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ON THE PROJECTIVE-INJECTIVE MODULES OVER CELLULAR ALGEBRAS

YONGZHI CAO

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Abstract. We show that the projective module P over a cellular algebra is injective if and only if the socle of P coincides with the top of P, and this is also equivalent to the condition that the mth socle layer of P is isomorphic to the mth radical layer of P for each positive integer m. This eases the process of determining the Loewy series of the projective-injective modules over cellular algebras.

1. Introduction

Cellular algebras, which have been introduced by Graham and Lehrer [GL] as a convenient axiomatization of Ariki-Koiki Hecke algebras and related algebras, are defined by the existence of a so-called cellular basis with very distinguished properties motivated by the Kazhdan-Lusztig basis of Hecke algebras. One of the main points of a cellular basis of a cellular algebra A is that it gives rise to a filtration of every projective A-module, with composition factors isomorphic to the cell modules of A. With this filtration, the projective modules over cellular algebras have many good features. The aim of this note is to present certain conditions which are necessary and sufficient for a projective module over cellular algebras to be injective. In fact, these conditions are also necessary and sufficient for an injective module over cellular algebras to be projective.

Throughout this paper we denote by K an arbitrary field and by A a finite-dimensional associative K-algebra with unit. By a module we mean usually a finitely generated left module.

Before stating our main result, we recall some basic notions. As usual, the **socle** of an A-module M, denoted by $\operatorname{soc}(M)$, is defined to be the submodule of M generated by all semisimple submodules of M. The **socle layers** of M are defined inductively by $\operatorname{soc}^0(M) = 0$, $\operatorname{soc}^n(M)/\operatorname{soc}^{n-1}(M) = \operatorname{soc}(M/\operatorname{soc}^{n-1}(M))$. The n**th socle layer** is $\operatorname{soc}^n(M)/\operatorname{soc}^{n-1}(M)$. The **radical** of M is the intersection of all the maximal submodules of M, and is written $\operatorname{rad}(M)$. The **radical series** of M is defined inductively by $\operatorname{rad}^0(M) = M$, $\operatorname{rad}^n(M) = \operatorname{rad}(\operatorname{rad}^{n-1}(M))$. The n**th radical layer** is $\operatorname{rad}^{n-1}(M)/\operatorname{rad}^n(M)$. For simplicity we denote by $\operatorname{top}(M)$ the first radical layer of M.

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Theorem 1.1. *Let A be a cellular algebra and P a projective A-module. Then the following conditions are equivalent:*

(a) *P is injective;*

(b) $\mathrm{soc}(P) \simeq \mathrm{top}(P)$*;*

(c) *The mth socle layer of P is isomorphic to the mth radical layer of P for each positive integer m.*

Recall that an algebra A is said to be **self-injective** if each projective A-module is injective as well, and A is called **weakly symmetric** if the projective cover of any given simple A-module is the injective envelope of the same simple A-module. As an immediate consequence of the above theorem we have the following corollary based upon certain conditions on all projective modules.

Corollary 1.2 ([KX2], Theorem 1.1)**.** *Let A be a cellular algebra over a field. If A is self-injective, then it is weakly symmetric.* □

Indeed, there are many important examples of cellular algebras, such as the algebras of blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [BGG] associated with semisimple complex Lie algebras, Schur algebras [G], Temperley-Lieb algebras, and partition algebras [M1] appearing in statistical mechanics, which are not self-injective in general. However, there exist certain projective-injective modules which turn out to play a major and natural role in the representation theory of these algebras (see [FNP], [X1], [X2], [W], [M2] and others). Thus our result as an extension of [KX2] increases the scope of applications to all of these classes of examples and to other classes of examples such as semigroup algebras and various algebras defined by quivers. Moreover, the condition (c) in the theorem sheds light on the rigidity of modules, namely their socle and radical filtration coincide, which has been intensively studied for the blocks of the BGG category $\mathcal{O}$ ([BGS], [I1], [I2], [KM]) and which may be of interest for the study of Schur algebras, partition algebras, and many others.

In the next section, we recall the definition of cellular algebras, and then we give the proof of the theorem and some applications; in particular, we shall give a dual version of Theorem 1.1 on injective modules.

2. Proof of the theorem

We first recall the equivalent definition of cellular algebras given by König and Xi in [KX1]. One can refer to [GL] for the original definition of cellular algebras depending on a cellular basis. Besides the above-mentioned algebras, many well-known algebras, such as Ariki-Koiki Hecke algebras, Brauer algebras, Jones' annular algebras [GL] and Birman-Wenzl algebras [X3], have been shown to be cellular.

In the following, a K-linear anti-automorphism i of A with $i^2 = id$ will be called an **involution**.

Definition 2.1 (König and Xi [KX1])**.** Let A be a K-algebra with an involution i. A two-sided ideal J of A is called a **cell ideal** if and only if $i(J) = J$ and there exists a left ideal $W \subset J$ such that there is an isomorphism of A-bimodules $\alpha : J \simeq W \otimes_K i(W)$ (where $i(W) \subset J$ is the i-image of W) making the following

diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & W\otimes_K i(W) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle x\otimes y\mapsto i(y)\otimes i(x)} \\ J & \xrightarrow{\alpha} & W\otimes_K i(W) \end{array}$$

The algebra A (with the involution i) is called **cellular** if and only if there is a vector space decomposition $A = J'_m \oplus J'_{m-1} \oplus \cdots \oplus J'_1$ (for some m) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{l=j}^m J'_l$ gives a chain of two-sided ideals of $A : 0 = J_{m+1} \subset J_m \subset J_{m-1} \subset \cdots \subset J_1 = A$ (each of them fixed by i) and for each j ($j = m, m-1, \ldots, 1$) the quotient J_j/J_{j+1} is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j+1}.

The above chain in A is called a **cell chain**, and the modules $W(j)$, $1 \le j \le m$, which are obtained from the sections J_j/J_{j+1} of the cell chain, are called **cell modules** of the cellular algebra A. It has been known that there is a natural bijection between isomorphism classes of simple A-modules and the set $\Lambda_0 := \{\lambda \mid 1 \le \lambda \le m$ such that $J_\lambda^2 \not\subseteq J_{\lambda+1}\}$. The inverse of this bijection is given by sending such a λ to the top of the cell module $W(\lambda)$ (see [GL], [KX1]).

Assume that the cardinality of Λ_0 is n, which equals the number of nonisomorphic simple A-modules. For convenience in the proofs, later on, we relabel the original cell chain as follows:

$$0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(i+1,0)} = J_{(i,s(i)+1)} \subset J_{(i,s(i))} \subset \\ \cdots \subset J_{(i,1)} \subset J_{(i,0)} = J_{(i-1,s(i-1)+1)} \subset \cdots \subset J_{(1,0)} = A,$$

where the ideals $J_{(i,0)}, 1 \le i \le n$, are just those ideals J_λ in the original cell chain with $\lambda \in \Lambda_0$, and $s(i)$ denotes the number of ideals J_μ in the original cell chain satisfying $J_{(i+1,0)} \subsetneq J_\mu \subsetneq J_{(i,0)}$. The cell module associated to the cell ideal $J_{(i,j)}/J_{(i,j+1)}$, in which $(i,j) \in \Lambda := \{(i,j) | 1 \le i \le n,\ 0 \le j \le s(i)\}$, will be denoted by $W(i,j)$. Note that for each cell ideal $J_{(i,0)}/J_{(i,1)}$, there is a primitive idempotent e_i of A such that $J_{(i,0)} = Ae_iA + J_{(i,1)}$ and, moreover, $W(i,0) \simeq Ae_i/J_{(i,1)}e_i$. The latter has a simple top, which is denoted by $S(i)$. Thus $S(1),\ S(2),\ \ldots,\ S(n)$ form a complete set of nonisomorphic simple A-modules. For each i, denote by $P(i)$ and $I(i)$ the projective cover and the injective envelope of $S(i)$, respectively.

We denote by $[X : S(k)]$ the Jordan-Hölder multiplicity of $S(k)$ in an A-module X. It is well known that $[X : S(k)] = \dim_K \mathrm{Hom}_A(P(k), X)$ if K is a splitting field for A. For a cellular algebra A, define $d_{(i,j)k} = [W(i,j) : S(k)]$ for all $(i,j) \in \Lambda$ and $k \in \Lambda_0$.

The following lemma collects some known facts from [GL] on cellular algebras which will be used in the sequel.

Lemma 2.2. *Let A be a cellular K-algebra with cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$. Then we have the following:*

(a) $d_{(i,j)k} = 0$ unless $i \ge k$, and $d_{ii} := d_{(i,0)i} = 1$, where $(i,j) \in \Lambda$ and $k \in \Lambda_0$. Moreover, K is a splitting field for A.

(b) Let $P = Ae_k$, $1 \le k \le n$. Then P has a cell module filtration $0 = J_{(n+1,0)}e_k \subset J_{(n,s(n))}e_k \subset \cdots \subset J_{(1,0)}e_k = P$ such that the factor modules $J_{(i,j)}e_k/J_{(i,j+1)}e_k$ are isomorphic to the modules $\bigoplus_{d_{(i,j)k}} W(i,j)$, in which we put $J_{(i,s(i)+1)} = J_{(i+1,0)}$. $\square$

We remark that the factor module $J_{(i,j)}e_k/J_{(i,j+1)}e_k$ appearing in the above lemma may be zero, and this occurs if and only if $d_{(i,j)k}$ is also zero.

In order to introduce another lemma we need one more notation. Let A be a cellular algebra with respect to an involution i and X an A-module. Following [KX1], we define the **dual** X^* of X to be the A-module $\mathrm{Hom}_K(i(X), K)$, where $i(X)$ is equal to X as a vector space, but with the right A-module structure given by $x{\cdot}a{=}i(a)x$ for all $x \in X$ and $a \in A$.

Observe that the functor $*$ is a self-dual functor, and furthermore, it has the following properties.

Lemma 2.3. *Let A be a cellular K-algebra with involution i. Then:*

(*a*) *For any simple A-module $S(j)$, we have that $S(j)^* \simeq S(j)$ and $P(j)^* \simeq I(j)$.*

(*b*) $\dim_K Hom_A(X, Y) = \dim_K Hom_A(Y^*, X^*)$ *for any A-modules X and Y.*

(*c*) *For any A-module M and any positive integer m we have that* $(M/\operatorname{rad}^m(M))^* \simeq \operatorname{soc}^m(M^*)$. *Furthermore, the mth radical layer of M is isomorphic to the mth socle layer of M^*.*

Proof. The assertions (a) and (b) are consequences of the dual functor and the known fact that $Ae_k \simeq Ai(e_k)$ (see [GL]). We need only to prove (c). It is not difficult to show by induction on m that $(M/\mathrm{rad}^m(M))^* \simeq \mathrm{soc}^m(M^*)$. To prove the second part of the assertion (c), it suffices to show that both the mth radical layer of M and the mth socle layer of M^* have the same composition factors. Applying the previous two assertions in this lemma and the fact that $(M/\mathrm{rad}^m(M))^* \simeq \mathrm{soc}^m(M^*)$, we have the following equality:

$$\dim_K \mathrm{Hom}_A(M/\mathrm{rad}^m(M), I(j)) = \dim_K \mathrm{Hom}_A(P(j), \mathrm{soc}^m(M^*)).$$

Since K is a splitting field for A by Lemma 2.2, the left-hand side of the equality is equal to the multiplicity of $S(j)$ as a composition factor in the first m radical layers of M, and the right-hand side is equal to the multiplicity of $S(j)$ as a composition factor in the first m socle layers of M^*. The second part of the assertion (c) follows by subtraction. □

Based on the above two lemmas we now can prove Theorem 1.1.

Proof of Theorem 1.1. In the following we fix a cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(i+1,0)} = J_{(i,s(i)+1)} \subset J_{(i,s(i))} \subset \cdots \subset J_{(i,1)} \subset J_{(i,0)} = J_{(i-1,s(i-1)+1)} \subset \cdots \subset J_{(1,0)} = A$. We first consider the case of P being indecomposable projective. Thus we can write P as $P(i) = Ae_i$ for some primitive idempotent $e_i \in A$.

(a)⇒(b). Since $P(i)$ is also an indecomposable injective module, we can assume that $\mathrm{soc}(P(i)) \simeq S(j)$ for some j; then $P(i) \simeq I(j)$. This means by Lemma 2.3 that $\mathrm{soc}(P(j)) \simeq \mathrm{top}(P(j)^*) \simeq \mathrm{top}(I(j)) \simeq S(i)$. Suppose that $J_{(s,t)}$ is the minimal ideal (on inclusion) in the cell chain satisfying $J_{(s,t)}e_j \neq 0$. Then by Lemma 2.2, $P(j)$ has a cell module filtration: $0 \subset J_{(s,t)}e_j \subset J_{(s,t-1)}e_j \subset \cdots \subset Ae_j$. Since $J_{(s,t)}e_j \simeq \bigoplus_{d_{(s,t)j}} W(s,t)$, we have that $W(s,t)$ is a submodule of Ae_j. Thus we have that $\mathrm{soc}(W(s,t)) \simeq S(i)$ and $d_{(s,t)i} \geq 1$. The latter implies that $J_{(s,t)}e_i \neq 0$. Let $J_{(s',t')}$ be the minimal ideal in the cell chain such that $J_{(s',t')}e_i \neq 0$. Obviously, $J_{(s',t')} \subset J_{(s,t)}$. There is a cell module filtration of $P(i)$ as follows: $0 \subset J_{(s',t')}e_i \subset J_{(s',t'-1)}e_i \subset \cdots \subset Ae_i$, which means that $W(s',t')$ is a submodule of Ae_i; furthermore, $\mathrm{soc}(W(s',t')) \simeq S(j)$. Hence, we have that $d_{(s',t')j} \neq 0$. Thus $J_{(s',t')}e_j/J_{(s',t'+1)}e_j \neq 0$; in particular, $J_{(s',t')}e_j \neq 0$. It follows

that $J_{(s,t)} \subset J_{(s',t')}$ since $J_{(s,t)}$ is the minimal ideal in the cell chain such that $J_{(s,t)}e_j \neq 0$. Therefore, we get that $J_{(s,t)} = J_{(s',t')}$ and $W(s,t) = W(s',t')$. Thus $S(j) \simeq \operatorname{soc}(W(s',t')) = \operatorname{soc}(W(s,t)) \simeq S(i)$; that is, $\operatorname{soc}(P) \simeq \operatorname{top}(P)$, which is condition (b).

(b)⇒(a). Since $\operatorname{soc}(P(i)) \simeq \operatorname{top}(P(i)) \simeq S(i)$, we have a short exact sequence

$$0 \longrightarrow P(i) \longrightarrow I(i) \longrightarrow C \longrightarrow 0 \tag{1}$$

with $P(i) \longrightarrow I(i)$ an injective envelope of $P(i)$. Applying the functor $*$, the above exact sequence induces an exact sequence

$$0 \longrightarrow C^* \longrightarrow I(i)^* \longrightarrow P(i)^* \longrightarrow 0.$$

It therefore follows from Lemma 2.3 that the sequence

$$0 \longrightarrow C^* \longrightarrow P(i) \longrightarrow I(i) \longrightarrow 0 \tag{2}$$

is exact. Comparing (1) and (2), we have that $P(i) \simeq I(i)$, which means that P is also an injective module.

(c)⇒(b). This is obvious.

(a)⇒(c). Since (a) and (b) are equivalent, we have that $P(i) \simeq I(i) \simeq P(i)^*$ according to the condition (a). Thus (c) follows by Lemma 2.3.

We now consider the case of an arbitrary projective module P. Write $P = P(1)^{r_1} \oplus P(2)^{r_2} \oplus \cdots \oplus P(n)^{r_n}$, where r_i, $1 \leq i \leq n$, are nonnegative integers. Since the direct summands of an injective module are injective, it follows that (a) implies (c) by the previous proof. The condition (c) evidently implies (b). Suppose now that $\operatorname{soc}(P)$ is isomorphic to $\operatorname{top}(P)$. Then there is an exact sequence $0 \longrightarrow P \longrightarrow I \longrightarrow C' \longrightarrow 0$ with $I = I(1)^{r_1} \oplus I(2)^{r_2} \oplus \cdots \oplus I(n)^{r_n}$, which is similar to (1). By the same argument as the above (b)⇒(a), we have that P is injective as well. This finishes the proof. □

As an application of Theorem 1.1, we readily get the following dual statement on injective modules.

Theorem 1.1*. *Let A be a cellular algebra and I an injective A-module. Then the following conditions are equivalent:*

(*a*) *I is projective;*

(*b*) $\operatorname{soc}(I) \simeq \operatorname{top}(I)$*;*

(*c*) *the mth socle layer of I is isomorphic to the mth radical layer of I for each positive integer m.*

Proof. Notice that if I is injective, then I^* is projective, and vice-versa. Thus the theorem follows directly from Theorem 1.1 and Lemma 2.3. □

To state the following result, we need one more notation. Let A be an algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. For any A-A-bimodule M, we define a Cartan matrix $C(M) = (c_{ij})$, where $c_{ij} = [Me_i : S(j)]$. Note that if A is viewed as an A-A-bimodule, then $C(A)$ coincides with the usual definition of the Cartan matrix of the algebra A.

Corollary 2.4. *Let A be a cellular algebra over a field and P a projective-injective A-module. Then $[P : S(j)] = [P/\operatorname{rad}^m(P) : S(j)] + [P/\operatorname{soc}^m(P) : S(j)]$. In particular, if A is a self-injective cellular algebra, then $C(A) = C(A/\operatorname{rad}^m(A)) + C(A/\operatorname{soc}^m(A))$, where m is any nonnegative integer and $C(M)$ denotes the Cartan matrix of an A-A-bimodule M.*

Proof. The first part follows directly from Theorem 1.1, and the second part is an easy consequence of the first part. □

For a self-injective cellular algebra, we also have the following property.

Proposition 2.5. *Let A be a self-injective cellular algebra with a cell chain $0 \subset J_m \subset J_{m-1} \subset \cdots \subset J_1 = A$. Then the cell module $W(m)$ corresponding to J_m is simple.*

Proof. Since A is a self-injective cellular algebra, each indecomposable projective A-module P has a simple socle that is isomorphic to the top of P. Suppose that $S(i)$ is a composition factor of $W(m)$. Then $J_m e_i$, which is isomorphic to $\bigoplus_{d_{m,i}} W(m)$, is a nonzero submodule of Ae_i. This forces that $d_{m,i} = 1$ and $\mathrm{soc}(W(m)) \simeq S(i)$. Hence the cell module $W(m)$ is simple. □

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Department of Mathematics, Beijing Normal University, 100875 Beijing, People's Republic of China

E-mail address: yongzhic@263.net

Current address: State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, Beijing 100084, People's Republic of China