

REGULARITY, PARTIAL ELIMINATION IDEALS AND THE CANONICAL BUNDLE

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ABSTRACT. We present partial elimination ideals, which set-theoretically cut out the multiple point loci of a generic projection of a projective variety, as a way to bound the regularity of a variety in projective space. To do this, we utilize a combination of initial ideal methods and geometric methods. We first define partial elimination ideals and establish through initial ideal methods the way in which, for a given ideal, the regularity of the partial elimination ideals bounds the regularity of the given ideal. Then we explore the partial elimination ideals as a way to compute the canonical bundle of the generic projection of a variety and the canonical bundles of the multiple point loci of the projection, and we use Kodaira Vanishing to bound the regularity of the partial elimination ideals.

1. BACKGROUND

1.1. Regularity, initial and generic initial ideals. In this section, we review the key results and definitions needed for the proof in section 2. None of the results in this section are new, and a good place to start for proofs not presented here is [Gre98]. We will work exclusively over the complex numbers \mathbf{C} throughout this paper.

Note 1.1. A multiplicative monomial order is a refinement of the degree order and the variable order $x_0 < x_1 < x_2 < \dots < x_r$ that respects multiplication. The most common orders are the **lexicographic order**, abbreviated **lex**, and the **reverse lexicographic order**, abbreviated **rlex**. In addition, we will have need of the **h -elimination order**: for our purposes, this is an amalgamation of the first two orders, and in other literature may also be called the **$\{\mathbf{h}, \mathbf{r} + \mathbf{1} - \mathbf{h}\}$ -product order**. The h -elimination order uses the lex order on the first h variables, and breaks lexicographic ties by using rlex on the remaining variables.

We let $U = (u_0, u_1, \dots, u_r)$, $u_i \in \mathbf{Z}$, and write x^U for $a^{u_0} x_1^{u_1} \dots x_r^{u_r}$.

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Definition 1.2. For a fixed monomial order and a homogeneous polynomial F , write

$$(1) \quad F = \sum c_U x^U,$$

$c_U \in \mathbf{C}$ nonzero. We define the **initial term** of F , $\text{in}(F) = c_V x^V$, where $c_V x^V$ is a nonzero monomial appearing in the expression for F above, and $x^V > x^U$ for every other U appearing in (1). We also have the **lead term** of F , $\text{LeadTerm}(F) = x^V$. The **initial ideal** of an ideal I , written $\text{in}(I)$, is generated by $\{\text{in}(F) \mid F \in I\}$.

Definition 1.3. An **elementary move** e_k , for $0 \leq k \leq r-1$, is defined by $e_k(x^U) = x^{\hat{U}}$, where $\hat{U} = (u_0, \dots, u_{k-1}, u_k + 1, u_{k+1} - 1, u_{k+1}, \dots, u_r)$, and we say $x^U = 0$ if some $u_i < 0$.

Proposition 1.4. *The following are equivalent for a monomial ideal I :*

- (1) *(I is closed under elementary moves.) If $x^U \in I$, then for every elementary move e_k , $e_k(x^U) \in I$.*
- (2) *$g(I) = I$ for every g belonging to the Borel subgroup B .*
- (3) *$\text{in}(g(I)) = I$ for every g in some open neighborhood of the identity in B .*

Definition 1.5. A monomial ideal I is **Borel-fixed** if any (hence all) of the conditions of the proposition hold for I .

We collect several needed but well-known results into the following.

Theorem 1.6.

(1) *Galligo's Theorem. For any multiplicative monomial order and any homogeneous ideal I , there is a Zariski open subset $W \subset GL$ such that $\text{in}(g(I))$ is constant and Borel-fixed for $g \in W$. Galligo's theorem allows us to define, for a homogeneous ideal I , the **generic initial ideal** of I , $\text{gin}(I) = \text{in}(g(I))$ for $g \in W$.*

(2) *For a Borel-fixed monomial ideal I , $(I : x_r) = (I : m)$.*

(3) *For a Borel-fixed monomial ideal I , $\text{sat}(I)$ is the degree of the largest generator of I involving x_r . In particular, I is saturated if no generator of I involves x_r .*

(4) *For a Borel-fixed monomial ideal I , $\text{reg}(I)$ is the maximal degree of a generator of I .*

(5) *For a homogeneous ideal I , using any order,*

$$\text{reg}(I) \leq \text{reg}(\text{in}(I)).$$

In particular, $\text{reg}(I)$ is bounded by the maximal degree of a generator of $\text{gin}(I)$.

(6) *Bayer-Stillman [BS87]. For a homogeneous ideal I , using the *rlx* order, $\text{sat}(I) = \text{sat}(\text{gin}(I))$, which is in turn equal to the maximal degree of a generator of $\text{gin}(I)$ involving x_r . In particular, I is saturated if and only if no generator of the *rlx* $\text{gin}(I)$ involves x_r . The regularity of I is equal to the regularity of the *rlx* $\text{gin}(I)$, or equivalently to the maximal degree of a generator of the *rlx* $\text{gin}(I)$.*

1.2. Partial elimination ideals. Now let I be a homogeneous ideal in the graded ring $S = \mathbf{C}[a, x_1, x_2, \dots, x_r]$ with maximal ideal m . I corresponds to a variety $X \subset \mathbf{P}^r$. We also write $\bar{S} = \mathbf{C}[x_1, x_2, \dots, x_r]$ with maximal ideal \bar{m} .

For any $0 \neq F \in I$, there is a maximal k for which we may write

$$F = a^k f_k + a^{k-1} f_{k-1} + a^{k-2} f_{k-2} + \cdots + f_0,$$

with $f_k \neq 0$ and with each $f_i \in \bar{S}$.

Definition 1.7. For fixed k , set $K_k(I) = \{f_k \mid F \in I \text{ as above}\}$. Then $K_k(I)$ is the k^{th} **-partial elimination ideal** of the ideal I . We will often write K_k if the ideal I is understood. The sheafification of the K_k will be written \mathcal{K}_k .

Remark 1.8. Geometrically, the 1-elimination order corresponds to the projection $\pi : \mathbf{P}^r \rightarrow \mathbf{P}^{r-1}$ from $[1, 0, \dots, 0]$. The ideal K_0 is the ideal of the projection; that is, it is the ideal of $\pi(X)$. We set $\pi(X) = \bar{X}$.

Definition 1.9. The **Jacobi ideal** of $I, J(I)$, is the ideal with generators $\{\partial F/\partial x_i$ (and $\partial F/\partial a \mid F \in I, 1 \leq i \leq r\}$.

The important results we need about partial elimination ideals are collected in the following.

Proposition 1.10.

- (1) *Let π be the projection of $X \subset \mathbf{P}^r$ from $[1, 0, 0, \dots, 0]$ to \mathbf{P}^{r-1} . Then the $K_k(I)$ are the ideals of $\{p \in \pi(X) \mid \text{mult}_p(\pi(X)) > k\}$.*
- (3) *In general coordinates, the Jacobi ideal of K_{k-1} is contained in K_k .*
- (3) *The partial elimination ideal K_0 , the ideal of the projection, is saturated.*

2. PARTIAL ELIMINATION IDEAL SHEAVES BOUND REGULARITY

2.1. A bound on regularity. Suppose $X \subset \mathbf{P}^r$ is a reduced, irreducible nondegenerate variety of degree d and dimension n . Recent work on regularity has sought to achieve the conjectured bound [EG84]:

$$\text{reg}(X) \leq d - (r - n) + 1.$$

In dimensions $n = 1, 2$, the bound has been proven to hold; results are incomplete in higher dimensions (see [GLP83], [Laz87], [Pin86] and see [Kwa98] for a summary of known results). It is hoped that the following theorem may be of use in the pursuit of results on regularity in higher dimensions.

Theorem 2.1. *Let X be a reduced, irreducible, nondegenerate variety of degree d and dimension n in \mathbf{P}^r , defined by the ideal I , with sheafification \mathcal{I}_X . Let $\pi : \mathbf{P}^r \rightarrow \mathbf{P}^{r-1}$ be a generic projection of X . Set*

$$i_0(X) = \max\{i \mid \mathcal{K}_i \text{ defines a nonempty variety}\}.$$

Then

$$\text{reg } \mathcal{I}_X \leq \max_{i \leq i_0} \{\text{reg } \mathcal{K}_i + i\}.$$

Proof of the Theorem. We will use generic initial ideal methods. Choose coordinates $[a, x_1, \dots, x_r]$ for \mathbf{P}^r so that $[1, 0, \dots, 0] \notin X$ and so that the projection from $[1, 0, \dots, 0]$ is generic, and corresponds to elimination of the variable a from the ideal I . Let K_i be the partial elimination ideals, and let \mathcal{K}_i be the sheafification of the K_i . Fix as the monomial order the $\{1, r\}$ -product order, which we also call the 1-elimination order. By the Bayer-Stillman Theorem, we know that

$$\text{reg } I \leq \max\{\text{deg}(f) \mid f \text{ is a generator of } \text{gin}(I)\},$$

where the gin is taken in any monomial order. Thus, for the 1-elimination order, we have

$$\begin{aligned} \text{gin}(I) = & (a^i f_{i,1}, a^i f_{i,2}, \dots, a^i f_{i,t_i}, a^{i-1} f_{i-1,1}, a^{i-1} f_{i-1,2}, \dots, a^{i-1} f_{i-1,t_{i-1}}, \dots, \\ (2) \quad & a f_{1,1}, a f_{1,2}, \dots, a f_{1,t_1}, f_{0,1}, \dots, f_{0,t_0}), \end{aligned}$$

where the $f_{i,j} \in \bar{S} = \mathbf{C}[x_1, \dots, x_r]$, and, for fixed k , the elements of the set

$$(3) \quad \{f_{k,j} \mid 1 \leq j \leq t_k\}$$

generate the (not necessarily saturated) rlex $\text{gin}(\mathbf{K}_k)$.

The maximal degree of a generator of $\text{gin}(I)$ involving a^k is thus equal to the maximal degree of a generator of $\text{gin}(\mathbf{K}_k)$ plus k .

Claim 2.2. The generators of $\text{gin}(I)$ of maximal degree do not involve x_r to a positive exponent.

Once the claim is proved, we will have proved the theorem. To see this, let f be a generator of $\text{gin}(I)$ of maximal degree, say $\text{deg}(f) = k + h$, where $f = a^k f_{k,j}$, and which does not involve x_r . We must have that h is the regularity of \mathbf{K}_k . Notice that the satiety of \mathbf{K}_k is at most h , because all terms of $\text{gin}(\mathbf{K}_k)$ are of degree at most h . The set in (3) is the set of rlex generators of $\text{gin}(\mathbf{K}_k)$, and so the maximal degree of a generator is equal to $\text{reg}(\mathbf{K}_k)$, and then $\text{reg}(\mathbf{K}_k) = \text{reg}(\mathcal{K}_k)$.

We now prove Claim 2.2 by contradiction. Assume that the claim is false, that is, that some generator of maximal degree in $\text{gin}(I)$ has the form

$$(4) \quad a^k x_0^{h_0} \cdot \dots \cdot x_r^{h_r}, \text{ with } h_r > 0.$$

Let T be the unique reduced Gröbner basis with respect to the 1-elimination order for I . Let P be an element of T of maximal degree and involving x_r , say $\text{deg}(P) = k + h$,

$$P = a^k x_1^{h_1} x_2^{h_2} \cdot \dots \cdot x_{r-1}^{h_{r-1}} x_r^{h_r} + p \in T,$$

where $\sum h_i = h$ and $\text{deg}(p) = k + h$, $h_r > 0$.

Now $p \neq 0$, since a product of monomials cannot be a generator of the nondegenerate and irreducible ideal I . Furthermore, x_r cannot divide p , because then it divides all of P , and since I is irreducible and nondegenerate, we would have $\frac{1}{x_r}P \in I$. So P would not be a minimal generator of I .

By the Borel-fixedness of $\text{gin}(I)$, we may perform the elementary move

$$e_{r-1}(\text{LeadTerm}(P)) = a^k x_1^{h_1} \cdot x_{r-2}^{h_{r-2}} x_{r-1}^{h_{r-1}+1} x_r^{h_r-1} \in \text{gin}(I).$$

So there is a $Q \in T$,

$$Q = a^{b_0} x_1^{b_1} x_2^{b_2} \cdot \dots \cdot x_{r-2}^{b_{r-2}} x_{r-1}^{b_{r-1}} x_r^{b_r} + q,$$

with $b_0 \leq k$, $b_{r-1} = h_{r-1} + 1$ (if not, $\text{LeadTerm}(Q)$ would divide $\text{LeadTerm}(P)$, which is a generator of the $\text{gin}(I)$), $b_r < h_r$, $\sum b_i \leq k + h$, $q \neq 0$.

Set

$$(5) \quad G_1 = x_{r-1}P - a^{k-b_0} x_1^{h_1-b_1} \cdot \dots \cdot x_{r-2}^{h_{r-2}-b_{r-2}} x_r^{h_r-b_r} Q$$

$$(6) \quad = x_{r-1}p - a^{k-b_0} x_1^{h_1-b_1} \cdot \dots \cdot x_{r-2}^{h_{r-2}-b_{r-2}} x_r^{h_r-b_r} q.$$

Notice that $G_1 \in I$, $\text{deg}(G_1) = k + h + 1$. Notice that $G_1 \neq 0$ because, by (5), that would imply that x_r divides P . Set $g_1 = \text{LeadTerm}(G_1)$. We now examine candidates for the lead term of G_1 , based on (5). Notice that

$$g_1 \leq \max\{\text{LeadTerm}(x_{r-1}P), \text{LeadTerm}(a^{k-b_0} x_1^{h_1-b_1} \cdot \dots \cdot x_{r-2}^{h_{r-2}-b_{r-2}} x_r^{h_r-b_r} Q)\}.$$

Since the two lead terms on the right side of the above inequality are equal and, by the construction of G_1 , cancel, the inequality is in fact strict.

From G_1 we will inductively construct an infinite sequence G_i of polynomials of degree $k + h + 1$ with strictly decreasing leading terms, thus leading to a contradiction.

Now assume inductively that we have defined

$$G_i, \text{ with } \deg(G_i) = k + h + 1, G_i \neq 0, \text{LeadTerm}(G_i) = g_i.$$

Let $F_i \in T$ be the Gröbner basis element that has lead term $f_i \in \text{gin}(I)$ so that f_i divides g_i . We examine the possible cases for g_i .

If g_i appears as a monomial term of $x_{r-1}p$, then we have an equation

$$(7) \quad f_i v = g_i = x_{r-1} \hat{p},$$

where \hat{p} is a monomial term of p . Now by assumption $\deg(f_i) \leq k + h$, and $\deg(g_i) = k + h + 1$; so $\deg(v) > 0$. Since T is a *reduced* Gröbner basis, f_i does not divide \hat{p} , but $\deg(f_i) \leq \deg(\hat{p})$ by assumption. Together with (7), we see then that

$$\begin{aligned} \deg_{x_{r-1}}(f_i) &= \deg_{x_{r-1}}(\hat{p}) + 1, \\ \deg_{x_j}(f_i) &\leq \deg_{x_j}(\hat{p}) \text{ for all } j, \\ \deg_a(f_i) &\leq \deg_a(\hat{p}), \end{aligned}$$

and so

$$(8) \quad \deg_{x_j}(f_i) < \deg_{x_j}(\hat{p}) \text{ for some } j \neq r - 1 \text{ or } \deg_a(f_i) < \deg_a(\hat{p}).$$

If, in (8), $j < r - 1$, then by the Borel-fixedness of $\text{gin}(I)$, a sequence of elementary moves produces

$$(9) \quad f' = e_j(e_{j+1}(\dots(e_{r-2}(f_i))\dots)) \in \text{gin}(I).$$

The elementary move sequence has the effect of replacing an x_{r-1} with an x_j . If $\hat{f} = \text{LeadTerm}(\hat{F}), \hat{F} \in T$, is a generator of $\text{gin}(I)$ dividing f' , then by referring back to (7) we see that \hat{f} divides \hat{p} , contradicting the fact that T is a reduced Gröbner basis. The case of $\deg_a(f_i) < \deg_a(\hat{p})$ is similar.

Thus, if (8) holds, it must be that $j = r$. Then x_r divides v , say $x_r v_i = v$, and so we can define

$$(10) \quad G_{i+1} = G_i - c_i x_r v_i F_i,$$

with c_i a constant, chosen so that $\text{in}(G_i)$ cancels with $\text{in}(-c_i x_r v_i F_i)$. G_{i+1} is nonzero since x_r does not divide P . G_{i+1} is still of degree $k + h + 1$, and the lead term of G_{i+1} is necessarily later in the elimination order than G_i .

If g_i does not appear as a term of $x_{r-1}p$, then g_i is a term of

$$(11) \quad a^{k-b_0} x_1^{h_1-b_1} \dots x_{r-2}^{h_{r-2}-b_{r-2}} x_r^{h_r-b_r} q - x_r v_{i-1} F_{i-1},$$

and we have

$$(12) \quad f_i v = g_i = x_{r-1} \hat{q},$$

where (since by assumption $\deg(f_i) \leq k + h$) $\deg(v) > 0$, and \hat{q} is some term of (11). If x_r divides v , then we again write $v = x_r v_i$, and define G_{i+1} as in (10). If x_r does not divide v , then we use Borel-fixedness of $\text{gin}(I)$ and a sequence of elementary moves to produce

$$f' = e_j(e_{j+1}(\dots(e_{r-1}(f_i))\dots)) \in \text{gin}(I),$$

which has the effect of replacing an x_r with an x_j . There is then an $\hat{F} \in T$ that has lead term \hat{f} dividing f' , so that, returning to (12), we have $\hat{f}x_r\hat{v}_i = g_i$. Redefine $F_i = \hat{F}$ and define G_{i+1} as in (10). Again, $G_{i+1} \neq 0$, G_{i+1} is still of degree $k+h+1$, and the lead term of G_{i+1} is necessarily later in the elimination order than G_i .

Thus our induction produces an infinite sequence of G_i with successively later lead terms (the degree of the G_i stays fixed for all i), but all the G_i are nonzero. This is the desired contradiction.

Thus it cannot be that any terms of $\text{gin}(I)$ of maximal degree involve x_r . So terms of maximal degree do not involve x_r . Hence we have proved the claim and the theorem.

Note 2.3. If X is smooth, there is an approach to possibly improving Theorem 2.1 yielding the bound

$$(13) \quad \text{reg}(\mathcal{I}_X) \leq \max\{\text{reg}(\mathcal{K}_0), \text{reg}(\mathcal{K}_1) + 1\}.$$

The exact sequence

$$(14) \quad 0 \rightarrow \mathcal{K}_0 \rightarrow \pi_*(\mathcal{I}_X) \rightarrow \mathcal{K}_1(-1) \rightarrow 0$$

may be able to give us an immediate improvement on Theorem 2.1. Let

$$\text{reg}(\mathcal{K}_0) = m, \quad \text{reg}(\mathcal{K}_1) = n, \quad p = \max\{m, n + 1\}.$$

Part of the associated cohomological long exact sequence of (14),

$$(15) \quad \dots \rightarrow H^i(\mathcal{K}_0(p-i)) \rightarrow H^i(\pi_*\mathcal{I}_X(p)) \rightarrow H^i(\mathcal{K}_1(p-i-1)) \rightarrow \dots,$$

is exact for each $i > 0$. By the choice of p , both the cohomology group on the left and the one on the right vanish; hence so does the one in the middle. Thus

$$\text{reg}(\pi_*(\mathcal{I}_X)) \leq p \leq \max\{\text{reg}(\mathcal{K}_0), \text{reg}(\mathcal{K}_1) + 1\}.$$

The aim is then to use this to prove

$$\text{reg}(\mathcal{I}_X) \leq \max\{\text{reg}(\mathcal{K}_0), \text{reg}(\mathcal{K}_1) + 1\}.$$

To do this, one may look at the exact sequence

$$0 \rightarrow \pi_*\mathcal{I}_X \rightarrow \pi_*\mathcal{O}_{\mathbf{P}^r} \rightarrow \pi_*\mathcal{O}_X \rightarrow R^1\pi_*\mathcal{I}_X \rightarrow R^1\pi_*\mathcal{O}_{\mathbf{P}^r} \rightarrow R^1\pi_*\mathcal{O}_X \rightarrow \dots$$

If $R^1\pi_*\mathcal{I}_X$ vanishes, a spectral sequence argument shows that the regularity of $\pi_*\mathcal{I}_X$ matches with the regularity of \mathcal{I}_X .

Remark 2.4. While this bound is no different than the theorem if X is a curve, in higher dimensions the partial elimination ideals \mathcal{K}_k , $k \geq 2$, of the generic projections usually define nonempty varieties.

3. PARTIAL ELIMINATION IDEALS AND THE CANONICAL BUNDLE

In this section we use the partial elimination ideal sheaves to compute the canonical bundle of a variety as well as the canonical bundles of the multiple point loci, under the right conditions. This information is used to bound the regularity of the partial elimination ideal sheaves and, in conjunction with Theorem 2.1, to obtain regularity bounds for curves X . Many of the questions surrounding partial elimination ideals remain unanswered, and in particular we are limited by what is known about the singularities of a generic projection.

For low-dimensional varieties where the Regularity Conjecture bound has been established, or something near to it, our methods will not provide any new information. However, these methods may shed some light on the possibility of near $\text{deg}(X)$ -regularity in arbitrary dimension. This chapter will hopefully pique interest in exploring the elimination order, partial elimination ideals, and their geometry.

3.1. Curves. Much effort has been made to understand multiple point loci. It has not always been clear what sheaf structure should be accorded to the loci. This is a broad subject unto itself, and it is beyond the scope of this paper to give even a thorough list of references, but see for instance the survey of Kleiman ([Kle82]).

We establish here the fact that for a generic projection of a space curve of degree d into \mathbf{P}^2 , the canonical bundle can be computed as a twist of \mathcal{K}_1 . d -Regularity for curves follows by examining some cohomological exact sequences by using the Kodaira Vanishing Theorem and Theorem 2.1 and its corollary.

Let $X_0 \subset \mathbf{P}^r$ be a nondegenerate smooth reduced irreducible curve of degree d . Let $\pi_0 : \mathbf{P}^r \rightarrow \mathbf{P}^3$ be a generic projection, with X_0 isomorphic to its image, which we call X . Let $\pi : \mathbf{P}^3 \rightarrow \mathbf{P}^2$ be a generic projection, so that the only singularities of π are ordinary double points, and write $\pi(X) = \bar{X}$. Let \mathcal{K}_1 be the first partial elimination ideal of the map π . Let \bar{D}_2 be the variety defined by \mathcal{K}_1 . By Proposition 1.10, \bar{D}_2 is also the set-theoretic double point locus.

We aim to prove

Theorem 3.1. *The curve X_0 is d -regular.*

Remark 3.2. This is a weaker bound for curves than what is already known.

Proof. We will first compute the canonical bundle. Following the method of [Xu94], we consider the blowup

$$f : \tilde{X} \subset V = \text{Bl}_{\bar{D}_2}(\mathbf{P}^2) \rightarrow \bar{X} \subset \mathbf{P}^2$$

along \bar{D}_2 , with exceptional divisor $E = \sum_{q_i \in \bar{D}_2} E_i$. This gives us

$$\begin{aligned} K_{\tilde{X}} &= (K_V + \tilde{X})|_{\tilde{X}} \\ &= (f^*(K_{\mathbf{P}^2}) + E)|_{\tilde{X}} + (f^*(\bar{X}) - 2E)|_{\tilde{X}} \\ &= (f^*(K_{\mathbf{P}^2} + \bar{X}) - E)|_{\tilde{X}} \\ &= (f^*(\mathcal{O}_{\bar{X}}(d-3)) - E)|_{\tilde{X}}, \end{aligned}$$

where $\mathcal{O}_{\bar{X}} = \mathcal{O}_{\mathbf{P}^2}/\mathcal{K}_0$. This computation enables us to prove

Lemma 3.3. $K_{\tilde{X}} \cong \mathcal{K}_1 \otimes \mathcal{O}_{\tilde{X}}(d-3)$.

Proof of the Lemma. Suppose g is a section of $H^0(\mathcal{K}_1 \otimes \mathcal{O}_{\bar{X}}(d-3))$. Then since g vanishes on \bar{D}_2 , it induces a section of $K_{\tilde{X}}$.

Conversely, fix $q_{i_0} \in \bar{D}_2$, and an open neighborhood U of q_{i_0} not meeting \bar{D}_2 except at q_{i_0} . Take local analytic coordinates (u_1, u_2) on U with $q_{i_0} = (0, 0)$, $\bar{X} = \{u_1 u_2 = 0\}$, and hence so that the local sections of \mathcal{K}_1 are u_1 and u_2 . Consequently, the blowup is defined locally near the exceptional divisor E_{i_0} as a subset of $\mathbf{A}^2 \times \mathbf{P}^1$ by coordinates $((u_1, u_2), [R_1, R_2])$ by $u_1 R_2 = u_2 R_1$. \tilde{X} is given by $\{u_1 = R_1 = 0\}$, which has local coordinate u_2 , and $\{u_2 = R_2 = 0\}$, which has local coordinate u_1 .

From the equation defining \bar{X} down in U , we get the relation $u_1 du_2 + u_2 du_1 = 0$. Hence an element of the canonical bundle of \bar{X} takes the form

$$\frac{rdu_1}{u_1} = -\frac{rdu_2}{u_2},$$

where r must be of degree $d - 3$ since, recalling that \bar{X} is a hypersurface of degree d in \mathbf{P}^2 , the differential $du_1/u_1 = du_2/u_2$ has a pole of degree $d - 3$. Thus, $r \in \mathcal{O}_{f^{-1}(U) \cap X}(d - 3)$. Now

$$\begin{aligned} \mathcal{O}_{f^{-1}(U) \cap X} &= \bigoplus_{i=1,2} (u_{3-i}) \mathcal{O}_{f^{-1}(U)} / (u_i) \\ &\cong (u_1, u_2) \mathcal{O}_U / (u_1 u_2) \\ &\cong \mathcal{K}_1 \otimes \mathcal{O}_{\bar{X} \cap U}. \end{aligned}$$

Hence r corresponds to a section of $\mathcal{K}_1 \otimes \mathcal{O}_{\bar{X}}(d - 3)$. This proves the lemma.

In order to prove that \mathcal{I}_{X_0} is d -regular, we will show that \mathcal{I}_X is d -regular. Then, since the map π_0 is an isomorphism and a generic projection, it can be decomposed into a sequence of maps $\pi_i' : \mathbf{P}^{r-i} \rightarrow \mathbf{P}^{r-i-1}$ which are generic projections and in which X_0 is isomorphic to its successive images under the π_i' . Then, by Theorem 2.1, we have

$$\text{reg}(X_0) \leq \text{reg}(X).$$

In order to prove that \mathcal{I}_X is d -regular, we will show that \mathcal{K}_0 is d -regular and \mathcal{K}_1 is $(d - 1)$ -regular. Then, since π has at most double points, so that the \mathcal{K}_k , $k \geq 2$, define empty varieties, Theorem 2.1 now gives us

$$\begin{aligned} \text{reg}(\mathcal{I}_X) &\leq \max\{\text{reg}(\mathcal{K}_0), \text{reg}(\mathcal{K}_1) + 1\} \\ &\leq \max\{d, (d - 1) + 1\} \\ &\leq d. \end{aligned}$$

Since \bar{X} is an irreducible hypersurface of degree d , \mathcal{K}_0 is generated by a single equation in degree d , i.e., $\mathcal{K}_0 \cong \mathcal{O}_{\mathbf{P}^2}(-d)$. Also, $\mathcal{O}_{\mathbf{P}^2}(-3) = K_{\mathbf{P}^2}$. We now utilize the exact sequence

$$(16) \quad 0 \rightarrow \mathcal{K}_0 \rightarrow \mathcal{K}_1 \rightarrow \mathcal{K}_1/\mathcal{K}_0 \rightarrow 0.$$

Recalling the lemma, we know that $\mathcal{K}_1/\mathcal{K}_0 \cong K_{\bar{X}}(-(d - 3))$. Hence, if we twist (16) by $(d - 3)$ and take the cohomological exact sequence, we get, in part,

$$\cdots \rightarrow H^1(K_{\bar{X}}) \rightarrow H^2(K_{\mathbf{P}^2}) \rightarrow H^2(\mathcal{K}_1(d - 3)) \rightarrow H^2(K_{\bar{X}}) \rightarrow \cdots .$$

Now the map $H^1(K_{\bar{X}}) \rightarrow H^2(K_{\mathbf{P}^2})$ is the Gysin map, and is an isomorphism in this case. Also, $H^2(K_{\bar{X}}) = 0$ because X is a curve. Thus $H^2(\mathcal{K}_1(d - 2)) = 0$. By twisting instead by $(d - 2)$, we have

$$\cdots \rightarrow H^1(K_{\mathbf{P}^2}(1)) \rightarrow H^1(\mathcal{K}_1(d - 2)) \rightarrow H^1(K_{\bar{X}}(1)) \rightarrow \cdots .$$

Here the Kodaira Vanishing Theorem gives us that the cohomology groups on the left and right are both zero; hence $H^1(\mathcal{K}_1(d - 2)) = 0$. So \mathcal{K}_1 is $(d - 1)$ -regular. Since \mathcal{K}_0 is generated in degree d , it is d -regular. This proves Theorem 3.1.

3.2. Higher dimensions. The canonical bundle of the image of a generic projection can be computed as a twist of the ideal sheaf \mathcal{K}_1 under some mild assumptions.

We need to expand a definition: Let X be a smooth, nondegenerate, reduced, irreducible variety of dimension n and degree d in \mathbf{P}^r . Take generic coordinates $a_1, a_2, \dots, a_{r-s}, x_0, x_1, \dots, x_s$ on \mathbf{P}^r so that $\Lambda = \{x_0 = 0, x_1 = 0, \dots, x_s = 0\}$ does not meet X and so that projection from Λ is generic. Fix $\pi : \mathbf{P}^r \rightarrow \mathbf{P}^s$ as the projection from Λ . Let $\bar{X} = \pi(X)$. Fix as the monomial order on \mathbf{P}^r the $(r - s)$ -elimination order on the variables $a_1, a_2, \dots, a_{r-s}, x_0, x_1, \dots, x_s$. This elimination corresponds to the projection of \mathbf{P}^r from Λ to \mathbf{P}^s . The coordinates on \mathbf{P}^s will then be x_0, x_1, \dots, x_s . Let X be defined by the saturated ideal $I \subset \mathbf{C}[a_1, a_2, \dots, a_{r-s}, x_0, x_1, \dots, x_s]$ with sheafification \mathcal{I} .

Definition 3.4. The $(1, r - s)$ -th partial elimination ideal of I is defined as

$$K_1(I) = \{f \mid a_{r-s}f + g \in I \text{ with } f, g \in \mathbf{C}[x_0, x_1, \dots, x_s]\}.$$

Remark 3.5. If we decompose the map π into generic projections $\pi_i' : \mathbf{P}^{r-i} \rightarrow \mathbf{P}^{r-i-1}$, the $(1, r - s)$ -elimination ideal will be identical to the first partial elimination ideal of the projection π_{r-s+1}' .

Now fix $s = n + 1$.

We now prove

Proposition 3.6. *Let X be a smooth, irreducible, reduced, and nondegenerate variety in \mathbf{P}^r . Let S be the set of singularities of the generic projection of X to \mathbf{P}^{n+1} other than the ordinary double points. Assume S is codimension 2 in X . Then*

$$K_{\bar{X}} \cong \mathcal{K}_1 \otimes \mathcal{O}_{\bar{X}}(d - n - 2).$$

Proof. Let \bar{D}_2 be the variety defined by \mathcal{K}_1 . By blowing up along \bar{D}_2 , and computing the canonical bundle of \bar{X} on the blowup, a computation similar to Lemma 3.3 shows that the canonical bundle \bar{X} is isomorphic to $\mathcal{K}_1 \otimes \mathcal{O}_{\bar{X}}(d - n - 2)$ away from S . Since S is codimension 2 in X , there is a unique analytic extension so that the sheaves match globally. This proves the proposition.

Corollary 3.7. *Let X be a smooth, irreducible, reduced, nondegenerate variety in \mathbf{P}^r . If X is an n -dimensional variety in \mathbf{P}^{n+2} , and the set S , defined in the proposition, of the generic projection of X to \mathbf{P}^{n+1} has codimension 2 in \bar{X} , then \mathcal{K}_1 is $(d - 1)$ -regular.*

Proof. In the situation of the generic projection $\pi : \mathbf{P}^{n+2} \rightarrow \mathbf{P}^{n+1}$, we can compute a regularity bound for \mathcal{K}_1 . We have an exact sequence

$$(17) \quad 0 \rightarrow \mathcal{K}_0 \rightarrow \mathcal{K}_1 \rightarrow \mathcal{K}_1/\mathcal{K}_0 \rightarrow 0.$$

Since \bar{X} is irreducible and codimension 1 in \mathbf{P}^{n+1} , $\mathcal{K}_0 \cong \mathcal{O}_{\mathbf{P}^{n+1}}(-d)$. So, in conjunction with Proposition 3.6, (17) twisted by $d - n - 2$ gives us

$$(18) \quad 0 \rightarrow K_{\mathbf{P}^{n+1}} \rightarrow \mathcal{K}_1(d - n - 2) \rightarrow K_{\bar{X}} \rightarrow 0.$$

Various twists of segments of the cohomological long exact sequence of (18) give us

$$\begin{aligned} \cdots &\rightarrow H^i(K_{\mathbf{P}^{n+1}}(n + 1 - i)) \rightarrow H^i(\mathcal{K}_1(d - 1 - i)) \rightarrow H^i(K_{\bar{X}}(n + 1 - i)) \\ &\rightarrow \cdots \end{aligned}$$

By the Kodaira Vanishing Theorem, we have

$$H^i(K_{\bar{X}}(m)) = 0 \text{ and } H^i(K_{\mathbf{P}^{n+1}}(m)) = 0$$

for all $i > 0, m > 0$. Moreover, as a sheaf on \mathbf{P}^{n+1} , $H^i(\mathcal{K}_1(m)) = 0$ for all $i > n + 1$ and all m . Thus we have

$$H^i(\mathcal{K}_1(d - 1 - i)) = 0 \text{ for } 0 < i < n + 1.$$

It remains to show that $H^{n+1}(\mathcal{K}_1(d - n - 2)) = 0$. For this we examine part of the cohomological exact sequence as follows:

$$\cdots \rightarrow H^n(K_{\bar{X}}) \rightarrow H^{n+1}(K_{\mathbf{P}^{n+1}}) \rightarrow H^{n+1}(\mathcal{K}_1(d - n - 2)) \rightarrow H^{n+1}(K_{\bar{X}}) \rightarrow \cdots$$

Here, the Gysin map $H^n(K_{\bar{X}}) \rightarrow H^{n+1}(K_{\mathbf{P}^{n+1}})$ is an isomorphism, and since X is n -dimensional, $H^{n+1}(K_{\bar{X}}) = 0$. Thus $H^{n+1}(\mathcal{K}_1(d - n - 2))$ also vanishes. So \mathcal{K}_1 is $(d - 1)$ -regular. This proves the corollary.

4. THE REGULARITY OF HIGHER-DIMENSIONAL VARIETIES AND THE SHARP REGULARITY BOUND

Ideally, one hopes to prove that each successive k -fold locus can be computed as a twist or sum of twists of \mathcal{K}_k , and then to use this to bound the regularity of the \mathcal{K}_k . However, this involves discerning the types of singularities beyond ordinary k -fold points and reconciling the behavior of the canonical bundle there with its behavior near ordinary multiple points. With this information, the next obstacle is to use Kodaira Vanishing or other geometric or cohomological tools to get the regularity of the \mathcal{K}_k . From there, an application of Theorem 2.1 would yield a bound on the regularity of \mathcal{I}_X .

To improve Theorem 2.1, note that while the maximum degree of a generator of the reverse lexicographic gin gives the exact regularity of the ideal, the elimination order may not. The examples seem to indicate that for each generic projection, we have $\text{reg}(\mathcal{I}_X) < \text{reg}(\mathcal{K}_0)$. Using this, a smooth nondegenerate projective variety $X \subset \mathbf{P}^r$ of dimension n and degree d could be projected generically to $\bar{X} \subset \mathbf{P}^{n+1}$, which is cut out by a single equation of degree d , and so $\text{reg}(\bar{X}) = d$. Since we have a chain of $r - n - 1$ strict inequalities between the regularity of X and the regularity of \bar{X} , we get that

$$\text{reg}(\mathcal{I}_X) \leq \text{reg}(\bar{X}) - (r - n - 1) = d - r + n + 1,$$

which is the conjectured bound.

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