) \quad (vx = 0 \implies x = 0) , \quad bv = va \quad \text{and} \quad \sigma(b) \subseteq \sigma(a) .\]
This in turn interacts with our idea of a “polaroid” element:

2. Definition. We shall call the element \( a \in A \) polaroid iff there is an implication, for arbitrary \( \lambda \in \mathbb{C} \),
\[
\begin{align*}
(2.1) \quad a - \lambda \text{ quasipolar } & \implies a - \lambda \text{ polar }, \\
(2.2) \quad a - \lambda \text{ quasipolar } & \implies a - \lambda \text{ simply polar }.
\end{align*}
\]

and simply polaroid if the implication is
\[
(2.3) \quad \text{normaloid } \implies \text{simply polaroid } \implies \text{reguloid }.
\]

For example, the argument of Stampfli \([8\text{, Theorem }14]\) says that for \( a = T \in B(X) \),
\[
\begin{align*}
(2.4) \quad \text{normaloid } & \implies \text{simply polaroid } \implies \text{reguloid }.
\end{align*}
\]

The quasi-affine comparison of (1.5) transmits polaroid and simple polaroid properties:

3. Theorem. If \( a, b \in A \), then
\[
\begin{align*}
(3.1) \quad a \prec_{\text{left}} b \text{ polaroid } & \implies a \text{ polaroid }, \\
(3.2) \quad a \prec_{\text{left}} b \text{ simply polaroid } & \implies a \text{ simply polaroid }.
\end{align*}
\]

Proof. Begin by checking that if \( a \prec_{\text{left}} b \), then
\[
\begin{align*}
(3.3) \quad \text{iso } \sigma(a) & \subseteq \text{iso } \sigma(b),
\end{align*}
\]

so that
\[
\begin{align*}
(3.4) \quad a \prec_{\text{left}} b, a \text{ quasipolar } & \implies b \text{ quasipolar }.
\end{align*}
\]

Thus iso \( (a) \), giving (3.3).

Towards (3.1) we can now argue, with \( c = (a - \lambda)^n \) and \( d = (b - \lambda)^n \),
\[
\begin{align*}
(3.5) \quad v(c - cc^* c) & = (d - dd^* d)v = 0 \implies c = cc^* c;
\end{align*}
\]

then, in particular, (3.2) is the case \( n = 1 \).

For operators it is easy to see \([8\text{, Theorem }9; 9\text{, Theorem }2]\) that “Browder’s theorem holds” for \( T \) in the sense of (0.2) if and only if
\[
\begin{align*}
(3.6) \quad \text{acc } \sigma(T) & \subseteq \omega_{\text{ess}}(T).
\end{align*}
\]
the polaroid condition is close to the reverse inclusion:

4. Theorem. If \( T \in B(X) \) for a Banach space \( X \), then the inclusion

\[ \omega_{\text{ess}}(T) \subseteq \text{acc}(\sigma(T)) \tag{4.1} \]

is sufficient for the polaroid condition (2.1), which in turn is sufficient for the disjointness (0.3). If \( T \) is polaroid and if \( T - \lambda I \) has for arbitrary \( \lambda \in \mathbb{C} \) either finite ascent or finite descent, then Weyl’s theorem holds for \( T \).

Proof. If (4.1) holds, then if \( T - \lambda I \) is quasipolar we have

\[ \lambda \in \text{iso}(\sigma(T)) \implies \lambda \in \text{iso}(\sigma(T) \setminus \omega_{\text{ess}}(T)), \]

which \([6]\), Theorem 9.8.4) by the punctured neighbourhood theorem makes it a “Riesz point” for \( T \), so that \( T - \lambda I \) is polar. Conversely, if \( T \) is polaroid and \( \lambda \in \pi_0^{\text{left}}(T) \), then \( T - \lambda I \) is polar with \( 0 < \dim (T - \lambda I)^{-1}(0) < \infty \). By the punctured neighbourhood theorem again this gives also \( \dim X/(T - \lambda I)X = \dim (T - \lambda I)^{-1}(0) \), excluding \( \lambda \) from \( \omega_{\text{ess}}(T) \). For the last part, if an operator \( T \) is Fredholm of index zero, then finite ascent and finite descent are equivalent.

Neither of the implications in the first part of Theorem 4 is reversible: for example, Weyl’s theorem holds for the Volterra operator \( x(t) \mapsto \int_{s=0}^{t} x(s)ds \) on \( C[0,1] \) while (4.1) fails. If (0.3) holds for \( T \) and fails for the dual operator \( T^* \), then \( T \in B(X) \) is not polaroid: for a specific example, take \( T = UW \) to be the product of the standard weight \( W : (x_n) \mapsto (\frac{1}{n}x_n) \) and the forward shift \( U \). For Weyl operators the ascent/descent condition is equivalent to the “single valued extension property” of Finch \([3]\).

References


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