A SYMMETRY COMPLETENESS CRITERION
FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract. A simple criterion for the completeness of an infinitesimal automorphism of a second-order differential equation is given.

Given a second-order differential equation $Y$ on a connected manifold $M$, and a vector field $X$ on $M$, we say that $X$ is an infinitesimal automorphism of $Y$ if $[\tilde{X}, Y] = 0$, where $\tilde{X}$ is the natural lift of $X$ to $TM$. It is of interest to know when $X$ has a complete flow, which is to say that $X$ is the infinitesimal generator of a one-parameter group of diffeomorphisms of $M$, and hence generates a one-parameter group of automorphisms of $Y$.

Example 1. Consider the second-order equation $Y$ on $M = \mathbb{R}$ with

$$Y = v\partial_v + (1 + 2x)(1 + x^2)^{-1}v^2 \partial_x.$$ 

Then $X = (1 + x^2)\partial_x$ satisfies $[\tilde{X}, Y] = 0$, but $X$ is not complete.

In this example, the flow of $Y$ is not complete. However, $Y$ can be incomplete with $X$ complete as in

Example 2. Let $g$ be any left-invariant Lorentz metric on the affine group $G = A(1, \mathbb{R})$. Then the geodesic flow is incomplete, but each left-invariant vector field $X$ on $G$ generates a one-parameter group of symmetries of the geodesic vector field.

In light of these examples, it is perhaps somewhat surprising that the following theorem holds.

Theorem. Let $Y$ be a second-order differential equation on a manifold $M$ with a complete flow $\psi_t$. Let $X$ be an infinitesimal automorphism of $Y$. Then $X$ is necessarily complete; that is, it generates a one-parameter group of diffeomorphisms of $M$.

Proof. First observe that in order to prove that $X$ is complete, it suffices to prove the uniform time $\tau$ existence for the local flow. Let $\pi : TM \to M$ be the projection, and let $\phi_t(x)$, $|t| < \tau$ be the local flow of $X$ about $x \in M$. Let $\bar{\phi}_t(p)$ be the lifted local flow about $p \in TM$ with $\pi(p) = x$. Then $\phi_t \circ \pi = \pi \circ \bar{\phi}_t$. Since $[\tilde{X}, Y] = 0$, $\psi_{t_s}X = \tilde{X}$ for all $t$. This says that if $\psi_s(p) = q$ for some $s$, then $\bar{\psi}_t(q) = \psi_s \circ \bar{\phi}_t(p)$.
Hence we may conclude existence of both a \( \tau > 0 \) and a local flow of \( \tilde{\phi}_t(q) \) for all time \( t \) with \( |t| < \tau \), and thus that of the local flow of \( \tilde{\phi}_t(\tau(q)) \) for the same time \( |t| < \tau \) as well.

It follows that we have time \( \tau \) existence of the local flow of \( \phi_t \) for all points \( x' \in M \) such that there is an integral curve of \( Y \) that projects to a curve connecting \( x \) and \( x' \). Continuing in this manner we may also conclude the same time \( \tau \) existence for all points \( x'' \) that may be so connected to \( x' \).

We will be done if we can show that any point \( y \in M \) may be connected to \( x \) by a finite chain of such points \( x, x', x'', \ldots, y \), all with the same time \( \tau \) existence. For this, it suffices to show that each point \( z \in M \) is connected to every other point in some open neighbourhood of \( z \). This suffices because a connected manifold is path connected, and so we can find a smooth curve \( \gamma : [0, 1] \to M \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \). Compactness then allows us to select a finite number of points \( x, x', x'', \ldots, y \) on the image of \( \gamma \) from the open covering provided by the following lemma to make the connection. This provides the uniform time \( \tau \) estimate that we require.

**Lemma.** Let \( Y \) be a second-order differential equation on a manifold \( M \). Then there is an open neighbourhood of \( x \in M \) in which every point \( x' \) is connected to \( x \) by the projection \( \pi \) of an integral curve of \( Y \).

In other words, for all points \( x' \) in some open neighbourhood of \( x \), we may choose an initial velocity \( v_x \) at \( x \) so that the projection of the integral curve of \( Y \) through \( v_x \) contains \( x' \). The proof of the lemma follows easily from the fact that the exponential map is onto. \( \square \)

This proof is patterned after the discussion of completeness for first-order \( G \)-structures in [1] and [2]. One should also note that in [3] an infinitesimal automorphism is defined to be complete in order to have a pretty statement about the maximal dimension of the symmetry group.

**References**


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