A GENERALIZATION OF A RESULT OF KAZHDAN AND LUSZTIG

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Abstract. Kazhdan and Lusztig showed that every topologically nilpotent, regular semisimple orbit in the Lie algebra of a simple, split group over the field \( \mathbb{C}(t) \) is, in some sense, close to a regular nilpotent orbit. We generalize this result to a setting that includes most quasisplit \( p \)-adic groups.

1. Introduction

Suppose \( G \) is the group of \( \mathbb{C}(t) \)-rational points of a simple, split, algebraic \( \mathbb{C}(t) \)-group. In [7, Corollary 4.1], Kazhdan and Lusztig show that if \( Z \) is a topologically nilpotent, regular semisimple element of the Lie algebra of \( G \) and \( x \) is a special vertex in the Bruhat-Tits building of \( G \), then there exists \( g \in G \) such that the image of \( gZ = \text{Ad}(g)Z \) in the complex Lie algebra associated to \( x \) is regular nilpotent. We generalize this result to a setting that includes most quasisplit groups over \( p \)-adic fields.

Motivation. Suppose \( k \) is a \( p \)-adic field, and \( G \) is the group of \( k \)-rational points of a connected reductive \( k \)-quasisplit group.

Our motivation for considering this problem came from harmonic analysis on \( p \)-adic groups; we were interested in what role the Bruhat-Tits building might play in stability questions. For example, we learned from lectures of Kottwitz that if \( S \) denotes a Kostant section [9, §2.4] in \( g \), then the map \( G \times S \to g \) given by \((g, Z) \mapsto gZ\) is a submersion. This implies that for a fixed regular nilpotent element \( X \) in \( g \) there is a neighborhood \( U \subset g \) of \( X \) with the following property: every stable regular semisimple orbit that intersects \( U \) nontrivially contains a unique \( G \)-orbit that intersects \( U \) nontrivially. The main result of this paper is the determination of a natural neighborhood of \( X \) that has this property with respect to all regular semisimple topologically nilpotent elements.

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This paper. Let $k$ denote a complete field with nontrivial discrete valuation and residue field $\mathfrak{f}$. We suppose that $\mathfrak{f}$ is perfect. All extensions of $k$ that we consider will lie in a fixed algebraic closure $\overline{k}$ of $k$. Let $K$ denote the maximal unramified extension of $k$.

Let $\mathbf{G}$ denote a connected, reductive group with Lie algebra $\mathfrak{g}$. Let $G = \mathbf{G}(k)$ and $\mathfrak{g} = \mathfrak{g}(k)$. Let $\mathfrak{g}^{s,ss}$ denote the set of regular, semisimple elements in $\mathfrak{g}$. For any algebraic extension $E/k$ of finite ramification index, and any facet $F$ in the Bruhat-Tits building $\mathcal{B}(\mathbf{G}, E)$ of $\mathbf{G}(E)$, one can define a parahoric subgroup $\mathbf{G}(E)_F$ of $\mathbf{G}(E)$ [3]. Let $\mathbf{G}(E)_{F,\mathfrak{p}}$ denote the pro-unipotent radical of $\mathbf{G}(E)_F$. As in [2], we define lattices $\mathfrak{g}(E)_F$ and $\mathfrak{g}(E)_{F,\mathfrak{p}}$ in $\mathfrak{g}(E)$: for any $x \in \mathcal{B}(\mathbf{G}, E)$, we define $\mathfrak{g}(E)_x$ and $\mathfrak{g}(E)_{x,\mathfrak{p}}$ to be $\mathfrak{g}(E)_F$ and $\mathfrak{g}(E)_{F,\mathfrak{p}}$, respectively, where $F$ is the unique facet in $\mathcal{B}(\mathbf{G}, E)$ containing $x$, and we define the $\mathfrak{g}(E)$-invariant set $\mathfrak{g}(E)_{0,\mathfrak{p}}$ to be $\bigcup_F \mathfrak{g}(E)_{F,\mathfrak{p}}$, where the union is taken over all facets $F$ in $\mathcal{B}(\mathbf{G}, E)$. When $E = k$, we denote the building by $\mathcal{B}(\mathbf{G})$, the reduced building by $\mathcal{B}^{red}(\mathbf{G})$, and the above objects by $G_F$, $G_{F,\mathfrak{p}}$, $G_{F,\mathfrak{p}}$, $G_{F,\mathfrak{p}}$, and $G_{0,\mathfrak{p}}$. This last object is the set of topologically nilpotent elements in $\mathfrak{g}$. We can identify $G_{F}/G_{F,\mathfrak{p}}$ with the group of $\mathfrak{f}$-points of a connected reductive $\mathfrak{f}$-group $G_{F}$. Let $L_F := \text{Lie}(G_F)$. We identify $L_F(\mathfrak{f})$ with $\mathfrak{g}(E)/\mathfrak{g}(E)_{0,\mathfrak{p}}$.

Suppose $\mathbf{G}$ is $k$-quasisplit and let $X \in \mathfrak{g}$ be regular nilpotent (see [3]). Suppose that we can complete $X$ to an $sl_2(k)$-triple $(Y, H, X)$. Under the hypotheses of Corollary 4.5 we can find a unique minimal facet $F \subset \mathcal{B}(\mathbf{G})$ such that the image of $F$ in $\mathcal{B}^{red}(\mathbf{G})$ is a special vertex and $Y, H, X \in \mathfrak{g}(F)$. The main result of this paper is:

**Proposition 1.** Suppose that all of the hypotheses of [3] and of Corollary 4.5 are valid, that $G, X, F$ are as above, and that $Z \in \mathfrak{g}^{s,ss}$. Then $Z \in \mathfrak{g}_{0,\mathfrak{p}}$ if and only if there is some $g \in G(\overline{k})$ such that $gZ \in X + \mathfrak{g}(E)_{0,\mathfrak{p}}$. Moreover, if $g' \in G(\overline{k})$ is such that $g'Z \in X + \mathfrak{g}(E)_{0,\mathfrak{p}}$, then $g'Z = \ell gZ$ for some $\ell \in G_{F,\mathfrak{p}}$.

In other words, the coset $X + \mathfrak{g}(E)_{0,\mathfrak{p}}$ picks out a unique $G$-orbit in every topologically nilpotent, regular semisimple, stable orbit in $\mathfrak{g}$. Note that when $k = \mathbb{C}(\!(t)\!)$ and $G$ is split and simple, we recover Corollary 4.1 of [7].

To prove Proposition 1, we will need some additional notation. Given a maximal $k$-split torus $\mathbf{S}$ of $\mathbf{G}$, we have the torus $S = \mathbf{S}(k)$ in $G$ and the corresponding apartment $\mathcal{A}(S) = \mathcal{A}(S, k)$ in $\mathcal{B}(G)$. For any subgroup $H \subset G$, let $C_G(H)$ denote the centralizer of $H$ in $G$. For $g \in G$, let $\gamma_H = g^{-1}hg$. For $Y \in G$, we denote the centralizer of $Y$ in $\mathfrak{g}$ by $C_\mathfrak{g}(Y)$. For $Z \in \mathfrak{g}$, let $O_Z$ denote the $G$-orbit of $Z$ in $\mathfrak{g}$. Recall that for every algebraic extension $E/k$,

$$O_Z(E) = G(E)\cap Z \in \mathfrak{g}(E) = \bigcup_{Z'} G(E)_{Z'},$$

where $Z' \in \mathfrak{g}(E)$ ranges over the elements of $O_Z(E)$ up to $G(E)$-conjugacy.

2. Hypotheses

In this section, we list various properties which we require. Under some restrictions on $\mathbf{G}$ and $k$, all of these hypotheses are valid. In particular, they are all true when $\mathfrak{f}$ has characteristic zero.
The first hypothesis is valid whenever the characteristic of \( k \) is either zero or larger than some constant that can be determined by looking at the absolute root datum of \( G \). For more information, see [6].

**Hypothesis 1.** Let \( X \in \mathfrak{g} \) be regular nilpotent. We can complete \( X \) to an \( \mathfrak{sl}_2(k) \)-triple \( (Y,H,X) \), produce a maximal \( k \)-split torus \( S \) in \( G \), and find a point \( x \in A(S,k) \), such that \( H \in \text{Lie}(S)(k) \), \( Y,H,X \in \mathfrak{g}_r \), and, for all finite extensions \( E/k \),

\[
G^{(E)}(X + C_{\mathfrak{g}(E)}^+(Y)) = X + \mathfrak{g}(E)_r^+.
\]

When \( G = \text{SL}_2(k) \), this hypothesis fails, and so does Proposition [1].

The remaining hypotheses are valid whenever \( k \) has characteristic zero.

**Hypothesis 2.** Let \( X \) and \( S \) be as in Hypothesis [1]. Let \( Z = C_G(S) \). (Since \( G \) is \( k \)-quasisplit, \( Z \) is a maximal \( k \)-torus.) For any algebraic extension \( E/k \) over which \( Z \) splits, if \( Z \in \text{Lie}(Z)(E) \) is regular semisimple, then \( X + Z \) is \( G(E) \)-conjugate to \( Z \).

**Hypothesis 3.** Suppose \( Z \in \mathfrak{g}^{r.s.s.} \). For all \( g \in G(K) \) such that \( gZ \in \mathfrak{g}(K) \), there exists \( g' \in G(K) \) such that \( g'Z = gZ \).

When the characteristic of \( k \) is not a “torsion” prime for \( G \), then the centralizer of \( Z \) in \( G \) is connected [15]. So this hypothesis follows immediately from Theorem III.2.3.1’ of [11]. (See also Remark 1 in loc. cit.)

**Hypothesis 4.** Let \( X \) and \( Y \) be as in Hypothesis [1]. For any algebraic extension \( E/k \), if \( Z \in \mathfrak{g}(E)^{r.s.s.} \), then the set \((X + C_{\mathfrak{g}(E)}(Y)) \cap \text{O}_Z(E)\) consists of one element.

This last hypothesis asserts the existence of a Kostant section (see, for example, [3, Theorem 0.10] and [21, §2.4 and §4.3]).

3. Some notation and results in a general setting

In this section only, let \( k \) be any field.

**General definitions.** The term Levi subgroup will mean a rational Levi factor of a rational parabolic subgroup; a Levi subalgebra means the Lie algebra of a Levi subgroup.

If \( L \) is a Levi subgroup of \( G \), let \( (L) \) denote the set of all subgroups of \( G \) that are \( G \)-conjugate to \( L \). If \( M \) is another Levi subgroup of \( G \), we write \( (L) \leq (M) \) provided that \( gL \subseteq M \) for some \( g \in G \).

We call an element of the Lie algebra of a reductive group distinguished provided that it is nilpotent and does not lie in any proper Levi subalgebra. Similarly, an orbit in such a Lie algebra is said to be distinguished if some (hence any) element of it is distinguished.

For any \( k \)-group \( H \), let \( X^k(H) \) denote the set of one-parameter subgroups of \( H \), and let \( X^k_1(H) \) denote the subset of \( k \)-rational elements.

If \( H \) is connected and reductive, then \( \lambda \in X^k_1(H) \) determines a rational parabolic subgroup \( P_\lambda \) with rational Levi decomposition \( P_\lambda = M_\lambda N_\lambda \). Specifically, \( P_\lambda \) (resp. \( M_\lambda, N_\lambda \)) consists of those elements \( g \in H \) such that \( \lim_{t \to 0} \lambda(t) g \) exists (resp. \( = g \), \( = 1 \)). Note that \( M_\lambda = C_H(\lambda) \), which is connected (by [13, Theorem 6.4.7]).

We will call an element \( u \in H(k) \) unipotent if there is some \( \lambda \in X^k_1(H) \) such that \( \lim_{t \to 0} \lambda(t) u = 1 \). Similarly, we will call an element \( X \in \text{Lie}(H)(k) \) nilpotent if
there is some \( \lambda \in X^*_c(H) \) such that \( \lim_{t \to 0} \lambda(t)X = 0 \). The terms “unipotent” and “nilpotent” are sometimes given other definitions. See §2.5 of [2] for a discussion.

**Comments on regular nilpotent elements.** In this subsection, we discuss some results concerning regular nilpotent elements in \( g \). Undoubtedly, these results are well known to the experts, but we could not find a reference.

Suppose \( B \subset G \) is a rational Borel subgroup and \( S \subset G \) is a maximal \( k \)-split torus contained in \( B \). Then we have associated sets \( \Phi^+(S, B, G) \) and \( \Delta(S, B, G) \) of positive and simple roots, respectively. Fixing an order on \( \Phi^+(S, B, G) \), we may write each \( u \) in the unipotent radical uniquely in the form \( u = \prod_{\alpha \in \Phi^+(S, B, G)} u_\alpha \), where each \( u_\alpha \) belongs to the root group corresponding to \( \alpha \). For any such element \( u, u_\alpha \) will always denote the factor of \( u \) associated to \( \alpha \). No statement that we make concerning \( u_\alpha \) will depend on the ordering of the roots. Similarly, for any \( X \) in the Lie algebra of the unipotent radical of \( B \), let \( X_\alpha \) denote the projection of \( X \) onto the \( \alpha \)-eigenspace in \( g \).

**Lemma 3.1.** Suppose \( G \) is a \( k \)-quasisplit group and \( u \in G \) is unipotent. If \( u \) is regular, then there is a unique rational Borel subgroup \( B \subset G \) such that \( B = B(k) \) contains \( u \). Moreover, if \( B \) is a rational Borel subgroup such that \( u \in B \), then \( u \) is regular if and only if \( u_\alpha \neq 1 \) for all \( \alpha \in \Delta(S, B, G) \), where \( S \) is any maximal \( k \)-split torus in \( B \).

**Proof.** Since \( u \) belongs to the group of \( k \)-rational points of the derived group of \( G \), it is enough to assume that \( G \) is semisimple. From [13], Lemma 3.2 and Theorem 3.3], \( u \) is regular if and only if it is contained in exactly one Borel subgroup \( B \) of \( G \).

Suppose \( u \) is regular. Let \( B \) denote the unique Borel subgroup of \( G \) that contains \( u \). By our definition of unipotent, \( u \) is contained in the unipotent radical of some rational parabolic subgroup \( P \). Since \( G \) is \( k \)-quasisplit, there exists a rational Borel subgroup \( B' \subset P \) such that \( u \in B'(k) \). By uniqueness, \( B = B' \).

We now consider the final statement of the lemma. Suppose that \( B \) is a rational Borel subgroup of \( G \) such that \( u \in B(k) \). Let \( S \) be a maximal \( k \)-split torus of \( G \) in \( B \) and let \( T = C_G(S) \). If \( U \) denotes the unipotent radical of \( B \), then \( B = TU \) is a rational Levi factorization of \( B \) and \( T \) is a maximal \( k \)-torus in \( G \). From Lemma 3.2 of [14], \( u_\beta \neq 1 \) for all simple \( \beta \in \Delta(T, B, G) \) if and only if \( u \) is regular. Let \( E \) be a Galois splitting field for \( G \) over \( k \). Then each \( \alpha \in \Delta(S, B, G) \) corresponds to a \( \text{Gal}(E/k) \)-orbit in \( \Delta(T, B, G) \). Thus, \( u \) is regular if and only if \( u_\alpha \neq 1 \) for all \( \alpha \in \Delta(S, B, G) \). \( \square \)

To transfer the above result to the Lie algebra, we will need to assume the following hypothesis.

**Hypothesis E.** There is a \( G \)-equivariant \( k \)-isomorphism from the unipotent variety in \( G \) to the nilpotent variety in \( g \).

It follows from work of Springer [12] that Hypothesis E holds under certain mild restrictions on \( G \) and \( k \).

**Corollary 3.2.** Suppose \( G \) is a \( k \)-quasisplit group, \( X \in g \) is nilpotent, and Hypothesis E is true for \( G \) and \( k \). If \( X \) is regular, then there is a unique rational Borel subgroup \( B \subset G \) such that the nilradical of the Lie algebra of \( B = B(k) \) contains \( X \). Moreover, if \( B \) is a rational Borel subgroup such that the nilradical of the Lie algebra of \( B \) contains \( X \), then \( X \) is regular if and only if \( X_\alpha \neq 0 \) for all \( \alpha \in \Delta(S, B, G) \), where \( S \) is any maximal \( k \)-split torus in \( B \). \( \square \)
Corollary 3.3. Suppose $X$ is a regular nilpotent element in $\mathfrak{g}$ and $\lambda \in X_0^\delta(G)$ is a one-parameter subgroup such that $\lim_{t \to 0} \lambda(t)X = 0$. If Hypothesis E is true for $G$ and $k$, then $\lambda \in X_0^\delta(B)$, where $B$ is the unique rational Borel subgroup containing $X$ in its Lie algebra. In particular, $C_G(\lambda)$ is a maximal $k$-torus.

Proof. As usual, $\lambda$ determines a rational parabolic subgroup $P_\lambda$ with rational Levi decomposition $P_\lambda = M_\lambda N_\lambda$, where $M_\lambda = C_G(\lambda)$. Since $\lim_{t \to 0} \lambda(t)X = 0$, we must have $X \in \text{Lie}(N_\lambda)$. Since there is a unique rational Borel subgroup containing $X$ in its Lie algebra, it follows that $P_\lambda$ is this Borel subgroup. \hfill $\square$

4. SOME COMMENTS ON THE PARAMETRIZATION OF NILPOTENT ORBITS

We now return to our assumption that $k$ is complete with respect to a nontrivial discrete valuation and has perfect residue field $\mathfrak{f}$.

Remark 4.1. The hypotheses of [6] §4.2 are enough to guarantee that Hypothesis E holds for $G$ and $k$ and also for $G_F$ and $\mathfrak{f}$ for every facet $F$.

Some notation. Let

\[ I^d := \{ (F, e) \mid F \text{ is a facet in } B(G), \text{ and } e \in L_F(\mathfrak{f}) \text{ is distinguished} \}. \]

Under some hypotheses on $k$ and $G$ (see [6] §4.2), to each pair $(F, e) \in I^d$ we can associate a nilpotent orbit $O(F, e) \subset \mathfrak{g}$ such that $O(F, e)$ is the unique nilpotent orbit of minimal dimension that intersects $e$ nontrivially [6, Lemma 5.3.3].

For any $\mathfrak{sl}_2(k)$-triple $(Y, H, X)$ in $\mathfrak{g}$, we have the set

\[ B(Y, H, X) := \{ x \in B(G) \mid Y, H, X \in \mathfrak{g}_F \}. \]

This set is closed, convex, nonempty, and a union of facets [6, §5.1].

Suppose $S$ is a maximal $k$-split torus in $G$. Following [10], we associate to any facet $F \subset \mathcal{A}(S, k)$ the Levi subgroup $M(F, S)$ that is generated by $C_G(S)(k)$ and the root groups $U_{\psi}(k)$, where $\psi$ is the gradient of an affine root $\psi$ of $G$ with respect to $S$, $k$, and a nontrivial discrete valuation of $k$ such that the restriction of $\psi$ to $F$ is constant. Since $(M(F, S))$ does not depend on $S$, we may write $(M_F)$ instead.

Nilpotent orbits and Levi subalgebras. In this subsection, we associate to $(F, e) \in I^d$ a unique conjugacy class (namely, $(M_F)$) of Levi subgroups that are minimal with respect to the property that $O(F, e)$ intersects the Lie algebra of some (hence any) element of this class nontrivially (compare to §5 of [3]). This answers a question of D. Kazhdan.

Proposition 2. Suppose the hypotheses of [6] §4.2 hold. Suppose that $(F, e) \in I^d$ and that $L$ is a Levi subgroup of $G$. If $O(F, e) \cap \text{Lie}(L) \neq \emptyset$, then $(M_F) \leq (L)$. Moreover, for every maximal $k$-split torus $S$ of $G$ with $F \subset \mathcal{A}(S, k)$, we have $O(F, e) \cap \text{Lie}(M(F, S)) \neq \emptyset$.

Proof. The last claim follows immediately from [6, Corollary 4.3.2 and Lemma 5.3.3(2)].

Suppose $L$ is a Levi subgroup of $G$ for which $O(F, e) \cap \text{Lie}(L) \neq \emptyset$. From [6 Lemma 5.3.3], we can produce an $\mathfrak{sl}_2(k)$-triple $(Y, H, X)$ in $\mathfrak{g}$ such that $Y, H, X \in \mathfrak{g}_F$ and $X \in O(F, e)$. From [6 §5.5], $F$ is a maximal facet in $B(Y, H, X)$. Since $O(F, e) \cap \text{Lie}(L) \neq \emptyset$, without loss of generality, we assume that $Y, H, X \in \text{Lie}(L)$.

From the last paragraph of the proof of Theorem 5.6.1 of [3], there exists $(F', e')$ in the analogue of $I^d$ for $L$ such that $Y, H, X \in \text{Lie}(L)_{F'}$, and $X \in O(F', e')$. Note
that $F'$ is a facet in $B(L)$. If $F''$ is maximal among those facets of $B(G)$ that lie in $F'$, then $(M_{F''}) \leq (L)$.

On the other hand, $F'' \subset B(Y, H, X)$ and $F$ is a maximal facet in $B(Y, H, X)$; so $(M_F) \leq (M_{F''})$.

\[ \square \]

**Remark 4.2.** With suitable changes, the above result remains valid in the context of generalized $r$-facets.

**Some consequences.**

**Corollary 4.3.** Suppose the hypotheses of [6, §4.2] hold. Suppose $(F, e) \in I^d$. The orbit $O(F, e)$ is distinguished if and only if $F$ is a minimal facet in $B(G)$.

**Corollary 4.4.** Suppose the hypotheses of [6, §4.2] hold. If $X \in g$ is a distinguished element and $(Y, H, X)$ is an $s\ell_2(k)$-triple completing $X$, then there exists a unique point $x \in B^\text{red}(G)$ such that $Y, H, X \in g_x$. Moreover, $x$ is a vertex.

**Proof.** Let $F$ be a maximal facet in $B(Y, H, X)$ and let $(f, h, e)$ denote the $s\ell_2(f)$-triple in $L_F(f)$ that is the image of $(Y, H, X)$. From [6, §5.5], we have that $(F, e) \in I^d$. From [6] Lemma 5.3.3(2), $X \in O(F, e)$. From Corollary 4.3, $F$ is a minimal facet in $B(G)$.

**Corollary 4.5.** Suppose $G$ is $k$-quasisplit and the hypotheses of [6, §4.2] hold. Suppose $(F, e) \in I^d$. The orbit $O(F, e)$ is regular if and only if $e \in L_F(f)$ is regular and the image of $F$ in $B^\text{red}(G)$ is a special vertex for which (a choice of) the simple $f$-roots of $G_F$ may be naturally identified with (a choice of) the simple $k$-roots of $G$.

**Proof.** Suppose $O(F, e)$ is regular. Then $O(F, e)$ is distinguished. From Corollary 4.3, we may assume that $F$ is a minimal facet in $B(G)$. Let $(f, h, e)$ be an $s\ell_2(f)$-triple completing $e$, and let $(Y, H, X)$ be an $s\ell_2(k)$-triple lifting $(f, h, e)$ (see [6, 5.3.3(1)]). From [6] 5.3.3(2), $X \in O(F, e)$. Let $\lambda$ be the one-parameter subgroup adapted (see Definition 4.5.6 of [6]) to $(Y, H, X)$. Let $M = C_G(\lambda)$, and $M = M(k)$. From [6] Corollary 4.5.9, we have $F \subset B(M)$. From Corollary 4.2 Corollary 3.3 and Remark 4.1, $X$ lies in the Lie algebra of a unique rational Borel subgroup $B$ of $G$, and $M$, a Levi factor of $B$, is a maximal $k$-torus of $G$. Let $B$ denote the Borel $f$-subgroup of $G_F$ corresponding to the image of $B(k) \cap G_F(f)$. Note that $e$ belongs to the Lie algebra of $B(f)$.

Let $S$ denote the maximal $k$-split torus in $M$ and let $S$ denote the corresponding maximal $f$-split torus in $G_F$. Let $g(2)$ denote the 2-eigenspace for the action of $\lambda$ on $g$. From [6] Corollary 4.3.2 and Lemma 5.3.3(2), $e \cap g(2) \subset O(F, e)$. Hence, any element of $e \cap g(2) = X + (g^{+} \cap g(2)) \subset \text{Lie}(B)(k)$ is regular. Thus, from Corollary 4.2 we must have that for all $Z \in e \cap g(2)$ and for all $\alpha \in \Delta(B, S, G_F)$, $Z_\alpha \neq 0$. This implies that every such $\alpha$, considered as a character of $S$, must be a root in $G_F$. Thus, we have an embedding of $\Delta(S, B, G)$ into $\Phi^+(S, B, G_F)$, and by comparing dimensions we see that $\Delta(S, B, G)$ can be identified with $\Delta(S, B, G_F)$. In particular, the image of $F$ in $B^\text{red}(G)$ is special. Thus, $e_\alpha \neq 0$ for all $\alpha \in \Delta(S, B, G_F)$; so $e$ is regular.

To prove the converse, suppose that the image of $F$ in $B^\text{red}(G)$ is a special vertex, and $e \in L_F(f)$ is regular. Let $(e, h, f)$, $(Y, H, X)$, $\lambda$, and $M$ be as above. From Corollary 4.3, $X$ is distinguished. Pick a maximal $k$-torus $T \subset M$ so that $T$ contains a maximal $k$-split torus $S$ with $F \subset A(S) \subset B(M)$. Since $T \subset M$, we have $\lambda \in X^k(T)$. Thus, there exists a rational Borel subgroup $B$ with Levi
factorization $B = TU$ such that $X \in \text{Lie}(U)$. Let $B, S,$ and $U$ denote the objects in $G_F$ corresponding to $B, S,$ and $U$, respectively. Note that $e \in \text{Lie}(U)(f)$. Since for all $\alpha \in \Delta(S, B, G_F)$ we have $e_\alpha \neq 0$, we have that for all $\alpha \in \Delta(S, B, G)$, $X_\alpha \neq 0$. Hence, $X$ is regular.

\section{Proof of Proposition 1}

Produce an $\mathfrak{s}$-triple $(Y, H, X)$ completing $X$, a maximal $k$-split torus $S$, and a point $x \in A(S, k)$ as in Hypothesis 1. From Corollary 1.5, the image of $x$ in $B_{\text{red}}(G)$ is special. Let $Z$ denote the maximal $k$-split torus $C_G(S)$. Fix $Z \in \mathfrak{g}_{\text{ss}}$. From Hypotheses 1 and 3 it is enough to prove that

$$Z \in \mathfrak{g}_{0+} \iff O_Z(k) \cap (X + C_{\mathfrak{g}_+^*}(Y)) \neq \emptyset.$$

"$\iff$": Suppose there exists $g \in G(k)$ such that $^gZ \in X + C_{\mathfrak{g}_+^*}(Y)$. (From 2 Corollary 3.2.6, this latter set is contained in $\mathfrak{g}_{0+}$.) From Hypothesis 3 we may assume $g \in G(K)$. From 1 Lemma 2.2.5], $\mathfrak{g}_{0+} = (g(K)_0)^+ \text{Gal}(K/k)$, the set of $\text{Gal}(K/k)$-fixed points in $g(K)_+$. Therefore, $Z \in (g^{-1}g(K)_0)^+ \text{Gal}(K/k) = \mathfrak{g}_{0+}$.

"$\implies$": Suppose $Z \in \mathfrak{g}_{0+}$. Let $E/k$ be a finite extension over which $Z$ splits and for which there exists $g \in G(E)$ such that $^gZ \in \text{Lie}(Z)(E)$. Since $Z \in \mathfrak{g}_{0+}$, we must have that $Z \in \mathfrak{g}_F^*$ for some $y \in B(G)$. Thus, $Z \in \mathfrak{g}(E)_0^* \subset \mathfrak{g}(E)_{0+}$; so $^gZ \in \mathfrak{g}(E)_{0+}$. From 2 Theorem 3.1.2(1) or 10 Lemma 8.2, $^gZ \in \text{Lie}(Z)(E)_{0+} \subset \mathfrak{g}(E)_{+}$.

From Hypothesis 2 there exists $h \in G(E)$ such that $^hZ = ^gZ + X$. From Hypothesis 1 there exists $\ell \in G(E)_{+}$ such that $\ell^hZ \in X + C_{\mathfrak{g}_+^*}(Y)$. Since $O_Z(k)$ (resp. $O_Z(E)$) intersects $X + C_{\mathfrak{g}_+}(Y)$ (resp. $X + C_{\mathfrak{g}(E)}(Y)$) exactly once (from Hypothesis 4), we conclude that $\ell^hZ \in X + C_{\mathfrak{g}_+^*}(Y) \subset \mathfrak{g}$. Therefore, $O_Z(k) \cap (X + C_{\mathfrak{g}_+^*}(Y)) \neq \emptyset$.

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