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COHOMOLOGY AND FINITE SUBGROUPS OF PROFINITE GROUPS

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Abstract. We prove two theorems linking the cohomology of a pro-p group G with the conjugacy classes of its finite subgroups.

The number of conjugacy classes of elementary abelian p-subgroups of G is finite if and only if the ring $H^*(G,\mathbb{Z}/p)$ is finitely generated modulo nilpotent elements.

If the ring $H^*(G,\mathbb{Z}/p)$ is finitely generated, then the number of conjugacy classes of finite subgroups of G is finite.

1. Pro-p groups and elementary abelian pro-p subgroups

Let G be a profinite group. Fix a fundamental system $\mathcal{U}$ of open neighborhoods U of $1 \in G$ such that $\bigcap_{U\in\mathcal{U}} U = 1$ and each U is an open normal subgroup of G. For convenience, write $V \le U$ for $V \subset U$ and $U, V \in \mathcal{U}$. For $U \in \mathcal{U}$, set $G_U = G/U$. Denote by $\varphi_{U,V} : G_V \to G_U$ the projection map with $V \le U$. $\{G_U, \varphi_{U,V}\}$ is an inverse system and

$$G = \varprojlim G_U.$$

Let $\varphi_U : G \to G_U, U \in \mathcal{U}$, be the projection map. For a given prime p, let $\mathcal{E}$ (resp. $\mathcal{E}_U$) be the set of finite elementary abelian p-subgroups of G (resp. G_U). (A finite elementary abelian p-group is one isomorphic to $(\mathbb{Z}/p)^n$, for some n, possibly 0.) It is clear that $\varphi_U(E)$ and $\varphi_{U,V}(F)$ are elements of $\mathcal{E}_U$, for every $E \in \mathcal{E}, F \in \mathcal{E}_V, V \le U$. Denote by $\psi_U : \mathcal{E} \to \mathcal{E}_U$ (resp. $\psi_{U,V} : \mathcal{E}_V \to \mathcal{E}_U, V \le U$) the map induced from φ_U (resp. $\varphi_{U,V}$).

Proposition 1. *Given $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ with $W \le V$ such that, for every $E \in \mathcal{E}_W$, $\psi_{V,W}(E) \in \operatorname{Im}\psi_V$.*

Proof. Let F be an element of $\mathcal{E}_V$ of rank n. Notice that if $F \in \operatorname{Im}\psi_{V,U}$ for $U \le V$, then in fact $F = \psi_{V,U}(E)$ for E of rank n. For each open $U \le V$, let $\mathcal{F}_U$ denote the set of homomorphisms $\eta : F \to G_U$ such that $\psi_{V,U}\eta = \mathrm{Id}_F$, and consider the inverse system of the $\mathcal{F}_U$ under the maps ψ.

If all the $\mathcal{F}_U$ are nonempty, then, since the inverse limit over a directed system of nonempty finite sets is always nonempty, $F \in \operatorname{Im}\psi_V$. Otherwise, let $U_F \le V$ be such that $\mathcal{F}_{U_F}$ is empty. Set $W = \bigcap_{F \notin \operatorname{Im}\psi_V} U_F$. □

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The following notation will be used. A prime p will be fixed, and all groups will be pro-p groups unless otherwise indicated. For any profinite group K, $H^*(K)$ will always denote the (continuous) cohomology of K with coefficients $\mathbb{Z}/p$. Denote by $I(K)$ the ideal of $H^*(K)$ consisting of elements that restrict trivially to every elementary abelian p-subgroup of K, and by $\mathfrak{R}_K$ the nilradical of $H^*(K)$. Set

$$H(K) = \begin{cases} H^{ev}(K) & \text{for } p > 2, \\ H^*(K), & \text{for } p = 2, \end{cases}$$

and let $H^+(K)$ be the ideal of $H^*(K)$ consisting of elements of positive degrees. We have

Proposition 2. *Given $\eta \in I(G)$, there exist $W \in \mathcal{U}$ and $\xi \in I(G_W)$ such that $(\varphi_W)^*(\xi) = \eta$.*

In particular, if $H^*(G)$ is finitely generated, there exists $U \in \mathcal{U}$ such that $I(G) \subset (\varphi_U)^*(I(G_U))$.

Proof. Let η be an element of $I(G)$. Since $H^*(G) = \varinjlim H^*(G_U)$, there exist V and $\zeta \in H^*(G_V)$ such that $\varphi_V^*(\zeta) = \eta$. Set $\xi = \varphi_{V,W}^*(\zeta)$ with W given in Proposition 1. Also $\varphi_W^*(\xi) = \eta$. We now prove that $\xi \in I(G_W)$.

Let E be an element of $\mathcal{E}_W$, and set $K = \psi_{V,W}(E)$. By Proposition 1, there exists $M \in \mathcal{E}$ such that $\psi_V(M) = K$. Consider the commutative diagram

$$\begin{array}{ccccc} H^*(G_W) & \xleftarrow{\varphi_{V,W}^*} & H^*(G_V) & \xrightarrow{\varphi_V^*} & H^*(G) \\ {\scriptstyle \mathrm{Res}}\downarrow & & {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ H^*(E) & \xleftarrow[(\varphi_{V,W}|_E)^*]{} & H^*(K) & \xrightarrow[(\varphi_V|_M)^*]{} & H^*(M) \end{array}.$$

Since $\varphi_V|_M$ is a monomorphism and $\eta|_M = 0$, the right square of the diagram tells us that $\zeta|_K = 0$. So, by the commutativity of the left square, $\xi|_E = 0$. Hence $\xi \in I(G_W)$.

If $H^*(G)$ is finitely generated, V can be chosen such that $H^*(G) = \operatorname{Im}(\varphi_V)^*$. It follows from what we just proved that $I(G) \subset (\varphi_W)^*(I(G_W))$. □

We can now deduce the profinite case of the following theorem of Quillen from the finite case, where many fairly elementary proofs are now known.

Theorem 1 (Quillen [8])**.** *For any profinite group G, $I(G) \subset \mathfrak{R}_G$. In other words, every element of $I(G)$ is nilpotent.*

Proof. Straightforward from Proposition 2, by noting that $I(G_W) \subset \mathfrak{R}_{G_W}$, by the finite case, since G_W is finite. □

The statement of Quillen's Stratification Theorem involves infinite elementary abelian subgroups, but these are not necessary for the detection result above.

This result also appears in [9].

Proposition 3. *Suppose that there exist $U \in \mathcal{U}$ and $a \in G \setminus U$ with* $\operatorname{ord}(a) = p$. *Set $b = \varphi_U(a) \in G_U$. Then there exists $\xi \in H^+(G_U)$ satisfying:*

(a) $\xi|_{\langle b\rangle} \neq 0$;

(b) $\varphi_U^*(\xi)|_{\langle a\rangle} \neq 0$. *In particular, $\varphi_U^*(\xi)$ is not nilpotent in $H^+(G)$.*

Proof. By [7, Theorem 2.7], there exists $\xi \in H^+(G_U)$ such that $\xi|_{\langle b\rangle}$ is not nilpotent. From the commutative diagram

$$\begin{array}{ccc} H^*(G_U) & \xrightarrow{\varphi_U^*} & H^*(G) \\ {\scriptstyle\text{Res}}\downarrow & & \downarrow{\scriptstyle\text{Res}} \\ H^*(\langle b\rangle) & \xrightarrow[(\varphi_U|_{\langle a\rangle})^*]{} & H^*(\langle a\rangle) \end{array},$$

since $H^*(\langle b\rangle) \overset{(\varphi_U|_{\langle a\rangle})^*}{\to} H^*(\langle a\rangle)$ is an isomorphism, it follows that $\varphi_U^*(\xi)|_{\langle a\rangle}$ is not nilpotent. The proposition follows. □

Theorem 2. *A pro-p group A is torsion-free if and only if $H^+(A) \subset \mathfrak{R}_A$.*

Proof. If A is not torsion-free, it follows from Proposition 3 that $H^+(A)$ contains a non-nilpotent element. If A is torsion-free, then $I(G) = H^+(A)$; hence, by Theorem 1, any element of $H^+(A)$ is nilpotent. □

Corollary 1. (i) *A subgroup A of G is torsion-free if and only* $\mathrm{Im}(H^+(G) \overset{\mathrm{Res}}{\to} H^+(A)) \subset \mathfrak{R}_A$.
(ii) *If $H^*(G)/\mathfrak{R}_G$ is finitely generated, then G contains an open, normal, torsion-free subgroup U.*

Proof. (i) If A is torsion-free, it follows from Theorem 2 that $H^+(A)$ consists of nilpotent elements; hence any element of $\mathrm{Im}(H^+(G) \overset{\mathrm{Res}}{\to} H^+(A))$ is nilpotent. Conversely, suppose that A contains an element a of order p. By Proposition 3, there exists $\zeta \in H^+(G)$ such that $\zeta|_{\langle a\rangle}$, hence $\zeta|_A$, is not nilpotent.
(ii) Suppose that $H^*(G)/\mathfrak{R}_G$ is finitely generated. Since $H^*(G) = \varinjlim H^*(G_V)$, there exists $U \in \mathcal{U}$ such that $H^*(G_U)/\mathfrak{R}_{G_U} \overset{(\varphi_U)^*}{\to} H^*(G)/\mathfrak{R}_G$ is surjective. Hence $\mathrm{Im}\,(H^+(G) \overset{\mathrm{Res}}{\to} H^+(U)) \subset \mathfrak{R}_U$. By (i), U is torsion-free.
The corollary follows. □

Remark 1. The following example shows that the converse of Corollary 1(ii) does not hold. For every $n \in \mathbb{N}$, define $\mathfrak{A}_n$ to be the procyclic group $\mathbb{Z}_2$ generated by e_n. Set $\mathfrak{A} = \hat{\bigoplus}_n \mathfrak{A}_n$ and let $\langle a\rangle \cong \mathbb{Z}/2$ act on $\mathfrak{A}$ by ${}^a e_n = e_n^{-1}$. Define $G = \mathfrak{A} \rtimes \mathbb{Z}/2$. For every n, $\langle e_n a\rangle \cong \mathbb{Z}/2$ and $\langle e_n a\rangle$ is not conjugate with $\langle e_m a\rangle$ if $m \neq n$. Hence there are infinitely many conjugacy classes of elementary abelian subgroups of G. According to Theorem 3 below, $H^*(G)/\mathfrak{R}_G$ is not finitely generated, although $\mathfrak{A}$ is open, normal and torsion-free in G.

We will give a necessary and sufficient condition for $H^*(G)$ to be finitely generated as a ring. First we prepare.

Lemma 1. *Suppose that $H^*(G)$ is finitely generated and M is a finite $\mathbb{F}_pG$-module. Then $H^*(G, M)$ is Noetherian over $H^*(G)$.*

Proof. We prove by induction on $n = \dim_{\mathbb{F}_p} M$. Suppose that the lemma holds for $n - 1$. Consider the exact sequence of $\mathbb{F}_pG$-modules

$$0 \to \mathbb{F}_p \overset{f}{\to} M \overset{g}{\to} N \to 0$$

with $N = M/f(\mathbb{F}_p)$. We then have the corresponding long exact sequence of cohomology

$$\dots H^{*-1}(G,N) \xrightarrow{\delta_*} H^*(G) \xrightarrow{f_*} H^*(G,M) \xrightarrow{g_*} H^*(G,N) \to \dots.$$

By the induction assumption, $H^*(G,N)$ is Noetherian over $H^*(G)$, hence so is $\operatorname{Im} g_*$. Pick elements $\xi_1, \dots, \xi_m$ of $H^*(G,M)$ so that $\{g_*(\xi_1), \dots, g_*(\xi_m)\}$ is a set of generators of $\operatorname{Im} g_*$. It follows that $H^*(G,M)$ is generated by $\xi_1, \dots, \xi_m$, as a module over $\operatorname{Im} f_*$. Hence $H^*(G,M)$ is Noetherian over $H^*(G)$. □

Corollary 2. *The following are equivalent:*

(a) *$H^*(G)$ is finitely generated;*
(b) *G contains an open normal, torsion-free subgroup U such that $H^*(U)$ is finite;*
(c) *there exists an open subgroup K of G such that $H^*(K)$ is finitely generated;*
(d) *$H^*(K)$ is finitely generated, for any open subgroup K of G.*

Proof. The implication (b) ⇒ (a) was proved by Quillen ([8, Proposition 13.5]) using a spectral sequence argument. It is clear that $(d) \Rightarrow (c)$ and $(d) \Rightarrow (a)$. Suppose that $H^*(G)$ is finitely generated and K is open in G. By the Eckmann-Shapiro lemma, $H^*(K) = H^*(G,M)$ with $M = \operatorname{Hom}_K(\mathbb{F}_pG, \mathbb{F}_p))$. Since M is Noetherian as an $\mathbb{F}_p$-module, $H^*(K)$ is Noetherian over $H^*(G)$, by Lemma 1. Since $H^*(G)$ is finitely generated, so is $H^*(K)$. In particular, if U is given as in Corollary 1 (ii), then $H^*(U)$ is finite dimensional, since $H^+(U) \subset \mathfrak{R}_U$. We then have $(a) \Rightarrow (b)$ and $(a) \Rightarrow (d)$.

Finally, suppose that K is open in G and $H^*(K)$ is finitely generated. It follows that K contains an open, normal, torsion-free subgroup U such that $H^*(U)$ is finite. Since U is also open in G, U contains an open, normal subgroup V of G. Since V is torsion-free and open in U, $H^*(V)$ is finite. So $H^*(G)$ is finitely generated. The implication $(c) \Rightarrow (a)$ is then proved. □

Remark 2. In [8], it was proved that, if $H^*(G)$ is finitely generated, then G has only finitely many conjugacy classes of elementary abelian p-subgroups. However, the converse does not hold: the group $\mathfrak{A}$ given above has only one elementary abelian subgroup, which is the one of rank 0, while $H^*(\mathfrak{A})$ is an exterior algebra with an infinite number of generators of degree 1, hence is not finitely generated.

From now on, fix $\mathcal{F}$ a set of representatives of conjugacy classes of elementary abelian p-subgroups of G. We now give a cohomological criterion for $\mathcal{F}$ to be finite, as follows.

Theorem 3. *The following are equivalent:*

(a) *There exists an open normal subgroup A of G such that $EA \neq FA$, for $E \neq F$ in $\mathcal{F}$;*
(b) *$H^*(G)/\mathfrak{R}_G$ is finitely generated (as a ring);*
(c) *$\mathcal{F}$ is finite.*

Proof. (a) ⇒ (c): Let $\mathcal{S}$ be the set of subgroups of G_A. The map $f : \mathcal{F} \to \mathcal{S}, E \mapsto EA/A$ is then injective. Since f maps $\mathcal{F}$ injectively into the finite set $\mathcal{S}$, $\mathcal{F}$ is finite.

The implication (c) ⇒ (a) is clear. We now prove (b) ⇒ (c). Suppose that $H^*(G)/\mathfrak{R}_G$ is finitely generated. Let U be the open, normal, torsion-free subgroup of G as given in Corollary 1(ii). It follows from the proof of the corollary that, for every $V \le U$, V is torsion-free and $H^*(G_V)/\mathfrak{R}_{G_V} \overset{(\varphi_V)^*}{\to} H^*(G)/\mathfrak{R}_G$ is surjective.

Let $\mathcal{M}$ be a set of representatives of conjugacy classes of maximal elementary abelian pro-p subgroups of G and let E, F be two different elements of $\mathcal{M}$. Since U is torsion-free, it follows that φ_V maps E (resp. F) isomorphically to $\varphi_V(E)$ (resp. $\varphi_V(F)$), for every $V \le U$. Furthermore, since $E \ne F$, there exists $W \le U$ such that each of $\varphi_W(E), \varphi_W(F)$ is not conjugate (in G_W) to any subgroup of the other. According to [7, Theorem 2.7], there exist $\xi, \eta \in H^*(G_W)$ such that $\xi|_{\varphi_W(E)}, \eta|_{\varphi_W(F)}$ are not nilpotent, and $\xi|_{\varphi_W(F)} = 0, \eta|_{\varphi_W(E)} = 0$. Set $\xi' = (\varphi_W)^*(\xi), \eta' = (\varphi_W)^*(\eta)$. It follows that $\xi'|_E, \eta'|_F$ are not nilpotent, and $\xi'|_F = 0, \eta'|_E = 0$; in particular, ξ' and η' are not nilpotent. Since $H^*(G_U)/\mathfrak{R}_{G_U} \overset{(\varphi_U)^*}{\to} H^*(G)/\mathfrak{R}_G$ is surjective, there exist $\zeta, \theta \in H^*(G_U)$ such that $\zeta|_{\varphi_U(E)}, \theta|_{\varphi_U(F)}$ are not nilpotent, and $\zeta|_{\varphi_U(F)} = 0, \theta|_{\varphi_U(E)} = 0$. Also, by [7, Theorem 2.7], it follows that each of $\varphi_U(E), \varphi_U(F)$ is not conjugate to any subgroup of the other; in particular, $\varphi_U(E) \ne \varphi_U(F)$. So ψ_U maps $\mathcal{M}$ injectively into $\mathcal{E}_U$. Hence $\mathcal{M}$ is finite, and so is $\mathcal{F}$.

Finally, let us prove (c) $\Rightarrow$ (b). Suppose that $\mathcal{F}$ is finite. It follows that there exists $T \in \mathcal{U}$ such that, for every $E \in \mathcal{E}$, $\varphi_T|_E$ is an isomorphism, and ψ_T maps $\mathcal{F}$ injectively into $\mathcal{E}_T$. For every $E \in \mathcal{F}$, the restriction map $\mathrm{Res}^{G_T}_{\varphi_T(E)}$ induces an action of $H(G_T)$ on $H^*(\varphi_T(E))$, hence on $H^*(E)$; furthermore, by [4, Corollary 7.4.7], $H^*(E)$ is a finitely generated $H(G_T)$-module. Besides, the inflation $(\varphi_T)^*$ also induces an action of $H(G_T)$ on $H^*(G)$ so that Res^G_E is an $H(G_T)$-homomorphism, for every $E \in \mathcal{F}$. Set $\mathfrak{J} = \prod_{E\in\mathcal{F}} H^*(E)$. Since $\mathcal{F}$ is finite, $\mathfrak{J}$ is a finitely generated $H(G_T)$-module. Since $H^*(G)/I(G)$ is isomorphic to a submodule of $\mathfrak{J}$, it is also a finitely generated $H(G_T)$-module. Furthermore, $I(G) \subset \mathfrak{R}_G$ implies that $H^*(G)/\mathfrak{R}_G$ is, in turn, a finitely generated $H(G_T)$-module. Since $H(G_T)$ is a finitely generated algebra, so is $H^*(G)/\mathfrak{R}_G$. □

It has been pointed out to us by Henn that such a result can also be deduced from material in [6].

Let $\mathcal{E}$ be the category with objects the elementary abelian p-subgroups of G and with morphisms from A to B defined to be the homomorphisms $\theta : A \to B$ of the form $\theta a = gag^{-1}$ for some $g \in G$. Set $\mathfrak{L} = \varprojlim_{E\in\mathcal{E}} H(E)$. It is clear that the product of restriction maps

$$\mathfrak{L} \to \mathfrak{J} = \prod_{E\in\mathcal{F}} H^*(E)$$

is a monomorphism of $H(G)$-modules. From the proof of the above theorem, if $\mathcal{F}$ is finite, then $\mathfrak{J}$ is Noetherian over $H(G)$. Thus $\mathfrak{L}$ is finitely generated over $H(G_T)$ and hence over $H(G)$. We now have

Proposition 4. *If $\mathcal{F}$ is finite, then $\mathfrak{L}$ is finitely generated over $H(G)$.*

According to [8, Proposition 13.4], if $\mathcal{F}$ is finite, the map

$$H(G) \overset{\mathrm{Res}^G}{\to} \mathfrak{L}$$

is an F-isomorphism. In other words, given $x \in I(G)$ and $y \in \mathfrak{L}$, there exists an integer $n = n_{x,y}$ such that $x^n = 0$ and $y^{p^n} \in \operatorname{Im} \mathrm{Res}^G$. We now give a sufficient condition for Res^G to be a uniform F-isomorphism (i.e., the integer n can be chosen independently of x and y).

Theorem 4. *If $H^*(G)$ is finitely generated, then Res^G is a uniform F-isomorphism.*

Proof. Suppose that $H^*(G)$ is finitely generated. $\mathcal{F}$ is then finite and the map $H(G) \stackrel{\mathrm{Res}^G}{\to} \mathfrak{L}$ is an F-isomorphism. By Proposition 4, $\mathfrak{L}$ is finitely generated over $H(G)$. So there exists an integer r such that $y^{p^r} \in \mathrm{Im}\ \mathrm{Res}^G$, for every $y \in \mathfrak{L}$.

By Proposition 2, there exists $U \in \mathcal{U}$ such that $I(G) \subset (\varphi_U)^*(I(G_U))$. Since G_U is finite, there exists an integer s such that $x^s = 0$, for every $x \in I(G)$. The theorem is then proved, by setting $n = n_{x,y} = \max(r, s)$. □

2. Pro-p groups and finite p-subgroups

Let G be a pro-p group. The purpose of this section is to prove the following.

Theorem 5. *If $H^*(G)$ is finitely generated, then G has only finitely many conjugacy classes of finite p-subgroups.*

An analogous result is known for discrete groups of type vFP (see [1], IX, 13.2) and for analytic pro-p groups (where the hypothesis is vacuous, see [2]).

Our proof depends on some deep results on unstable algebras over the Steenrod algebra.

We will need:

Lemma 2. *If $H^*(G)$ is finitely generated and C is a finite central subgroup of G, then $H^*(G/C)$ is finitely generated.*

Proof. Without loss of generality, we may suppose that C is of order p. Write $K = G/C$, and let $z \in H^2(K)$ be the cohomology class classifying the central extension

$$1 \to C \to G \to K \to 1.$$

There exists then an open, normal subgroup L of K such that $\mathrm{Res}^K_L(z) = 0$. Therefore, the preimage H of L in G is isomorphic to $C \times L$. In other words, L can be considered as an open subgroup of G. By Corollary 2, $H^*(L)$, and so $H^*(K)$, are finitely generated. □

For every subgroup P of G, denote by $N_G(P)$ (resp. $C_G(P)$) the normalizer (resp. centralizer) of P in G. We have

Lemma 3. *If P is a subgroup of G of order p, then:*

(i) *$N_G(P) = C_G(P)$;*

(ii) *$H^*(C_G(P))$ and $H^*(C_G(P)/P)$ are finitely generated, provided that so is $H^*(G)$.*

Proof. (i) follows from the fact that $N_G(P)/C_G(P)$ is embedded into $\mathrm{Aut}(P)$, which is of order $p-1$.

(ii) By Lemma 2, we need only prove that $H^*(C_G(P))$ is finitely generated. By Corollary 2, there exists U open, normal, torsion-free in G. Set $K = \langle U, P\rangle$. K is then of p-rank 1 (i.e., every elementary abelian subgroup of K is of rank at most 1), open in G and has finitely generated cohomology, by Corollary 2. Furthermore, $C_K(P)$ is open in $C_G(P)$. By Corollary 2, it suffices to prove that $H^*(C_K(P))$ is finitely generated. According to the theory in [3] (see also [6, Corollary 1.7]) the unstable algebra $T^V H^*(G)$ is Noetherian and also $T^V H^*(G) = \prod_{(\rho)} H^*(C_K(\mathrm{im}\rho))$, where the product is taken over conjugacy classes of homomorphisms $\rho : \mathbb{Z}/p \to K$. It follows that $H^*(C_K(P))$ is finitely generated. □

Proof of Theorem 5. Since $H^*(G)$ is finitely generated, it follows from Corollary 2 that there exists an open normal, torsion-free subgroup U of G. Define

$$n_G = \min\{n \mid |G/U| = p^n \text{ for some open, normal, torsion-free subgroup } U \text{ of } G\}.$$

We argue by induction on n_G. If $n_G = 1$, the conclusion follows, since every finite subgroup of G is elementary abelian of rank 1. Suppose that the theorem holds if $n_G < m$.

Assume that $n_G = m$. It is known that the number of conjugacy classes of elementary abelian p-subgroups of rank 1 of G is finite. Let $\{C_1, \dots, C_k\}$ be a set of representatives of such conjugacy classes. It is clear that, for any finite p-subgroup P of G, there exist $g \in G$ and i such that P^g contains C_i as a central subgroup. Let $\mathcal{N}_i$ be the set of finite subgroups of G containing C_i as a central subgroup, $1 \le i \le k$. It is then sufficient to prove that the number of conjugacy classes in $\mathcal{N}_i$ is finite, $1 \le i \le k$.

Fix such an i. Note that $\mathcal{N}_i$ coincides with the set of finite subgroups of $K = C_G(C_i)$ containing C_i, hence is in one-to-one correspondence with the set of finite subgroups of $H = C_G(C_i)/C_i$. Therefore, we need only prove that H has many finite conjugacy classes of finite subgroups. Let U be open, normal, torsion-free in G with $|G/U| = p^{n_G}$ and set $V = U \cap K$. V is then open, normal, torsion-free in K and $|K/V| \le |G/U| = p^{n_G}$. So $n_K \le n_G$. Since $n_H = n_K - 1 < n_G$, it follows from the induction hypothesis that H has many finite conjugacy classes of finite subgroups. The theorem is proved. □

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